INTEGRAL METHODS IN LOW-FREQUENCY ELECTROMAGNETICS

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Nowadays, most standard problems in low-frequency electromagnetics are modeled via Maxwell's equations and solved by suitable finite element methods (FEMs). This partial differential equations (PDEs)-based approach is used in virtually all modern commercial codes (OPERA, MagNet, FLUX, and others), and its theoretical background can be found in numerous books and other references. Less frequently, finite difference methods (FDMs) are also used to solve PDE-based models—however, these methods are restricted to very simple geometries and lack the option of automatic adaptivity (mesh refinement aimed at the improvement of local resolution), and thus they are not real competitors to finite element methods.

Regardless of the quality of the numerical method used, the PDE-based approach has generic limitations that make it impractical for various important problem classes. These problems, typically, are not widely advertised in the literature since they hardly can be tackled by means of existing commercial or academic software. We can give the following examples:

- Multiscale problems involving geometrically incommensurable subdomains such as, for example, thin conductors of one-dimensional nature, coils built of such conductors, two-dimensional charged surfaces, and/or three-dimensional objects. In such situations, the application of FEMs is problematic due to meshing and other problems.

- The above-mentioned difficulties escalate if some parts of the computational arrangement are moving. Then the computational domain changes in time, and the need for frequent remeshing makes the application of FEMs impractical. In contrast to this, integral methods typically do not require meshing in all parts of the computational
domain, such as in the air surrounding charged electrical objects, and thus they can handle motion naturally.

- Problems with uncertain geometries and/or uneasily implementable boundary conditions. As a simple example, let us mention the magnetic field of a time-variable current carrying massive conductor of an arbitrary cross section. In addition to the meshing problems mentioned above, the FEM requires either an appropriate choice of an artificial boundary at a sufficient distance from the solved system or the implementation of some suitable open-boundary technique. These problems are not present in integral models, as the boundary conditions are included in the kernel functions of the corresponding integrals.

Provided that the solved problems are linear and involve homogeneous media, the integral approach is able to avoid many difficulties of PDE-based methods. Historically, integral methods have been used much less frequently in computational electromagnetics compared to PDE-based models. For a long time PDE-based models have attracted more attention than the integral ones since the latter lead to large, fully populated (dense) matrices that are difficult to handle numerically. However, the situation in the domain is changing as progress is being made in the development of higher-order methods that lead to a significant reduction of the number of degrees of freedom, and thus the dense matrices become much smaller and easier to handle. The higher-order methods have a lot of computational potential that has not been explored yet.

The aim of our book is to summarize the current state-of-the-art knowledge on integral methods in low-frequency electromagnetics. It includes theory as well as a lot of examples, which we expect to be interesting for the electrical engineering community. We also expect that readers will appreciate our effort to present the field in a broader context of coupled problems with the dominance of electromagnetic fields, such as induction heating. All computations presented in the book are done by means of our own codes and a significant portion of our own original new results is included. At the end of the book we also discuss novel integral techniques of higher order of accuracy, which undoubtedly represent the future in this field.

We expect that this book will attract new attention to integral methods within the electrical engineering community.

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