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FOUNDATIONS *and* APPLICATIONS *of the* TIME VALUE *of* MONEY

PAMELA PETERSON DRAKE, FRANK J. FABOZZI

Foundations and Applications of the Time Value of Money

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Preface

We wrote this book for those who want to understand more about the math of the time value of money. We wrote this book for the novice by starting with the basics. We also take the time value of money mathematics to a more advanced level for readers who are brushing up on their finance skill in the time value of money, readers who want to learn how to perform these calculations using spreadsheets or financial calculators, and readers who are just curious how this all works.

You don't have to have a financial calculator or scientific calculator with financial functions to learn this stuff, but it helps. You don't have to have access to computer spreadsheet programs to perform the calculations, but it's not a bad idea to see how it all works in this software. You don't have to be a math genius to perform the time value of money skills. All it takes is being comfortable with basic math.

We've included some examples from personal finance situations so that you can see how you can apply these skills to help you make better financial decisions. Though this is not a personal finance book, we do at least arm you with the basic skills to approach most any financial problem that involves the time value of money.

About the Authors

PAMELA PETERSON DRAKE, PhD, CFA, is the J. Gray Ferguson Professor of Finance and Department Head, Department of Finance and Business Law in the College of Business at James Madison University. She received her PhD in finance from the University of North Carolina at Chapel Hill and her B.S. in Accountancy from Miami University. She earned the designation of Chartered Financial Analyst. Professor Drake previously taught at Florida State University (1981–2004), and was an Associate Dean at Florida Atlantic University (2004–2007). She has published numerous articles in academic journals, as well as authored and co-authored several books. Professor Drake's expertise is in financial analysis and valuation. She has been teaching students about the time value of money for almost 30 years. She has been teaching so long that she can remember when a spreadsheet was columnar yellow paper with green lines, and the only calculator was a 10-key adding machine.

FRANK J. FABOZZI, PhD, CFA, CPA, is Professor in the Practice of Finance and Becton Fellow at the Yale School of Management. Prior to joining the Yale faculty, he was a Visiting Professor of Finance in the Sloan School at MIT. Professor Fabozzi is a Fellow of the International Center for Finance at Yale University and on the Advisory Council for the Department of Operations Research and Financial Engineering at Princeton University. He is the editor of the *Journal of Portfolio Management* and an associate editor of the *Journal of Fixed Income* and *Journal of Structured Finance*. He earned a doctorate in economics from the City University of New York in 1972. In 2002, Professor Fabozzi was inducted into the Fixed Income Analysts Society's Hall of Fame and is the 2007 recipient of the C. Stewart Sheppard Award given by the CFA Institute. He earned the designation of Chartered Financial Analyst and Certified Public Accountant. He has authored and edited numerous books in finance. He has been writing and teaching about finance for over 35 years, so long that if he had invested $1 in an account earning 5% when he first started teach, it would be worth over $5.51 today. He recently tossed out the computer punch cards used for his doctoral dissertation.

Introduction

The most powerful force in the universe is compound interest.
—Albert Einstein

Understanding financial transactions, whether involving investing, borrowing, or lending, requires understanding the time value of money. The purpose of this book is to help you understand the time value of money and all the financial mathematics that go with it.

If we think about the most common transactions in someone's personal finance, we can see some of the basic financial mathematics. Consider a few examples:

- A home mortgage involves an annuity of mortgage payments to pay off the borrowed amount.
- Leasing a car involves an annuity of lease payments, along with a down payment, for the use of a vehicle for a specified period.
- Saving for retirement involves an annuity—typically in terms of the periodic saving for retirement and the periodic withdrawals from savings during retirement.
- Comparing loan terms among different financial arrangements involves determining the effective annual rate for each loan so that you can choose the lowest cost loan.

In this book, we cover the financial math that you need to address these and other financial transactions so that you can make the better, more informed financial decisions.

OUTLINE OF THE BOOK

In Part One, we cover the fundamental math. In Chapters 1 and 2, we deal with the compounding and discounting of lump sums—that is, translating single values through time. In Chapter 3 we show how valuing a series of cash flows is a simple extension of discounting or compounding lump sums. We show you how to value these cash flows today and at any point in time. In addition to valuing cash flows, we address how to calculate yields on financial transactions, which are useful when comparing different investment opportunities or financing arrangements. In Chapter 4, we demonstrate how to calculate the annual percentage rate, the effective annual rate, and the internal rate of return for financial transactions.

In Part Two, we look at different applications of the time value of money mathematics. In Chapter 5 we examine loans and how loans are amortized. We show you how you can take a payment on a loan and break it into the interest and principal repayment components. In Chapter 6, we focus on deferred annuities, which are typically associated with the retirement issue: How do we reach a specific goal? How do we save money to satisfy our needs in retirement?

In Chapter 7, we show you how you can value a bond using the mathematics involving lump sums and annuities. We also show you how you can calculate the yield on a bond and examine the sensitivity of a bond's value to a change in interest rates. In Chapter 8, we focus on the valuation of stock. We look at how simple models, which assume that dividends on a stock grow at a constant rate, can characterize the value of a stock. We also show you how to modify this simple model to capture other dividend patterns of stock to arrive at a valuation. We close the book in Chapter 9, where we look at how the time value of money mathematics has been applied to three scenarios: evaluating whether to get an MBA; deciding whether to lease or buy a car; and whether gold is a good investment. We chose these three applications so that we could demonstrate how the time value of money mathematics can be used to address personal financial planning issues. Although this is not a personal finance book—and we do not purport to give financial advice—we do want you to take away with you the basic tools and techniques that allow you to address financial problems that interest you.

We advise you to read through—and work through—Part One before attempting Part Two. Part One lays the foundation that you need for Part Two, and you may be a bit lost in Part Two without that foundation. Once you finish Part One, it won't be a problem if you skip around Part Two and take these chapters in any order that you wish.

OUR APPROACH

Throughout the book, we use alternative approaches to most every problem. We work examples using the pure mathematics—with all the gory formulas.

We do this because some readers may be able to look at the equation and realize "Oh, yeah, now I get it!" But then other readers may look at an equation and think "Oh, no, how do I stop the pain?!" For these readers, we offer calculations in table format, with calculators, with spreadsheets, and plenty of graphs. Hopefully, one of these methods will enable you to understand what is going on.

Calculators

We offer calculator and spreadsheet steps along with many of the examples. We do this because not everyone is a math purest and most everyone is practical: The calculators and spreadsheets are there to help us. We do suggest, however, that you do attempt to learn the math that lies behind the calculator program just in case a financial problem comes your way that does not fit neatly into a calculator or spreadsheet program.

We show the steps for two calculators throughout the book—the financial calculator, the Hewlett-Packard 10B and the scientific calculator, the Texas Instruments TI-83 (which is similar to the Texas Instruments TI-84 model, which is why we refer to these calculators as TI-83/84). The Hewlett-Packard 10B calculator (HP 10B) is the simplest to learn and the steps required to perform calculations are very similar to most of the other financial calculators. However, we do provide instruction in Appendix A on several other financial calculators so that you can find some instruction for your financial calculator or one similar. In addition to these financial calculators, we provide additional instruction on the TI-83/84 in this appendix as well.

If you encounter problems when using your financial calculator, check the Tips and Troubleshooting that we offer in this appendix. If you keep getting answers that disagree with ours, check to see that you set up your calculator properly and that you are executing the functions correctly.

Spreadsheets

In addition to the calculator explanations, we also provide information on how to use the financial functions in spreadsheets to perform the calculations. We refer to Microsoft Excel throughout the book, but as we explain in Appendix B, the functions operate the same as those in the free Google Docs' spreadsheet, which is available at www.google.com. We encourage you to learn how to use the spreadsheets for financial calculations because, when you begin to apply time value of money mathematics to your personal financial decision making, you may need some of the tools that these spreadsheets can provide, including graphing.

Formulas

We cannot avoid using formulas in a book that covers the time value of money mathematics. We include all the relevant formulas within the chapters—whether you want to work through these or not. We also include the formulas in Appendix C, which is a summary of the notation and of the formulas by chapter.

Problems and Examples

When you read a chapter, you will encounter three different types of problems:

- *Examples*, which are brief problems that demonstrate the calculation that was just discussed in the text. These problems are numbered sequentially throughout the chapter and the answers, including an explanation of how to get the answer, are included right there.
- *Try it! problems* are problems that you can work on your own. You can find detailed answers to these problems at the end of each chapter summary.
- *End-of-chapter problems*. There are 10 problems at the end of each chapter. We provide the detailed solutions to these problems in Appendix E, at the back of this book.

We encourage you to work all of these problems. You will notice that one chapter will build upon another chapter, so it is important for you to have a good understanding of a chapter before moving on to the next.

Glossary

Along with the math comes a bunch of terminology, so we've tried to sort it out for you. In Appendix D, we include a glossary of the terms used in this book.

THE KEYS TO LEARNING THE TIME VALUE OF MONEY

We would like to leave you with a couple of suggestions for learning the time value of money mathematics:

- **Focus on the basics.** The basic valuation equation, which we introduce you to in Chapter 1, is the heart of all of the time value of money. Learn this and you've got it made.

- **Learn at least two ways to do each problem.** You are more likely to learn the mathematics of the time value of money if you can see it from at least two angles. Learn to do each problem with at least two of the three approaches that we offer: basic math, financial calculator, or spreadsheet.
- **Practice, practice, and practice.** There is no substitute for this.

Foundations and Applications of the Time Value of Money

PART

One

The Basics of the Time Value of Money

CHAPTER 1

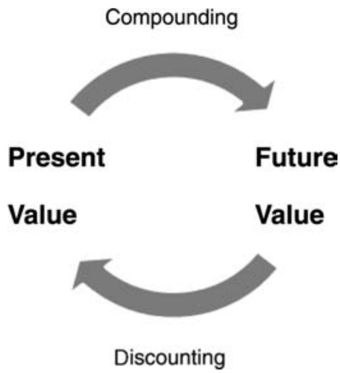
The Value of Compounding

Remember that time is money.
—Benjamin Franklin
Advice to a Young Tradesman (1748)

Most people are familiar with the Seven Wonders of the World: the Great Pyramid of Giza, the Hanging Gardens of Babylon, the Statue of Zeus at Olympia, the Temple of Artemis at Ephesus, the Mausoleum of Maussollos at Halicarnassus, the Colossus of Rhodes, and the Lighthouse of Alexandria. Supposedly, when Baron von Rothschild was asked if he could list the Seven Wonders, he said he could not. However, he did respond by saying that he could name the Eighth Wonder of the World: compound interest. Actually, labeling compound interest as the Eighth Wonder of the World has been attributed to other notable figures: Benjamin Franklin, Bernard Baruch, and Albert Einstein. Regardless of to whom we attribute this label, as you will see in this chapter, the label is appropriate.

One of the most important tools in personal finance and investing is the time value of money. Evaluating financial transactions requires valuing uncertain future cash flows; that is, determining what uncertain cash flows are worth at different points in time. We are often concerned about what a future cash flow or a set of future cash flows are worth today, though there are applications in which we are concerned about the value of a cash flow at a future point in time.

One complication is the *time value of money*: a dollar today is not worth a dollar tomorrow or next year. Another complication is that any amount of money promised in the future is uncertain, some riskier than others.



Moving money through time—that is, finding the equivalent value to money at different points in time—involves translating values from one period to another. Translating money from one period involves interest, which is how the time value of money and risk enter into the process.

Interest is the compensation for the opportunity cost of funds and the uncertainty of repayment of the amount borrowed; that is, it represents both the price of time and the price of risk. The price of time is compensation for the opportunity cost of funds—what someone could have done with the money elsewhere—and the price of risk is compensation for bearing risk. That is, the riskier the investment, the higher the interest rate.

Interest is *compound interest* if interest is paid on both the principal—the amount borrowed—and any accumulated interest. In other words, if you borrow $1,000 today for two years and the interest is 5% compound interest, at the end of two years you must repay the $1,000, plus interest on the $1,000 for two years and interest on the interest. The amount you repay at the end of two years is $1,102.50:

Repayment of principal		$1,000.00
Payment of interest on the principal—first year	5% of $1,000	50.00
Payment of interest on the principal—second year	5% of $1,000	50.00
Payment of interest in the second year on the interest from the first year	5% of $50	2.50
Total amount repaid at the end of the second year		$1,102.50

You can see the accumulation of values in Exhibit 1.1. The $2.50 in the second year is the interest on the first period's interest.

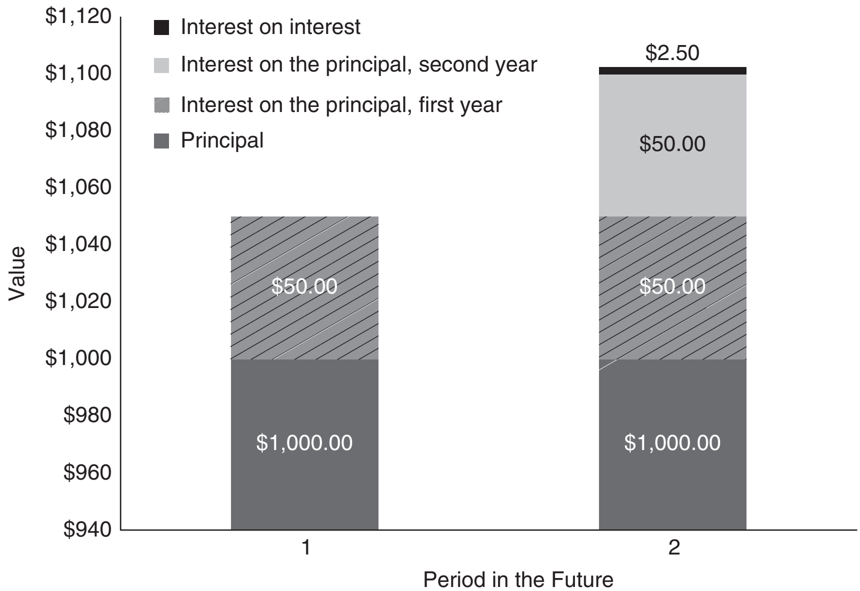


EXHIBIT 1.1 Components of the Future Value of $1,000 Invested at 5% for Two Years

We refer to translating a value today into a value in the future as *compounding*, whereas *discounting* is translating a future value into the present.

The future value is the sum of the present value and interest:

$$\text{Future value} = \text{Present value} + \text{Interest}$$

Most financial transactions involve compound interest, though there are a few consumer transactions that use *simple interest*. Simple interest is the financing arrangement in which the amount repaid is the principal amount and interest on the principal amount. That is, interest is paid only on the principal or amount borrowed. For example, if you borrow $10,000 at 5% simple interest and repay the loan after two years, you must repay the $10,000, plus two periods' interest at 5%:

$$\begin{aligned}\text{Repayment with simple interest} &= \$10{,}000 + [\$10{,}000 \times 2 \times 0.05] \\ &= \$11{,}000\end{aligned}$$

In the case of compound interest, the amount repaid has three components:

1. The amount borrowed
2. The interest on the amount borrowed
3. The interest on interest

The *basic valuation equation* is the foundation of all the financial mathematics that involves compounding, and if you understand this equation, you understand most everything in financial mathematics:

$$FV = PV(1+i)^n$$

where: FV = the future value
PV = the present value
i = the rate of interest
n = is the number of compounding periods

The term $(1 + i)^n$ is the *compound factor*. When you multiply the value today—the present value—by the compound factor, you get the future value.

We can rearrange the basic valuation equation to solve for the present value, PV:

$$PV = FV\left[\underbrace{\frac{1}{(1+i)^n}}_{\text{Discount factor}}\right] = \frac{FV}{(1+i)^n},$$

where $1 \div (1 + i)^n$ is the *discount factor*. When you multiply the value in the future by the discount factor, you get the present value.

In sum,

$$\text{Future value} = \text{Present value} \times \text{Compound factor}$$

$$\text{Present value} = \text{Future value} \times \text{Discount value}$$

The focus of this chapter is on compounding—that is, determining a value in the future. We look at discounting in the next chapter.

OF INTEREST

The word *interest* is from the Latin word *intereo,* which means "to be lost." Interest developed from the concept that lending goods or money results in a loss to the lender because he or she did not have the use of the goods or money that is loaned.

In the English language, the word *usury* is associated with lending at excessive or illegal interest rates. In earlier times, however, usury (from the Latin *usura,* meaning "to use") was the price paid for the use of money or goods.

COMPOUNDING

We begin with compounding because this is the most straightforward way of demonstrating the effects of compound interest. Consider the following example: You invest $1,000 in an account today that pays 6% interest, compounded annually. How much will you have in the account at the end of one year if you make no withdrawals? Using the subscript to indicate the year the future value is associated with, after one year you will have

$$FV_1 = \$1{,}000\,(1 + 0.06) = \$1{,}060$$

After two years, the balance is

$$\begin{aligned} FV_2 &= \$1{,}000\,(1 + 0.06)\,(1 + 0.06) = \$1{,}000\,(1 + 0.06)^2 \\ &= \$1{,}000\,(1.1236) = \$1{,}123.60 \end{aligned}$$

After five years, the balance is

$$FV_5 = \$1{,}000\,(1 + 0.06)^5 = \$1{,}000\,(1.3382) = \$1{,}338.23$$

After 10 years, the balance is

$$FV_{10} = \$1{,}000\,(1 + 0.06)^{10} = \$1{,}000\,(1.7908) = \$1{,}790.85$$

You can see the accumulation of interest from interest on the principal and interest on interest over time in Exhibit 1.2.

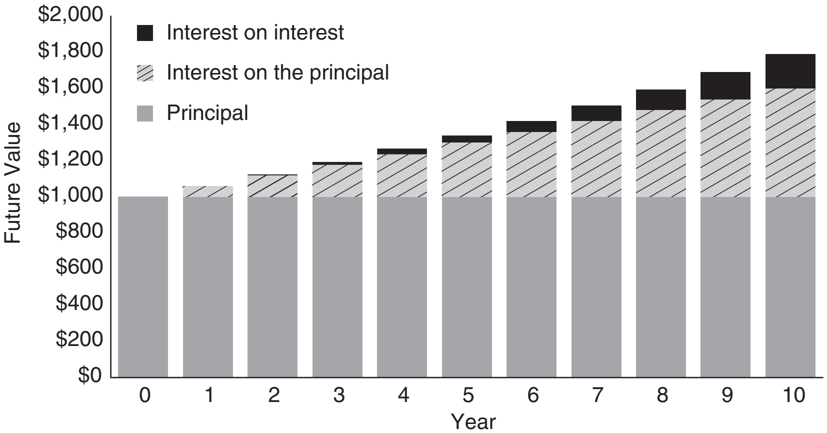


EXHIBIT 1.2 The Accumulation of Interest and Interest on Interest of a $1,000 Deposit with 6% Compound Annual Interest

If you invest $1,000 today and receive $1,790.85 at the end of 10 years, we say that you have a return of 6% on your investment. This return is an average annual return, considering compounding.

TRY IT 1.1: SAVINGS

Suppose you deposit $1,000 in an account that earns 5% interest per year. If you do not make any withdrawals, how much will you have in the account at the end of 20 years?

What if interest was not compounded interest, but rather simple interest? Then we would have a somewhat lower balance in the account after the first year. At the end of one year, with simple interest, you will have:

$$FV_1 = \$1{,}000 + [\$1{,}000\,(0.06)] = \$1{,}060$$

After two years:

$$\begin{aligned} FV_2 &= \$1{,}000 + [\$1{,}000\,(0.06)] + [\$1{,}000\,(0.06)] \\ &= \$1{,}000 + [\$1{,}000\,(0.06)\,(2)] = \$1{,}120 \end{aligned}$$