REVOLUTIONS OF GEOMETRY

MICHAEL O'LEARY
College of DuPage
Glen Ellyn, Illinois

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REVOLUTIONS OF GEOMETRY

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For my wife, Barb

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CONTENTS

Preface xi

Acknowledgments xiii

PART I FOUNDATIONS

1 The First Geometers 3

1.1 Egypt 6
1.2 Babylon 13
1.3 China 20

2 Thales 27

2.1 The Axiomatic System 29
2.2 Deductive Logic 35
2.3 Proof Writing 43

3 Plato and Aristotle 53

3.1 Form 56
3.2 Categorical Propositions 62

3.3 Categorical Syllogisms 72
3.4 Figures 77

PART II THE GOLDEN AGE

4 Pythagoras 87
4.1 Number Theory 91
4.2 The Pythagorean Theorem 98
4.3 Archytas 102
4.4 The Golden Ratio 110

5 Euclid 123
5.1 The Elements 124
5.2 Constructions 130
5.3 Triangles 138
5.4 Parallel Lines 147
5.5 Circles 159
5.6 The Pythagorean Theorem Revisited 167

6 Archimedes 173
6.1 The Archimedean Library 174
6.2 The Method of Exhaustion 182
6.3 The Method 193
6.4 Preliminaries to the Proof 204
6.5 The Volume of a Sphere 214

PART III ENLIGHTENMENT

7 François Viète 227
7.1 The Analytic Art 229
7.2 Three Problems 236
7.3 Conic Sections 246
7.4 The Analytic Art in Two Variables 257

8 René Descartes 267
8.1 Compasses 269
8.2 Method 274
8.3 Analytic Geometry 279

9 Gérard Desargues **293**

9.1 Projections 294
9.2 Points at Infinity 298
9.3 Theorems of Desargues and Menelaus 306
9.4 Involutions 312

PART IV A STRANGE NEW WORLD

10 Giovanni Saccheri **323**

10.1 The Question of Parallels 324
10.2 The Three Hypotheses 330
10.3 Conclusions for Two Hypotheses 337
10.4 Properties of Parallel Lines 340
10.5 Parallelism Redefined 349

11 Johann Lambert **353**

11.1 The Three Hypotheses Revisited 355
11.2 Polygons 360
11.3 Omega Triangles 373
11.4 Pure Reason 383

12 Nicolai Lobachevski and János Bolyai **393**

12.1 Parallel Fundamentals 397
12.2 Horocycles 404
12.3 The Surface of a Sphere 414
12.4 Horospheres 424
12.5 Evaluating the Pi Function 431

PART V NEW DIRECTIONS

13 Bernhard Riemann **443**

13.1 Metric Spaces 445
13.2 Topological Spaces 457
13.3 Stereographic Projection 464
13.4 Consistency of Non-Euclidean Geometry 471

14 Jean-Victor Poncelet **483**

14.1 The Projective Plane 486
14.2 Duality 492

14.3 Perspectivity 501
14.4 Homogeneous Coordinates 507

15 Felix Klein **519**

15.1 Group Theory 520
15.2 Transformation Groups 529
15.3 The Principal Group 535
15.4 Isometries of the Plane 543
15.5 Consistency of Euclidean Geometry 553

References 565

Index 573

PREFACE

One of the courses that had a strong influence on me as an undergraduate student was a class on the history of mathematics. It was a serious course with a calculus prerequisite. We used David M. Burton's *The History of Mathematics: An Introduction* and Morris Kline's *Mathematics: The Loss of Certainty*. I found the problems challenging and the introduction to the history and philosophy of mathematics interesting. Based on this experience from too many years ago, I believe that studying mathematics within a historical context can be very motivating for students. Although I certainly believe that *Revolutions of Geometry* is a text that any junior or senior mathematics student would find stimulating, my main concern is for those who plan to teach the subject as a profession. I believe that it is crucial that the mathematics teachers of the future are not only competent in the subject but also enjoy it.

For these reasons, this book is a chronological introduction to the history of geometry presenting the mathematics alongside the story of those who made the discoveries. With few exceptions, the text focuses on the original work of the mathematicians encountered. It strives to clarify and organize the arguments to make them more accessible. Since it is geometry, the work emphasizes proofs, so it will not be without its challenges.

Because the history of geometry dictated the topics that are included in the text, it has a wide range of material from which to choose for many different kinds of geometry courses. The text is divided into five parts.

- *Foundations.* After encountering pre-Greek geometry, we meet Thales, Plato, and Aristotle. We learn their philosophy and logic. Methods of proof that are needed later in the book are introduced here.

- *The Golden Age.* This is the mathematics of the classical Greeks. We read of the teachings of Pythagoras and his followers, review high school geometry through Euclid's *Elements*, and see firsthand how close Archimedes actually was to the calculus.

- *Enlightenment.* The focus is the work of three great French mathematicians: Viète's revolutionary contributions to algebra, Descartes' merging of algebra and geometry as he solves the Pappus Problem, and Desargues' development of projective geometry.

- *A Strange New World.* Saccheri and Lambert were close, but it took Bolyai and Lobachevski to finally answer the question of parallels. Their work is based on a new definition of what it means for lines to be parallel. Here we encounter Saccheri's three hypotheses, results concerning triangles and rectangles, horocycles, horospheres, and some trigonometry. The intellectual giants Gauss and Kant also make appearances.

- *New Directions.* As we approach the twentieth century, the redefinition of geometry continues. Riemann defined it as the study of manifolds. Poncelet returned to projective geometry, but now it is more abstract. Klein used group theory to characterize different geometries. Along the way questions of consistency become important. Through the work of Beltrami, Poincaré, Klein again, and Hilbert we learn that each of Saccheri's three hypotheses is without contradiction.

There are occasions when topics in the text use results in set theory, function theory, and linear algebra. This is due to the requirements of some of the geometry studied. These prerequisites did not prevent these topics from being included because they provide a very different view of geometry and a good review of the material.

The exercise sets at the end of each section are designed both to reinforce the material and to push the student beyond the contents presented. They are organized to coincide approximately with the order of topics in their section. No solutions for these problems are given in the text, but a student's solution manual is available, and instructors may contact Wiley for an instructor's manual. The text also has a web page that contains supplementary and review material. Its address is

www.revolutionsofgeometry.com.

Finally, this book was typeset in LaTeX and the illustrations were created with Adobe Illustrator, both by the author.

Michael O'Leary
College of DuPage
December 2009

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Thanks go the Princeton University Press for their permission to use quotations from Glen R. Morrow's translation of Proclus's *A Commentary on the First Book of Euclid's* Elements.

On a personal note, I would like to express my gratitude to my parents for their continued support and understanding; to my brother and his wife, who will teach my niece mathematics; to my dissertation advisor, Paul Eklof, who guided me through graduate school; to Robert Meyer, who introduced me to the history of mathematics; to Kenneth Mangels, who taught me non-Euclidean geometry; to David Elfman, who taught me about science and computers; to my new family, who have not seen me much lately; and to my wife, Barb, whose love and patience (and some proofreading) supported me as I played the hermit to finish this book.

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PART I

FOUNDATIONS

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CHAPTER 1

THE FIRST GEOMETERS

The exact origin of the ancient civilization of **Egypt** is unknown, but it definitely dates to sometime before the year 4000 B.C. There were two kingdoms along the Nile, an upper kingdom and a lower. It is said that somewhere between the years 3500 to 3000 B.C. the legendary Menes unified the two kingdoms and so formed the First Dynasty. He is further claimed to have founded the city of Memphis and dedicated a temple there to the creator-god called Ptah. For the next 500 to 1000 years Egypt grew in strength as a state and a society. The Third Dynasty, which lasted from 2700 to 2625 B.C, saw the reign of King Djoser. With the assistance of his chief architect and physician, Imhotep, they directed the construction of the first great stone building, the Step Pyramid at Saqqara. It sat in the middle of a great mortuary complex and was intended to see Djoser safely into the afterlife. During the Fourth Dynasty, approximately a half century later, Khufu (Cheops) reigned. Little is known about this pharaoh. The few texts that speak of Khufu range from portraying him as a amicable monarch interested in magic to a cruel slave master driven to build a magnificent monument to himself. This monument was the Great Pyramid. It currently stands the tallest of a group of three pyramids on the Giza plateau and is the only surviving member of the Seven Wonders of the Ancient World. It originally stood 481 feet tall but has lost 30 feet off its height. Its square base has sides that

Revolutions of Geometry. By Michael O'Leary

measure about 755 feet each with the difference between the longest and shortest sides being only 8 inches. Scholars estimate that it was built using 2.3 million stone blocks. The average weight of each is around 2.5 tons. In July 1798, Napoleon calculated that the stones from the three pyramids at Giza could be used to build a wall around France 1 foot wide and 12 feet high (Grimal 1992, 63–67, 389; Boardman 1982, 14, 145; Clayton 1995, 37, 46).

According to the ancient Greek historian Herodotus, there was a great king of Egypt whose name was Sesostris. It was claimed that he conquered vast portions of Asia and Europe and erected pillars on which were inscribed his name and country. After he had conquered the world, Sesostris returned home and focused on domestic issues. One of his major achievements was supposedly an extensive canal system that brought water from the Nile to his people. It is unlikely that this Sesostris ever existed. There were pharaohs who shared his name. Sesostris I (Senusret I) reigned at the beginning of the 12th Dynasty (*c.* 1990 B.C.), and he was succeed by his son, Amenemhat II, who was followed by his son, Sesostris II, and then his grandson, Sesostris III. However, neither of these can be the Sesostris of Herodotus. Instead, the pharaoh described by Herodotus was probably a legend, based on the reigns of Rameses II and Sety I, that later Egyptians created at a time when the power of Egyptian civilization began to wane (Herodotus 1987, II; Shaw 2000, 160–166, 295–302).

To the northeast of Egypt between the Tigris and Euphrates rivers was the land of Mesopotamia. This was the location of the fabled cities of Babylon, Ur, Susa, and Kish. About 4000 B.C. the **Sumerians** migrated to Mesopotamia and settled in its southern regions. Their land became known as Sumer and its capital was Ur. Around 2200 B.C. they lost their independence to the **Akkadians**, who were Semites from northern Mesopotamia. At the time the Akkadians were ruled by Sargon. His political life began in the court of King Ur-Zababa of Kish. Sargon rose to power and eventually gained control of Sumer. Many cuneiform tablets laud Sargon as a great ruler and warrior who supposedly conquered Asia Minor and the Mediterranean. Although the Sumerians and the Akkadians had there own identities, it is common to refer to them under the single term **Babylonian** (Leick 2003, 6, 102, 113; Crawford 2004, 29, 33).

Probably the best known ruler of Mesopotamia was Hammurabi. He reigned from 1792 to 1750 B.C. as king of Babylon during the First Dynasty. Beginning with a small region around the city, Hammurabi gradually extended his domain. This was accomplished by a combination of military conquests and temporary diplomatic alliances that were ended in betrayal by the king. By 1755 B.C., after 30 years of strengthening his position, Hammurabi had gained control of all of Mesopotamia, and Babylon would become the leading power in western Asia. As he accomplished this, Hammurabi established a strong central government, invested in irrigation, and strengthened the walls of the city of Babylon. In order to establish justice throughout his realm, a sequence of laws was published in all of his major cities. Known as the Code of Hammurabi, it contained 282 separate laws, covering everything from serious crimes such as murder to everyday business transactions. Several copies survive, including one found in 1902 at Susa on an 8-foot-tall stone monument of polished black diorite. The death of Hammurabi saw the beginning of the decline of the First Dynasty, which fell when Mursili, king of the Hittites, attacked Babylon by surprise in 1595 B.C. or 1499 B.C. (Leick 2003, 47–48, 57–58).

Although there was some contact between the Egyptian and Mesopotamian civilizations, on the other side of the continent in **China** there developed a society independent of Western contacts. The Chinese civilization began as a collection of small city-states along the banks of the Yellow River. Near the end of the twelfth century B.C., an ancient tribe began to grow in strength. It later became known as the Zhou dynasty and was led by King Wen, a well-respected leader who had ambitions to conquer the Shang dynasty, which was then dominant. King Wen died before he could finish his plan, but his son, Prince Wu, would later join forces with neighboring tribes to completely defeat the Shang (Shaughnessy 1999, 307–309).

The now King Wu established the Zhou dynasty with Gaojing as its capital. This dynasty lasted from 1122 to 256 B.C. This long time period is divided into two periods, the first being the Western Zhou (1122–771 B.C). The Western Zhou society was feudal in the sense that the king appointed rulers as his representatives over local jurisdictions. However, the relationship between subject and king was modeled on that of kinship as opposed to the impersonal, contractual relationship that arose in Europe. This system allowed the land over which the dynasty exercised control to expand, but it expanded too quickly and led to a collapse. This resulted in the Zhou capital's moving to Luoyi. This is the beginning of the Eastern Zhou period. This second period is divided further into the Spring and Autumn Period (771–481 B.C) and the Warring States Period (402–221 B.C). Although both periods saw many wars where states vied for power, significant cultural progress was made. For example, iron was introduced, new farming and military techniques were developed which included the use of horses, and writing was further developed (Roberts 1999, 9–12).

During the Warring States Period the Qin state, which had existed since the 8th century B.C., developed into a power that could rival the Zhou. The Qin were led by Qin Shi Huangdi who wished to unify all of China under his leadership. At the time there were six other states that were in a seemingly constant state of war. They were all defeated by Qin Shi Huangdi, ushering in the Qin dynasty and bringing order and unification to China. The dynasty only lasted from 221 B.C. to 206 B.C., but it was a significant period in Chinese history during which Chinese society was restructured so that the peasants now owned the land but were taxed by the government and a new code of laws was established which applied to all equally. In addition, Qin Shi Huangdi standardized weights, measures, and the font used in writing, and he led the construction of many civil projects, including a network of walls which led to construction of the Great Wall. However, to control knowledge the emperor ordered the destruction of many books and ordered the execution of any scholar who dissented. By the end of his reign Qin Shi Huangdi had became a tyrant. At the time of his death in 210 B.C., the people had grown dissatisfied living under the harsh conditions imposed by the emperor's construction projects. Millions had been drafted to build the projects, and this led to a rebelion. Qin Shi Huangdi's successor was killed, after which the Qin quickly surrendered. The year was 206 B.C., and civil war soon followed. The winner was a leader of the peasants named Liu Bang. His victory led to the establishment in 205 B.C. of the Han dynasty which lasted for 426 years. During this time a strong central government was established, yet many of the harsh policies of the Qin were removed and taxes reduced. This was also a time when the teachings of Confucius were made the state religion, the arts and literature flourished, and Chinese philosophy was at its peak (Roberts 1999, 19–29).

1.1 EGYPT

It is believed by most historians of mathematics that the Egyptians began studying geometry due to a need for a reliable method of measuring areas of land. This view is supported by the testimony of Herodotus.

> The priests also say that it was this king [Sesostris] who divided the land among all the Egyptians, giving to each man as an allotment a square, equal in size; for the king derived his revenues, as he appointed the payment therefore of a yearly tax. If the river should carry off a portion of the allotment, the man would come to the king himself and signify what had happened, whereupon the king sent men to inspect and remeasure by how much the allotment had grown less, so that for the future it should pay proportionally less of the assigned tax. I think it was from this that geometry was discovered and came to Greece. (Herodotus 1987, 2.109, 175)

The ancient Egyptians used lengths of rope with knots at regular intervals to measure land. By stretching the ropes along the needed dimensions and applying the appropriate formula the area would be found. From this practice the subject of geometry received its name. It is a combination of two Greek words: *geo*, meaning "earth," and *metron*, meaning "measure" (Burton 1985, 56–57).

What we know of Egyptian geometry can be traced to a surprising source. In 1798, Napoleon invaded Egypt in an attempt to protect French trade interests in the region and to weaken British ties to India. Napoleon brought a group of scholars along with his army. This was evidently a propaganda ploy to cast a favorable light on his intentions. Whatever his true motivation, his decision led to the discovery of the **Rosetta Stone** in 1799. This is a stone slab on which is carved a message in Greek, in Demotic, and in the Egyptian writing called **hieroglyphics**, a picture-based system that was used in formal settings. The meaning behind the Egyptian hieroglyphics had long eluded scholars. Finding the Rosetta Stone unlocked their meaning since they were now able to compare the hieroglyphics with ancient Greek (Boyer and Merzbach 1991, 10–11; Burton 1985, 36).

In 1858, Scotland native A. Henry Rhind made another discovery. In the tombs of Thebes in Luxor he found a scroll and purchased it for his collection. Scholars estimate that it was written about 1650 B.C., placing its creation during the reign of King Apophis I of the 17th Dynasty. The author of the scroll, Ahmes, claimed that his work was a copy of a document from the Third Dynasty, possibly originating from Imhotep. The scroll material was **papyrus**, sheets made by pressing the pith of the papyrus plant and then slicing off the pieces that were needed. Since it is organic, papyrus is not a good material for preservation purposes. Although the climate in Egypt is dry, which helped to preserve the scrolls, most finds have deteriorated to some degree. This was the case with Rhind's discovery. The scroll was 1 foot wide and 18 feet long but missing its middle portions. The content of the scroll that remained was seen to be written in the ancient Egyptian writing called **hieratic**. Beginning as a simple modification of hieroglyphics, hieratic later became a language in which each syllable is represented as a symbol and each symbol represents a particular idea. Because of this connection to hieroglyphics, the Rosetta Stone made possible the translation of the scroll that we know as the **Rhind Mathematical Papyrus** (Kline 1972, 15–16).

About four years after Rhind found his papyrus, the American Edwin Smith was studying in Egypt when he purchased what he believed to be a scroll on ancient Egyptian medicine. However, it was a fraud. Someone had taken pieces from a papyrus and created what looked like a scroll, but only the outside layers were ancient Egyptian. The inside was fake. Smith was an expert on Egypt, and it seems that it would be unlikely for someone in his position to be fooled by a forgery. We must remember, however, that the condition of any found papyri will be very fragile. It takes time and care to open an ancient scroll. Fortunately for posterity, Smith kept his find, and it went with the rest of his collection to the New York Historical Society after his death. In 1922 experts at the museum determined that the fraudulent scroll was assembled using the lost pieces of Rhind's papyrus! The pieces were shipped to the British Museum, where experts were able to fit together most of the missing section and complete its translation (Burton 1985, 36).

Other papyri discovered are the Golenischev Papyrus, usually called the **Moscow Papyrus** because it is now a part of the collection of the Museum of Fine Arts in Moscow, and a second purchase of Rhind's, which we know as the **Egyptian Mathematical Leather Scroll**. All of these finds have greatly aided our understanding of ancient Egyptian mathematics. The ancient Egyptian's wrote their mathematics in the form of questions followed by answers. The goal was to calculate values associated with government and commerce, specifically the computation of areas and volumes. They used specific numbers to solve problems, and they did not generalize their results. Any formulas that they did use were not deduced from first principles but were discovered empirically by trial and error. Because of this, some of their formulas were not accurate but only approximations, which, considering their needs, often sufficed. The greatest mystery regarding the mathematics of the ancient Egyptians is that for about two millennia they made very little progress beyond their initial work (Burton 1985, 36, 52, 57, 64).

Approximations

The mathematical scrolls show that the Egyptians did not distinguish between algebra and geometry. They viewed mathematics simply as a tool by which to obtain needed values. They had formulas for basic areas and volumes. This included the volumes of a cube and a cylinder. There is some debate in academic circles as to whether the Egyptians ever gave any justifications for any of the steps in their solutions. None have been discovered, but it is clear that the Egyptians did not consider geometry to be a logical system (Kline 1972, 19–20).

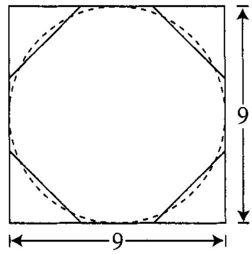

Figure 1.1 Using a 9 × 9 square to estimate the area of a circle.

The Rhind Mathematical Papyrus provides an example of an ancient Egyptian formula that did satisfactorily approximate its intended value. Take a square with sides of length 9. Trisect each of the sides, and then by joining consecutive points at the corners, create a regular octagon. This is illustrated in Figure 1.1, but we must note that the diagram found in the papyrus was a simple sketch of this diagram without the circle. The scribe of the papyrus considered the octagon a good estimate for the area of the circle. To find the octagon's area, we simply remove the four equilateral triangles from the square. Therefore, its area is

$$9^2 - 4 \cdot \frac{9}{2} = 63.$$

It is conjectured that this provided the ancient Egyptians a reason to believe that for a circle of diameter d,

$$A \approx \left(\frac{8d}{9}\right)^2. \tag{1.1}$$

The Egyptians could have relied on the octagonal estimate in support of Equation 1.1 because it yields an area of 64 square units when $d = 9$, which is close to the Rhind approximation. Furthermore, notice what Equation 1.1 implies about the Egyptian value of π. Since $A = \pi d^2/4$, we substitute Equation 1.1 to find

$$\pi \approx 3\frac{13}{81} = 3.16049\ldots. \tag{1.2}$$

This is very close to the common approximation of $3\frac{1}{7}$ (Burton 1985, 58).

An example of an equation from ancient Egypt that did not provide very good estimates is one that was found at the temple of Horus at Edfu. The priests of the temple received gifts in the form of patches of land. Each plot was in the form of a quadrilateral but was not necessarily rectangular. If we let a, b, c, and d represent the lengths of consecutive sides of the quadrilateral, their formula is

$$A = \frac{1}{4}(a + c)(b + d).$$

Notice that this amounts to multiplying the average of opposite sides together. It is not a bad estimate for regions that are nearly rectangular, but for most quadrilaterals it is an example of an instance when the Egyptians did try to generalize a result but did it poorly (Burton 1985, 57).

Pyramids

Herodotus recounts an encounter that he had with one of the priests (Burton 1985, 62–63). He claims that he was told that the Great Pyramid was designed so that the area of a square with side equal to the height of the pyramid is equal to the area of one of the faces of the pyramid. Using Figure 1.2, this is equivalent to writing

$$h^2 = \frac{1}{2}(2ba) = ab,$$

where $2b$ is the length of one of the sides of its square base, a is the altitude of one of the faces, and h is the height of the pyramid. From this we can calculate the ratio b/a by first noting that since

$$h^2 + b^2 = a^2,$$

we have

$$a^2 - b^2 = ab.$$

Dividing through by a^2 and a little rearranging yields

$$\left(\frac{b}{a}\right)^2 + \frac{b}{a} - 1 = 0.$$

This is a quadratic equation with b/a serving as the unknown. Thus, since both a and b are positive,

$$\frac{b}{a} = \frac{\sqrt{5} - 1}{2}. \tag{1.3}$$

Geometrically, a **pyramid** is a solid consisting of a collection of polygons joined at their edges. Recall that a **polygon** is a closed plane figure consisting of line segments called **sides**. An endpoint of a side is called a **vertex**. A polygon with n sides is often called an ***n*-gon**. If all the sides of the polygon are of the same magnitude and if the same can be said of the interior angles, the polygon is **regular**. In a pyramid one

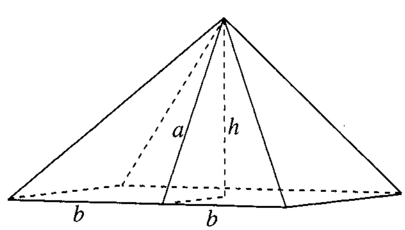

Figure 1.2 The Great Pyramid.

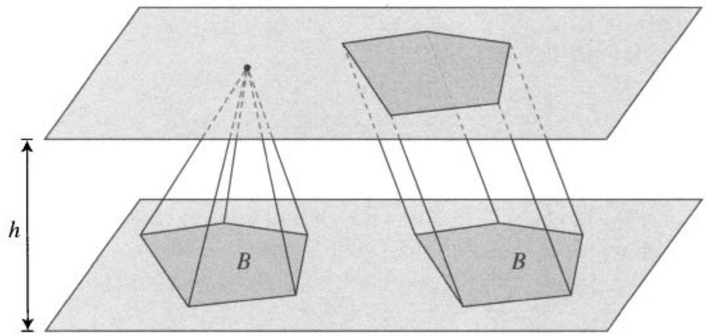

Figure 1.3 A pyramid and a prism of height h.

polygon serves as the **base** and the others are **faces**. The faces all share a common vertex called the **apex** of the pyramid. The distance between that apex and the plane containing the base is the **height** of the pyramid (Figure 1.3).

Similar in definition to a pyramid is the prism. A **prism** is a solid consisting of a collection of polygons joined at their edges. Two of the polygons are the **bases** of the prism. They are identical in shape and size (in other words, they are **congruent**) and lie in parallel planes. The other polygons are called **faces** and are formed by joining corresponding vertices of the bases (Figure 1.3). The **height** of a prism is the distance between the two parallel planes. In general, if B is the area of the base of a prism and h is its height,

$$\text{area(prism)} = Bh,$$

and the volume of a pyramid is one third the volume of a prism with the same base and height of the pyramid. This means that

$$\text{area(pyramid)} = \frac{Bh}{3}.$$

The formula for the volume of a **square pyramid** (one with a square base) quickly follows from this.

The Moscow Papyrus contains what is one of the most significant mathematical discoveries of the ancient Egyptians. The translation of Gillings (1972, 188) reads as follows:

> Method of calculating a truncated pyramid.
> If it is said to thee, a truncated pyramid of 6 [cubits] in height,
> Of 4 [cubits] of the base, by 2 on the top,
> Reckon thou with this 4, squaring. Result 16.
> Double thou this 4. Result 8.
> Reckon thou with this 2, squaring. Result 4.
> Add together this 16, with this 8, and with this 4. Result 28.
> Calculate thou [one third] of 6. Result 2.
> Calculate thou with 28 twice. Result 56.
> Lo! It is 56! Thou has found rightly.

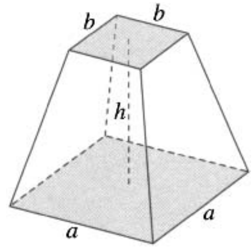

Figure 1.4 A frustum or truncated pyramid.

The Egyptians found an accurate formula for the area of a truncated square pyramid. Such a volume is called a **frustum** and is formed by taking the volume of a solid, such as a pyramid or a cone, that lies between two parallel planes. If a is the length of a side of the base of the truncated pyramid and b the length of the square at its top (Figure 1.4), then

$$V = \frac{h}{3}(a^2 + ab + b^2). \tag{1.4}$$

How Equation 1.4 was obtained is only conjecture, but a reasonable guess is that the ancient Egyptians checked such cases as when b is half of a or a third of a, and from these results concluded a formula that they could then check further in other cases (Gillings 1972, 190–191; Burton 1985, 59–60).

We can prove Equation 1.4, however, by setting up a coordinate system on a cross-section of the frustum with the origin at O (Figure 1.5). The slope of $\overline{BA}$ is $2h/(b-a)$, so the equation for the line through B and A is

$$y = \frac{2h}{b-a}x + \frac{ah}{a-b}.$$

Therefore, the height of the pyramid without the top removed is $ah/(a-b)$. The formula for the frustum is simply the volume of the large pyramid minus the volume

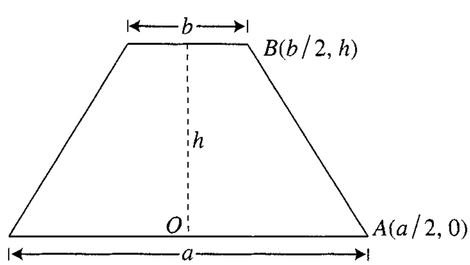

Figure 1.5 Deriving the formula for the volume of the truncated pyramid.

of the small pyramid at the tip. We calculate as follows:

$$\begin{aligned} V &= \frac{1}{3}\left(\frac{ha}{a-b}\right)a^2 - \frac{1}{3}\left(\frac{ha}{a-b} - h\right)b^2 \\ &= \frac{h}{3}\left(\frac{a^3}{a-b} - \frac{ab^2}{a-b} + b^2\right) \\ &= \frac{h}{3(a-b)}(a^3 - ab^2 + b^2[a-b]) \\ &= \frac{h}{3(a-b)}(a^3 - b^3) \\ &= \frac{h}{3}(a^2 + ab + b^2). \end{aligned}$$

Exercises

1. The Egyptians estimated the area of a quadrilateral that had consecutive sides of length a, b, c, and d with the formula $A = \frac{1}{4}(a+c)(b+d)$.
   (a) Show that if the quadrilateral is a rectangle, its area equals A.
   (b) Explain why A is equal to the product of the average of opposite sides of the quadrilateral.
   (c) Find the dimensions of a quadrilateral where the absolute value of the error between its true area and the estimate given by A is 25 percent.

2. Confirm that the ancient Egyptian approximation for π is as given in Equation 1.2.

3. Assuming $\pi = 3$, the ancients inaccurately used the formula

$$V = \frac{h}{12}\left(\frac{3}{2}[D+d]\right)^2$$

for the volume of a truncated cone where D is the diameter of the base, d is the diameter of the top, and h is the height. Derive the correct formula.

4. The **lateral surface area** of a solid is the total area of the sides of the solid excluding any bases. Find the lateral surface area of a truncated square pyramid of height h and bases with sides of lengths a and b.

5. From the Rhind Mathematical Papyrus we know that the Egyptians were able to calculate the slopes of the sides of a pyramid. This slope is called a **seked**. Its value is equal to the horizontal distance for every unit rise (Gillings 1972, 185–187).
   (a) Find the seked of a square pyramid of height 429 feet and with a base that is 618 feet long.
   (b) If the seked of a square pyramid is 5 feet per unit and its base is 400 feet long, what is the height of the pyramid?
   (c) Find an equation that expresses the seked in terms of the base and height of the pyramid.

6. Since the volume of a pyramid is equal to one third of the volume of a prism with base and height equal to that of the pyramid, it would have been simple for the ancient Egyptians to determine an empirical formula for a pyramid.
   (a) Describe an experiment that the Egyptians could reasonably have used to determine the formula for the volume of a pyramid.
   (b) Gillings (1972, 190) gives two conjectures concerning a mathematical approach to determining the formula. The first involves taking a right pyramid with a square base. Divide it into four congruent oblique pyramids by passing two planes through the vertex of the pyramid such that the planes are perpendicular to each other and the pyramid's base and divide the base into four equal squares. Show that the resulting pyramids can be arranged into a square from which the formula can be derived.
   (c) The second method is to start with a cube. Draw six pyramids such that each has as its base a side of the cube and each has a height that is equal to half the length of a side of the cube. It is best to draw these pyramids so that they share a common vertex on the interior of the cube. Each of these is called a **Juel's pyramid**. Use this construction to confirm the formula for the volume of a pyramid.

7. A **regular** pyramid has a regular base.
   (a) Find the lateral surface area of a regular hexagonal pyramid with height 10 and base with sides of length 4.
   (b) Find the pyramid's total area and volume.

8. Use calculus to confirm the formula for the volume of a square pyramid.

1.2 BABYLON

As the Rosetta Stone aided in the translation of the hieroglyphics of ancient Egypt, the 1870 discovery at the Behistun Cliff in Iran did the same for the Babylonian language. An inscription in the side of the cliff announced the ascension to power of Darius. It was written in the Persian, Elamitic, and Babylonian languages. The symbols were all various forms of **cuneiform**. Typically, cuneiform was written on clay tablets when they were still soft using a stylus with a triangular tip to impress wedges into the clay. This is how cuneiform received its name, for this word is from the Latin for "wedge." The clay was then dried. As opposed to papyrus, clay survives well over time, so there were many museums with large collections of tablets with cuneiform writing on them. However, they were not easily deciphered. In particular, there were large collections at the British Museum, the Louvre, Yale, Columbia, and the University of Pennsylvania, with many needing translation. Because of the Behistun Cliff and the later work of Otto Neugebauer and Paul Thureau-Dangin, the tablets were discovered to contain ancient Babylonian mathematics (Boyer and Merzbach 1991, 9–10; Kline 1972, 5).

Approximately two thirds of the clay tablets dated from 1800 to 1600 B.C, and they originated from the Sumerian and Akkadian civilizations. But for a few exceptions, the Babylonian mathematics proved to be significantly ahead of the Egyptians. Both civilizations used empirical methods to find their results, but the Babylonians were more theoretical and able to solve algebraic equations. They used symbols for unknown quantities and their equations would include one or more variables. They even knew a form of the Quadratic Formula. Despite this, their solutions were on a case-by-case basis. They had no concept of letting coefficients be represented as unknown parameters. More importantly, they had no concept of a proof. The closest they came would be a step-by-step explanation of how to solve an equation, yet this would be given without any justification (Burton 1985, 67, 69; Kline 1972, 13–14).

Pythagorean Triples

Although geometry as a subject on its own right played little role in Babylonian mathematics, there is one particular clay tablet that is of interest. Deciphered by Neugebauer and Abraham Sachs in 1945, it gave clear evidence that the Babylonians around the years 1900 to 1600 B.C. knew the Pythagorean Theorem. The tablet is called Plimpton 322 since it is housed in the G. A. Plimpton collection at Columbia University. The tablet contains a table displaying four columns of numbers, three of which are translated into decimal numbers as follows:

x	z	
119	169	1
3367	*4825	2
4601	6649	3
12709	18541	4
65	97	5
319	481	6
2291	3541	7
799	1249	8
*481	769	9
4961	8161	10
45	75	11
1679	2929	12
*161	289	13
1771	3229	14
56	*106	15

If we check some of the numbers, we find that the difference between the the square of a value from the z column and the square from the x column yields another perfect square: for instance,

$$169^2 - 119^2 = 120^2.$$

Four of the lines of the table did not follow this pattern (indicated by a *). These discrepancies have been shown to be due to errors on the part of the scribe. The right-most column served simply to enumerate the lines of the table (Burton 1985, 77–78; Kline 1972, 4–5).