Homogenization of Coupled Phenomena in Heterogeneous Media

Jean-Louis Auriault
Claude Boutin
Christian Geindreau
Homogenization of Coupled Phenomena in Heterogenous Media
This page intentionally left blank
Homogenization of Coupled Phenomena in Heterogenous Media

Jean-Louis Auriault
Claude Boutin
Christian Geindreau
Main notations ............................................. 17

Introduction ............................................. 21

PART ONE. UPSCALING METHODS .......................... 27

Chapter 1. An Introduction to Upscaling Methods .......... 29

1.1. Introduction ........................................... 29
1.2. Heat transfer in a periodic bilaminate composite .... 30
  1.2.1. Transfer parallel to the layers ....................... 31
  1.2.2. Transfer perpendicular to the layers ................. 33
  1.2.3. Comments ........................................ 35
  1.2.4. Characteristic macroscopic length ................... 35
1.3. Bounds on the effective coefficients .................... 36
  1.3.1. Theorem of virtual powers .......................... 36
  1.3.2. Minima in the complementary power and potential power 38
  1.3.3. Hill principle .................................... 39
  1.3.4. Voigt and Reuss bounds ............................ 40
    1.3.4.1. Upper bound: Voigt .......................... 40
    1.3.4.2. Lower bound: Reuss .......................... 42
  1.3.5. Comments ........................................ 44
  1.3.6. Hashin and Shtrikman’s bounds ....................... 45
  1.3.7. Higher-order bounds ................................ 46
1.4. Self-consistent method ................................ 46
  1.4.1. Boundary-value problem ............................ 47
  1.4.2. Self-consistent hypothesis .......................... 48
  1.4.3. Self-consistent method with simple inclusions .... 49
Chapter 2. Heterogenous Medium: Is an Equivalent Macroscopic Description Possible? ........................................ 55

2.1. Introduction ....................................... 55
2.2. Comments on techniques for micro-macro upscaling ............ 56
  2.2.1. Homogenization techniques for separated length scales .... 57
  2.2.2. The ideal homogenization method .......................... 59
2.3. Statistical modeling ................................... 60
2.4. Method of multiple scale expansions .......................... 61
  2.4.1. Formulation of multiple scale problems .................... 61
    2.4.1.1. Homogenizability conditions ........................ 61
    2.4.1.2. Double spatial variable ............................ 62
    2.4.1.3. Stationarity, asymptotic expansions ................ 64
  2.4.2. Methodology ....................................... 65
  2.4.3. Parallels between macroscopic models for materials with periodic and random structures ................ 68
    2.4.3.1. Periodic materials ............................... 68
    2.4.3.2. Random materials with a REV ....................... 68
  2.4.4. Hill macro-homogenity and separation of scales ............ 69
2.5. Comments on multiple scale methods and statistical methods .... 69
  2.5.1. On the periodicity, the stationarity and the concept of the REV 69
  2.5.2. On the absence of, or need for macroscopic prerequisites .... 70
  2.5.3. On the homogenizability and consistency of the macroscopic description ........................................ 71
  2.5.4. On the treatment of problems with several small parameters ... 72

Chapter 3. Homogenization by Multiple Scale Asymptotic Expansions ........................................ 75

3.1. Introduction ....................................... 75
3.2. Separation of scales: intuitive approach and experimental visualization 75
  3.2.1. Intuitive approach to the separation of scales .......... ... 75
  3.2.2. Experimental visualization of fields with two length scales ..... 78
    3.2.2.1. Investigation of a flexible net .................... 78
    3.2.2.2. Photoelastic investigation of a perforated plate .... 81
3.3. One-dimensional example .................................. 84
  3.3.1. Elasto-statics .................................... 85
3.3.1.1. Equivalent macroscopic description ............... 86
3.3.1.2. Comments .................................. 89
3.3.2. Elasto-dynamics ........................................ 91
3.3.2.1. Macroscopic dynamics: \( P_l = O(\varepsilon^2) \) ............. 92
3.3.2.2. Steady state: \( P_l = O(\varepsilon^3) \) .................. 95
3.3.2.3. Non-homogenizable description: \( P_l = O(\varepsilon) \) ....... 95
3.3.3. Comments on the different possible choices for spatial variables ........................................ 97
3.4. Expressing problems within the formalism of multiple scales .............. 100
3.4.1. How do we select the correct mathematical formulation based on the problem at hand? ......................... 100
3.4.2. Need to evaluate the actual scale ratio \( \varepsilon_r \) ................. 101
3.4.3. Evaluation of the actual scale ratio \( \varepsilon_r \) ..................... 102
3.4.3.1. Homogenous treatment of simple compression .......... 103
3.4.3.2. Point force in an elastic object ..................... 104
3.4.3.3. Propagation of a harmonic plane wave in elastic composites ........................................ 104
3.4.3.4. Diffusion wave in heterogenous media ................. 105
3.4.3.5. Conclusions to be drawn from the examples .......... 106

PART TWO. HEAT AND MASS TRANSFER .................................. 107

Chapter 4. Heat Transfer in Composite Materials ......................... 109

4.1. Introduction ............................................. 109
4.2. Heat transfer with perfect contact between constituents ........... 109
4.2.1. Formulation of the problem ................................ 110
4.2.2. Thermal conductivities of the same order of magnitude ........ 113
4.2.2.1. Homogenization .................................. 113
4.2.2.2. Macroscopic model ............................... 117
4.2.2.3. Example: bilaminate composite .................... 119
4.2.3. Weakly conducting phase in a connected matrix: memory effects ........................................ 121
4.2.3.1. Homogenization .................................. 122
4.2.3.2. Macroscopic model ............................... 124
4.2.3.3. Example: bilaminate composite .................... 125
4.2.4. Composites with highly conductive inclusions embedded in a matrix ........................................ 126
4.2.4.1. Homogenization .................................. 127
4.2.4.2. Macroscopic model ............................... 129
4.3. Heat transfer with contact resistance between constituents .......... 130
4.3.1. Model I – Very weak contact resistance .................... 132
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3.2. Model II – Moderate contact resistance</td>
<td>133</td>
</tr>
<tr>
<td>4.3.3. Model III – High contact resistance</td>
<td>135</td>
</tr>
<tr>
<td>4.3.4. Model IV – Model with two coupled temperature fields</td>
<td>138</td>
</tr>
<tr>
<td>4.3.5. Model V – Model with two decoupled temperature fields</td>
<td>140</td>
</tr>
<tr>
<td>4.3.6. Example: bilaminate composite</td>
<td>141</td>
</tr>
<tr>
<td>4.3.7. Choice of model</td>
<td>142</td>
</tr>
</tbody>
</table>

### Chapter 5. Diffusion/Advection in Porous Media

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1. Introduction</td>
<td>143</td>
</tr>
<tr>
<td>5.2. Diffusion-convection on the pore scale and estimates</td>
<td>143</td>
</tr>
<tr>
<td>5.3. Diffusion dominates at the macroscopic scale</td>
<td>146</td>
</tr>
<tr>
<td>5.3.1. Homogenization</td>
<td>146</td>
</tr>
<tr>
<td>5.3.1.1. Boundary value problem for $c^*(0)$</td>
<td>146</td>
</tr>
<tr>
<td>5.3.1.2. Boundary value problem for $c^*(1)$</td>
<td>147</td>
</tr>
<tr>
<td>5.3.1.3. Boundary value problem for $c^*(2)$</td>
<td>148</td>
</tr>
<tr>
<td>5.3.2. Macroscopic diffusion model</td>
<td>148</td>
</tr>
<tr>
<td>5.4. Comparable diffusion and advection on the macroscopic scale</td>
<td>149</td>
</tr>
<tr>
<td>5.4.1. Homogenization</td>
<td>149</td>
</tr>
<tr>
<td>5.4.1.1. Boundary value problems for $c^<em>(0)$ and $c^</em>(1)$</td>
<td>149</td>
</tr>
<tr>
<td>5.4.1.2. Boundary value problem for $c^*(2)$</td>
<td>149</td>
</tr>
<tr>
<td>5.4.2. Macroscopic diffusion-advection model</td>
<td>150</td>
</tr>
<tr>
<td>5.5. Advection dominant at the macroscopic scale</td>
<td>151</td>
</tr>
<tr>
<td>5.5.1. Homogenization</td>
<td>151</td>
</tr>
<tr>
<td>5.5.1.1. Boundary value problem for $c^*(0)$</td>
<td>151</td>
</tr>
<tr>
<td>5.5.1.2. Boundary value problem for $c^*(1)$</td>
<td>151</td>
</tr>
<tr>
<td>5.5.1.3. Boundary value problem for $c^*(2)$</td>
<td>153</td>
</tr>
<tr>
<td>5.5.2. Dispersion model</td>
<td>154</td>
</tr>
<tr>
<td>5.6. Very strong advection</td>
<td>154</td>
</tr>
<tr>
<td>5.7. Example: Porous medium consisting of a periodic lattice of narrow parallel slits</td>
<td>155</td>
</tr>
<tr>
<td>5.7.1. Analysis of the flow</td>
<td>156</td>
</tr>
<tr>
<td>5.7.2. Determination of the dispersion coefficient</td>
<td>157</td>
</tr>
<tr>
<td>5.8. Conclusion</td>
<td>159</td>
</tr>
</tbody>
</table>

### Chapter 6. Numerical and Analytical Estimates for the Effective Diffusion Coefficient

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1. Introduction</td>
<td>161</td>
</tr>
<tr>
<td>6.2. Effective thermal conductivity for some periodic media</td>
<td>162</td>
</tr>
<tr>
<td>6.2.1. Media with spherical inclusions, connected or non-connected</td>
<td>162</td>
</tr>
</tbody>
</table>
PART THREE. NEWTONIAN FLUID FLOW THROUGH RIGID POROUS MEDIA

Chapter 7. Incompressible Newtonian Fluid Flow Through a Rigid Porous Medium

7.1. Introduction .............................................. 197
7.2. Steady-state flow of an incompressible Newtonian fluid in a porous medium: Darcy’s law ........................................ 199
  7.2.1. Darcy’s law ........................................... 201
  7.2.2. Comments on macroscopic behavior ..................... 203
    7.2.2.1. Physical meaning of the macroscopic quantities ...... 203
    7.2.2.2. Structure of the macroscopic law .................... 204
    7.2.2.3. Study of the underlying problem .................... 205
    7.2.2.4. Properties of $K^*$ .................................. 205
    7.2.2.5. Energetic consistency ............................... 206
  7.2.3. Non-homogenizable situations .......................... 206
    7.2.3.1. Case where $Q_L = O(\varepsilon^{-1})$ .................. 207
7.2.3.2. Case where \( Q_L = O(\varepsilon^{-3}) \) ................................................. 208
7.3. Dynamics of an incompressible fluid in a rigid porous medium .... 209
  7.3.1. Local description and estimates ........................................... 209
  7.3.2. Macrocscopic behavior: generalized Darcy’s law ................. 211
  7.3.3. Discussion of the macroscopic description ............................... 213
    7.3.3.1. Physical meaning of macroscopic quantities .................. 213
    7.3.3.2. Energetic consistency ............................................. 213
    7.3.3.3. The tensors \( \mathbb{H}^* \) and \( \Lambda^* \) are symmetric ...... 215
    7.3.3.4. Low-frequency behavior .......................................... 215
    7.3.3.5. Additional mass effect ........................................... 215
    7.3.3.6. Transient excitation: Dynamics with memory effects ....... 216
    7.3.3.7. Quasi-periodicity ................................................. 216
  7.3.4. Circular cylindrical pores ............................................... 216
  7.4. Appearance of inertial non-linearities .................................. 220
    7.4.1. Macroscopic model .................................................. 221
    7.4.2. Macroscopically isotropic and homogenous medium ............... 224
    7.4.3. Conclusion .......................................................... 226
  7.5. Summary ............................................................................. 226

Chapter 8. Compressible Newtonian Fluid Flow Though a Rigid Porous
Medium ................................................................................................. 229

  8.1. Introduction ............................................................................. 229
  8.2. Slow isothermal flow of a highly compressible fluid ................. 229
    8.2.1. Estimates ....................................................................... 230
    8.2.2. Steady-state flow .......................................................... 231
    8.2.3. Transient conservation of mass ........................................ 235
  8.3. Wall slip: Klinkenberg’s law ................................................... 238
    8.3.1. Pore scale description and estimates ................................ 238
    8.3.2. Klinkenberg’s law .......................................................... 240
    8.3.3. Small Knudsen numbers ................................................... 241
    8.3.4. Properties of the Klinkenberg tensor \( \mathbb{H}^k \) ..................... 243
      8.3.4.1. \( \mathbb{H}^k \) is positive ............................................... 243
      8.3.4.2. Symmetries ............................................................. 244
  8.4. Acoustics in a rigid porous medium saturated with a gas .......... 245
    8.4.1. Harmonic perturbation of a gas in a porous medium ............. 246
    8.4.2. Analysis of local physics ............................................... 247
    8.4.3. Non-dimensionalization and renormalization ....................... 249
    8.4.4. Homogenization ............................................................ 251
      8.4.4.1. Pressure and temperature ......................................... 251
      8.4.4.2. Velocity field ........................................................ 252
8.4.4.3. Macroscopic conservation of mass .......................... 252
8.4.5. Biot-Allard model .................................................. 253

Chapter 9. Numerical Estimation of the Permeability of Some Periodic
Porous Media ................................................................. 257

9.1. Introduction ............................................................ 257
9.2. Permeability tensor: recap of results from periodic homogenization . 259
9.3. Steady state permeability of fibrous media .......................... 259
9.3.1. Microstructures ....................................................... 259
9.3.2. Transverse permeability ........................................... 260
9.3.2.1. Mesh, velocity fields and microscopic pressure fields ... 261
9.3.2.2. Transverse permeability \( K_T \) .................................. 262
9.3.3. Longitudinal permeability ......................................... 264
9.3.3.1. Mesh, velocity fields ............................................. 264
9.3.3.2. Longitudinal permeability \( K_L \) .............................. 264
9.4. Steady state and dynamic permeability of granular media .......... 267
9.4.1. Microstructures ....................................................... 267
9.4.2. Methodology .......................................................... 267
9.4.3. Steady state permeability .......................................... 269
9.4.4. Dynamic permeability .............................................. 269
9.4.4.1. Effect of frequency .............................................. 269
9.4.4.2. Low-frequency approximation ................................. 270
9.4.4.3. High-frequency approximation ................................. 272

Chapter 10. Self-consistent Estimates and Bounds for Permeability ...... 275

10.1. Introduction ............................................................ 275
10.1.1. Notation and glossary ............................................. 277
10.2. Intrinsic (or steady state) permeability of granular and fibrous media 278
10.2.1. Summary of results obtained through periodic
        homogenization ....................................................... 279
10.2.1.1. Global and local descriptions – energetic consistency ... 280
10.2.1.2. Connections between the micro- and macroscopic
        descriptions .......................................................... 281
10.2.2. Self-consistent estimate of the permeability of granular media . 281
10.2.2.1. Formulation of the self-consistent problem .................. 281
10.2.2.2. General expression for the fields in the inclusion ............ 283
10.2.2.3. Boundary conditions .......................................... 285
10.2.3. Solution and self-consistent estimates .......................... 288
10.2.3.1. Pressure approach: \( \varpi_p \) field .......................... 288
10.2.3.2. Velocity approach: \( \varpi_v \) field .......................... 289
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.6.1.1</td>
<td>Comparison between the trapping constant and intrinsic permeability</td>
<td>330</td>
</tr>
<tr>
<td>10.6.1.2</td>
<td>Self-consistent estimate of the trapping constant for granular media</td>
<td>331</td>
</tr>
<tr>
<td>10.6.2</td>
<td>Diffusion-trapping in the transient regime</td>
<td>332</td>
</tr>
<tr>
<td>10.6.3</td>
<td>Steady-state diffusion-trapping regime in media with a finite absorptivity</td>
<td>333</td>
</tr>
<tr>
<td>10.7</td>
<td>Conclusion</td>
<td>334</td>
</tr>
<tr>
<td>PART FOUR</td>
<td>Saturated Deformable Porous Media</td>
<td>337</td>
</tr>
<tr>
<td>Chapter 11</td>
<td>Quasi-statics of Saturated Deformable Porous Media</td>
<td>339</td>
</tr>
<tr>
<td>11.1</td>
<td>Empty porous matrix</td>
<td>340</td>
</tr>
<tr>
<td>11.1.1</td>
<td>Local description</td>
<td>340</td>
</tr>
<tr>
<td>11.1.2</td>
<td>Equivalent macroscopic behavior</td>
<td>342</td>
</tr>
<tr>
<td>11.1.2.1</td>
<td>Boundary-value problem for ( u^*(0) )</td>
<td>342</td>
</tr>
<tr>
<td>11.1.2.2</td>
<td>Boundary-value problem for ( u^*(1) )</td>
<td>343</td>
</tr>
<tr>
<td>11.1.2.3</td>
<td>Boundary-value problem for ( u^*(2) )</td>
<td>344</td>
</tr>
<tr>
<td>11.1.3</td>
<td>Investigation of the equivalent macroscopic behavior</td>
<td>345</td>
</tr>
<tr>
<td>11.1.3.1</td>
<td>Physical meaning of quantities involved in macroscopic description</td>
<td>345</td>
</tr>
<tr>
<td>11.1.3.2</td>
<td>Properties of the effective elastic tensor</td>
<td>346</td>
</tr>
<tr>
<td>11.1.3.3</td>
<td>Energetic consistency</td>
<td>348</td>
</tr>
<tr>
<td>11.1.4</td>
<td>Calculation of the effective coefficients</td>
<td>348</td>
</tr>
<tr>
<td>11.2</td>
<td>Deformable saturated porous medium</td>
<td>349</td>
</tr>
<tr>
<td>11.2.1</td>
<td>Local description and estimates</td>
<td>350</td>
</tr>
<tr>
<td>11.2.2</td>
<td>Diphasic macroscopic behavior: Biot model</td>
<td>352</td>
</tr>
<tr>
<td>11.2.2.1</td>
<td>Boundary-value problem for ( u^*(0) )</td>
<td>352</td>
</tr>
<tr>
<td>11.2.2.2</td>
<td>Boundary-value problem for ( p^<em>(0) ) and ( v^</em>(0) )</td>
<td>352</td>
</tr>
<tr>
<td>11.2.2.3</td>
<td>Boundary-value problem for ( u^*(1) )</td>
<td>353</td>
</tr>
<tr>
<td>11.2.2.4</td>
<td>First compatibility equation</td>
<td>354</td>
</tr>
<tr>
<td>11.2.2.5</td>
<td>Second compatibility equation</td>
<td>355</td>
</tr>
<tr>
<td>11.2.2.6</td>
<td>Macroscopic description</td>
<td>355</td>
</tr>
<tr>
<td>11.2.3</td>
<td>Properties of the macroscopic diphasic description</td>
<td>355</td>
</tr>
<tr>
<td>11.2.3.1</td>
<td>Properties of macroscopic quantities and effective coefficients</td>
<td>355</td>
</tr>
<tr>
<td>11.2.3.2</td>
<td>The coupling between (11.31) and (11.32) is symmetric, ( \alpha = \gamma )</td>
<td>356</td>
</tr>
<tr>
<td>11.2.3.3</td>
<td>The tensor ( \alpha^* ) is symmetric</td>
<td>356</td>
</tr>
<tr>
<td>11.2.3.4</td>
<td>The coefficient ( \beta^* ) is positive, ( \beta^* &gt; 0 )</td>
<td>357</td>
</tr>
<tr>
<td>11.2.3.5</td>
<td>Specific cases</td>
<td>357</td>
</tr>
</tbody>
</table>
11.2.3.6. Homogenous matrix material .................................. 357
11.2.3.7. Homogenous and isotropic matrix material and
macroscopically isotropic matrix .................................. 358
11.2.3.8. Diphasic consolidation equations: Biot model ........... 359
11.2.3.9. Effective stress ........................................... 361
11.2.3.10. Compressible interstitial fluid .......................... 361
11.2.4. Monophasic elastic macroscopic behavior: Gassman model . 362
11.2.5. Monophasic viscoelastic macroscopic behavior ............ 363
11.2.6. Relationships between the three macroscopic models ....... 365

Chapter 12. Dynamics of Saturated Deformable Porous Media ....... 367

12.1. Introduction .................................................... 367
12.2. Local description and estimates ................................ 368
12.3. Diphasic macroscopic behavior: Biot model .................. 370
12.4. Study of diphasic macroscopic behavior ....................... 374
  12.4.1. Equations for the diphasic dynamics of a saturated deformable
         porous medium ............................................. 374
  12.4.2. Rheology and dynamics ................................... 375
  12.4.3. Additional mass ......................................... 376
  12.4.4. Transient motion ....................................... 376
  12.4.5. Small pulsation \( \omega \) .................................. 376
  12.4.6. Dispersive waves ....................................... 376
12.5. Macroscopic monophasic elastic behavior: Gassman model ... 377
12.6. Monophasic viscoelastic macroscopic behavior ............... 378
12.7. Choice of macroscopic behavior associated with a given material and
      disturbance .................................................. 380
  12.7.1. Effects of viscosity ..................................... 382
    12.7.1.1. Transition from diphasic behavior to elastic behavior .. 382
    12.7.1.2. Transition from viscoelastic behavior to elastic behavior 383
  12.7.2. Effect of rigidity of the porous skeleton .................. 384
  12.7.3. Effect of frequency ..................................... 384
    12.7.3.1. Low-dispersion \( P_1 \) and \( S \) waves ................. 384
    12.7.3.2. Dispersive \( P_2 \) wave ............................. 385
  12.7.4. Effect of pore size ..................................... 385
  12.7.5. Application example: bituminous concretes ............... 385

Chapter 13. Estimates and Bounds for Effective Poroelastic Coefficients . 389

13.1. Introduction .................................................... 389
13.2. Recap of the results of periodic homogenization .......... 389
13.3. Periodic granular medium .................................... 391
13.3.1. Microstructure and material ......................................... 391
13.3.2. Effective elastic tensor $c$ ........................................ 392
  13.3.2.1. Methodology .................................................. 392
  13.3.2.2. Compressibility and shear moduli .......................... 394
  13.3.2.3. Degree of anisotropy ...................................... 396
  13.3.2.4. Young’s modulus and Poisson’s ratio ....................... 396
13.3.3. Biot tensor ......................................................... 398
13.4. Influence of microstructure: bounds and self-consistent estimates ... 398
  13.4.1. Voigt and Reuss bounds ....................................... 399
  13.4.2. Hashin and Shtrikman bounds ................................ 399
  13.4.3. Self-consistent estimates .................................... 400
  13.4.4. Comparison: numerical results, bounds and self-consistent 
          estimates ......................................................... 401
13.5. Comparison with experimental data .................................. 403

Chapter 14. Wave Propagation in Isotropic Saturated Poroelastic Media  . 407
14.1. Introduction .......................................................... 407
14.2. Basics ....................................................................... 408
  14.2.1. Notation ............................................................. 408
  14.2.2. Comments on the parameters .................................. 410
    14.2.2.1. Elastic coefficients ..................................... 410
    14.2.2.2. Dynamic permeability .................................. 410
  14.2.3. Degrees of freedom and dimensionless parameters .......... 411
14.3. Three modes of propagation in a saturated porous medium .......... 412
  14.3.1. Wave equations ................................................... 413
  14.3.2. Elementary wave fields: plane waves ......................... 416
    14.3.2.1. Homogeneous plane waves ................................ 416
    14.3.2.2. Inhomogenous plane waves ............................... 417
  14.3.3. Physical characteristics of the modes ......................... 419
    14.3.3.1. Low frequencies: $f \ll f_c$ .............................. 419
    14.3.3.2. High frequencies: $f \gg f_c$ ............................ 421
    14.3.3.3. Full spectrum .............................................. 423
14.4. Transmission at an elastic-poroelastic interface .................. 423
  14.4.1. Expression for the conditions at the interface ............... 426
  14.4.2. Transmission of compression waves ............................ 428
14.5. Rayleigh waves ......................................................... 430
14.6. Green’s functions ..................................................... 432
  14.6.1. Source terms ..................................................... 432
  14.6.2. Determination of the fundamental solutions ................. 433
  14.6.3. Fundamental solutions in plane geometry ..................... 437
14.6.4. Symmetry of the Green’s matrix, and reciprocity theorem . . . . 438
14.6.5. Properties of radiated fields . . . . . . . . . . . . . . . . . . . . 439
  14.6.5.1. Far-field – near-field – quasi-static regime . . . . . . . . 441
  14.6.5.2. Decomposition into elementary waves . . . . . . . . . . 442
14.6.6. Energy and moment sources: explosion and injection . . . . . 442
14.7. Integral representation . . . . . . . . . . . . . . . . . . . . . . . . 445
14.8. Dislocations in porous media . . . . . . . . . . . . . . . . . . . . 448

Bibliography . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 453

Index . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 473
Main notations

Operators

⊗  tensor product
:  doubly contracted tensor product
·  scalar product
×  vector product
△, △_x  Laplacian, Laplacian with respect to the variable x
div, div_ x  divergence, divergence with respect to the variable x
grad, grad_ x  gradient, gradient with respect to the variable x
curl, curl_ x  curl, curl with respect to the variable x
⟨a⟩  mean of a over the period Ω
⟨a⟩_Ω_ α  mean of a over the domain Ω_ α

Dimensions, bases, spatial variables

Ω  period
Ω_ α  domain occupied by constituent α
∂Ω  boundary of the period
Γ  interface between two constituents
l  microscopic length [m]
L  macroscopic length [m]
ε = l/L  separation of scales parameter
e_i  unit vector – orthonormal basis
n  unit vector normal to the interface Γ
X = X_i e_i  physical spatial variable [m]
x^*_ i = X / L_c = x^*_ i e_i  macroscopic dimensionless variable [-]
y^*_ i = X / l_c = y^*_ i e_i  microscopic dimensionless variable [-]
**Dimensionless numbers**

- \( B \) Biot number [-]
- \( \mathcal{K}n \) Knudsen number [-]
- \( Pe \) Péclet number [-]
- \( Re \) Reynolds number [-]
- \( Rt \) transient Reynolds number [-]
- \( S \) Strouhal number [-]

**Properties and physical quantities**

\[ a = a_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l \]

- elastic tensor [MPa]

\[ c = c_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l \]

- concentration [-]

\[ c_\alpha \]

- volume fraction of constituent \( \alpha \) [-]

\[ C_\alpha \]

- heat capacity of constituent \( \alpha \) [J/(Kg.K)]

\[ D(\nu) = D_{ij} e_i \otimes e_j \]

- strain rate tensor [s⁻¹]

\[ D^{\text{eff}} = D^{\text{eff}}_{ij} e_i \otimes e_j \]

- effective diffusion tensor [m²/s]

\[ D^{\text{dis}} = D^{\text{dis}}_{ij} e_i \otimes e_j \]

- effective dispersion tensor [m²/s]

\[ D^{\text{mol}} = D^{\text{mol}}_{ij} e_i \otimes e_j \]

- molecular diffusion tensor [m²/s]

\[ e(u) = e_{ij} e_i \otimes e_j \]

- strain tensor [-]

\[ E \]

- Young’s modulus [MPa]

\[ h \]

- inverse of the thermal contact resistance

\[ H = H_{ij} e_i \otimes e_j \]

- inverse of the intrinsic permeability tensor [m²]

\[ H(\omega) = H_{ij}(\omega) e_i \otimes e_j \]

- inverse of the dynamic permeability tensor \( K \) [m⁻²]

\[ H^R, H^I \]

- real and imaginary parts of the \( H \) tensor

\[ H(\omega) = K^{-1} = H^I + iH^R \]

- inverse of the dynamic permeability of an isotropic medium

\[ H^k(\omega) = H^k_{ij}(\omega) e_i \otimes e_j \]

- inverse of the dynamic hydraulic conductivity tensor \( \Lambda = \eta K(\omega) \)

\[ H^k \]

- Klinkenberg tensor [m²]

\[ I \]

- identity tensor

\[ K \]

- intrinsic permeability of an isotropic medium [m²]

\[ K^k \]

- intrinsic steady state permeability tensor [m²]

\[ K_k = K^k_{ij} e_i \otimes e_j \]

- Klinkenberg permeability tensor [m²]

\[ K(\omega) = K^R + iK^I \]

- dynamic permeability of an isotropic medium [m²]

\[ K(\omega) = K_{ij}(\omega) e_i \otimes e_j \]

- dynamic permeability tensor [m²]

\[ K^R, K^I \]

- real and imaginary parts of the \( K \) tensor

\[ M \]

- form factor

\[ p \]

- fluid pressure [Pa]

\[ t \]

- time [s]

\[ T_\alpha \]

- temperature of constituent \( \alpha \) [°K]

\[ u_s = u_{si} e_i \]

- solid displacement [m]
\( \mathbf{v} = v_i \mathbf{e}_i \)  
\( \alpha \)  
\( \delta \)  
\( \eta \)  
\( \lambda_{\alpha} \)  
\( \lambda_{\text{eff}} = \lambda_{ij}^{\text{eff}} \mathbf{e}_i \otimes \mathbf{e}_j \)  
\( \lambda \)  
\( \Lambda_v \)  
\( \Delta(\omega) = \Lambda_{ij}(\omega) \mathbf{e}_i \otimes \mathbf{e}_j \)  
\( \mu \)  
\( \nu \)  
\( \rho \)  
\( \sigma = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \)  
\( \tau_0, \tau_\infty \)  
\( \phi \)  
\( \omega \)

**Subscripts**

\( \alpha_c \)  
\( \alpha^* \)  
\( Q_l \)  
\( Q_L \)

**Definitions**

- Fluid velocity \([\text{m/s}]\)
- Biot tensor \([-]\)
- Boundary layer thickness (thermal, diffusion...)
- Dynamic fluid viscosity \([\text{Pa.s}]\)
- Thermal conductivity of constituent \(\alpha\) \([\text{W/(m.˚K)}]\)
- Effective thermal conductivity tensor \([\text{W/(m.˚K)}]\)
- Lamé coefficient \([\text{MPa}]\)
- Viscous length
- Hydraulic conductivity tensor
- Shear modulus \([\text{MPa}]\)
- Poisson’s ratio \([-]\)
- Fluid density \([\text{kg/m}^3]\)
- Cauchy stress tensor
- Low- and high-frequency tortuosity
- Porosity \([-]\)
- Pulsation (angular frequency) \([\text{rad/s}]\)

**Characteristics**

- Characteristic value of \(\alpha\)
- Dimensionless quantity \(\alpha = \alpha_c \alpha^*\)
- Dimensionless number \(Q\) estimated from the microscopic viewpoint
- Dimensionless number \(Q\) estimated from the macroscopic viewpoint
Introduction

The result of a fusion of mathematical and physical concepts, homogenization has established itself as a method of overcoming the usual framework based on a description of elementary phenomena in a homogenous medium, to achieve the objective of a global description of coupled phenomena in heterogenous media. This book aims to present the key methodological points and their relevance to engineering science in a pedagogical format.

What is the nature of the problem? Even brief observation of natural or industrial materials reveals that they often consist of a combination of different constituents in various structures, and they are therefore heterogenous.

For example, take the behavior of civil engineering materials. The descriptions of the properties that they exhibit – and consequently the design rules for structures built using these materials – are, for the most part, issues related to the mechanics of continuous media applied to homogenous media. This theory has been widely proven, and a huge number of constructions designed using these principles can attest to its success, as can the accuracy of modeling performed using this approach. This simple observation leads us to believe that heterogenous materials can, at least subject to certain constraints, be treated similarly. But why and to what extent is this concept useful?

Furthermore, although the heterogenous nature of the material may not be obvious for this apparently continuous medium, it is on the other hand clear that its behavior depends on the characteristics of the heterogenities. How then do we proceed if we are to account for the properties of the constituents when defining the behavior of an equivalent continuous homogenous medium?

These two points are of great practical importance. On one hand, understanding of the limits of a method is an important safety consideration, and on the other hand determination of the equivalent continuous medium allows us to better understand the
parameters that govern its behavior (for a natural material) or to adapt the constituent parts to achieve the desired performance (for an artificial material).

The homogenization methods have been developed to answer these questions. They make it possible – under well-specified conditions – to obtain a description of the behavior of heterogenous materials starting from the behavior of the heterogenities.

A condition essential for the existence of an equivalent continuum is that the physical mechanism under study should vary on a length scale which is very large compared to the scale of the heterogenities. This requirement for a difference in length scales gives rise to the expressions “upscaling method” and “method of multiple scales”. The term “homogenization” also arises from this, because considering the heterogenities to be of infinitesimal size compared to the effects under study naturally leads us to consider the medium as a homogenous or, more precisely, homogenized continuum.

Linking the large-scale observable behavior to microscopic mechanisms is an age-old preoccupation of physicists.

One famous example is that of elasticity, where Navier (1821) and Poisson (1829) obtained a single macroscopic isotropic elastic coefficient from a particular molecular model: the two Lamé coefficients are equal. Cauchy (1828) obtained a two-coefficient isotropic elastic model starting from a more sophisticated molecular model. From among these well-known names we also draw attention to the preliminary work of Rayleigh (1892) on the conductivity of media containing impurities present in a parallelopiped lattice, and that of Einstein (1906) on the viscosity of suspensions and sedimentation rates.

These attempts remained intermittent until the 1950s when the needs of industrial development demanded a detailed understanding of the behavior of natural materials (the oil industry), manufactured materials (in particular steels and alloys), and the design of new materials (mainly for aeronautics). In order to understand the significance of small scale mechanisms on global behavior, scientific approaches at the time involved phenomenological micromechanical models built on thermodynamic principles. The pioneering works of Biot (1941) and of Hill (1965) took this approach.

It was in the 1970s that a new school of thought was born, started by Keller who used a rather different angle to tackle the question of the change of scales. This involved the method of asymptotic expansions at multiple scales. Initially built on an approach which was more mathematical than mechanical, this method was a true conceptual leap forwards in terms of its rigor and formalism. The works of Bensoussan et al. (1978) and Sanchez-Palencia (1980) are still important references on this subject, with similar ideas developed in Russia by Bakhvalov and Panasenko (1989).
Initially confined to very specialist circles, these methods blossomed considerably in the 1990s. During this period their fields of application were broadly diversified across all traditional engineering fields, and also in life sciences, particularly biomechanics. This undeniable success is thanks to the effectiveness of asymptotic methods for treating complex physics on a microscopic scale, and their ability to include coupling between different phenomena.

However, while their use has become almost routine for some research groups, it remains poorly documented at present in a form suitable for engineers and researchers working in related fields. These considerations convinced us there was a demand for a book which would set out a coherent picture of these approaches, and render them accessible to a wider audience than just specialists in this research area.

Rather than be exhaustive (which would not be an easy task), we have chosen to pick a few problems where the main points can be presented in a simple manner. The aim is also – using a unified treatment – to illustrate the common thought processes connecting the issues addressed in this book. In keeping with this approach, the bibliography does not attempt to be exhaustive, but shows the reader the seminal works in the field, and the references corresponding to the main steps of the problems we consider.

This volume, which has grown out of the Mechanics of Heterogeneous Media course taught at the University of Grenoble by J.L. Auriault, is intended to be a basic course in upscaling methods, aimed at advanced students, engineers and graduate students. With this pedagogical aim, we have used a progressive approach to each subject, starting out with traditional problems and then following them with recent developments. We also thought it useful to illustrate the potential applications of the results of homogenization. With this in mind, for each of the themes we treat, the theoretical results are followed by an example of the development through homogenization which provides a concrete example of the advances in a particular field of application.

This book is divided into four parts.

Part 1 is an introduction to the philosophy of homogenization methods. We discuss methods aimed at periodic and random materials while emphasizing their physical significance and their potential applications to real materials, which are often neither perfectly periodic or perfectly random.

The basic examples given in Chapter 1 give an understanding of the fundamental tools underlying both methods. Chapter 2 goes into more detail on the techniques and discusses connections between them and the details which distinguish them. There is a detailed discussion of conditions for their application, delineating the range of validity of these approaches. This overview of methods underlines the power of the asymptotic
method at multiple scales for the treatment of complex physics with many coupled
effects in materials with simple or hierarchical morphologies. Combining rigorous
formalism and intuitive reasoning, Chapter 3 presents the methodology of the multiple
scale approach which will be used throughout the rest of the book. The emphasis is
on the systematic use of dimensional analysis combined with the separation of length
scales. We also detail the means of expressing a practical problem involving real
materials in the context of homogenization. This methodological basis is applied in
the following sections where we specifically treat the physical mechanisms involved
in coupled phenomena.

Part 2 presents a first field of application of homogenization. We study the physics
of transport by diffusion, convection and advection, phenomena which allow us to
apply the basic tools of upscaling methods to engineering problems.

Chapter 4 focuses on thermal transfer in heterogenous media. Going beyond
the classical model of thermal transfer in a composite, we find a diverse range of
macroscopic models depending on the level of contrast in the conductive properties
of the constituents and their interfaces. In particular, memory effects arise from
the presence of local non-equilibrium of a very weakly conducting phase, and two-
temperature models can be developed for quasi-insulating interfaces. The transport
of solutes in porous media is examined in Chapter 5. We highlight the different
descriptions associated with the local physics of pure diffusion and then with diffusion-
advection. This second situation, which is reached at high transport rates, results in
a macroscopic dispersion. The range of validity of each of these models is explicitly
specified. Chapter 6 makes use of, and extends, these results, focusing on specific
materials. The numerical procedure of periodic homogenization is illustrated by
determining the coefficients for fibrous and granular materials. By way of comparison,
we recall the classical self-consistent analytical estimates. Finally, comparison with
experimental results enables us to judge the appropriateness of these models for
describing the properties of materials.

Part 3 is dedicated to the modeling of Newtonian fluid flows in rigid porous media.

Chapter 7 discusses incompressible fluids using multiple-scale asymptotic
expansions. It starts with the canonical problem of Darcy’s law (in the regime of
steady-state laminar flow). It continues taking into account inertial effects, both in the
dynamic linear regime which leads to memory effects through visco-inertial coupling,
and in the steady-state advective regime, where the correction due to weak non-
linearities is established. The flow in porous media of compressible fluids such as
gases is the subject of Chapter 8. Using the asymptotic method, we treat in succession
high pressure steady-state flows, wall slip effects in rarefied gases and, in the dynamic
regime, the acoustic description under weak pressure perturbations with thermal
coupling. The transfer of theoretical results for homogenization to their numerical
formulation is illustrated in Chapter 9. The solution to local problems derived by
periodic homogenization is given for calculation of the Darcy permeability of granular and fibrous materials. Finally, Chapter 10 returns to the same problems, which are discussed in the context of a self-consistent approach. We use this to establish analytical estimates and bounds for steady state and dynamic permeabilities, thermal effects, wall slip corrections and – by analogy – for the trapping constant.

Part 4 focuses on the behavior of deformable saturated porous media.

Chapter 11 considers the behavior in the quasi-static regime, first examining that of the empty porous medium (a specific case of an elastic composite) and then that of the saturated medium, introducing the fluid-solid coupling. Depending on the level of contrast between the shear properties of the fluid and the solid, the asymptotic method of multiple scales leads to three distinct behaviors whose properties are discussed.

The study of poroelastic behavior is extended to the dynamic regime in Chapter 12. The characteristics of the three possible behaviors – including the Biot biphasic model – are analyzed in detail, particularly properties of the effective coefficients. The range of validity of each of the descriptions is specified. Chapter 13 puts the homogenization results to numerical use in order to carry out a parametric analysis of the elastic and coupling coefficients in the biphasic model. At the same time these results, obtained for cohesive granular media, are compared to traditional self-consistent estimates and to bounds. In Chapter 14, the homogenized biphasic behavior is used with the aim of describing the propagation of waves in saturated porous media. After specifying the properties of the three propagation modes, the transmission of waves across a poroelastic interface is examined. We also establish the expression for Green’s functions in the context of poroelasticity, the integral formulation, and the fields radiated by abrupt dislocations.

To complete our summary of this text, it is worth mentioning certain important subjects which are not treated here (or only discussed briefly).

One of these subjects is complex microstructures. In fact, we will only consider media whose local geometry is sufficiently simple that it can be characterized by a typical length scale of the heterogeneity, and whose local problems can be formulated in terms of continuous media. This choice means that we omit:

– Media whose architecture involves very different characteristic sizes (such as double porosity media). These can give rise to various interacting physical effects on each length scale. These many possible couplings vastly increase the diversity of the possible macroscopic behaviors, with some behaviors only being possible in such media as;

– Microstructure whose behavior can be reduced to that of various interacting points within the material (for example the nodes in trellis structures). For these it is preferable to use a locally discrete description, and to move to a continuum description through homogenization. This alternative approach will not be discussed here.
A second aspect only outlined is that of the corrections to macroscopic descriptions which have been established to first order. Indeed, for the most part, the results presented here are restricted to the first significant term, and lead to descriptions involving a continuous medium which is materially simple, and descriptions valid in the bulk of the heterogenous medium. There are two corrections which can usefully be applied to these descriptions:

– those which appear on the boundary of the medium. They lead to a boundary layer with a thickness of the order of the size of the representative elementary volume. This makes it possible to reconcile local boundary conditions and boundary conditions used at the macroscopic scale;

– those which make it possible to treat situations with weak separation of scales, obtained by including higher-order terms within the homogenized descriptions. These correctors to simple continuum models introduce non-local effects whose spatial extent is of the order of the size of the representative elementary volume.

Finally, we will not discuss the taking into account of non-linearities. All the cases that we present involve linear effects, or sometimes weakly non-linear ones where the non-linearity can be treated as a perturbation of the linear solution. Whether they have material or geometric origins, non-linearities introduce considerable theoretical difficulties compared to linear situations. While the establishment of criteria through local limit analysis – the rheology of elastic composites with a non-linear power law, or the flow of non-Newtonian power-law fluids in porous media – has been successfully achieved, in general non-linearities present a real challenge to upscaling methods.

These three omitted themes – complex microstructures, corrections and non-linearities – are very rich and interesting, and they deserve further discussion on their own. We hope that this volume will offer a sufficiently clear and solid basis to guide the reader who may wish to explore these fields.

This work is the fruit of a long collaboration between its authors. It has of course been supported by the work and suggestions of numerous friends, colleagues and research students, whom we are delighted to thank for the assistance that they have given us, and in particular: P. Adler, I. Andrianov, L. Arnaud, P. Y. Bard, J. F. Bloch, G. Bonnet, L. Borne, L. Dormieux, H. Ene, M. Lefik, T. Levy, J. Lewandowska, C.C. Mei, X. Olny, L. Orgéas, P. Royer, E. Sanchez-Palencia, T. Strzelecki.

We extend particular gratitude to P. Adler, whose sound advice and criticism has added a great deal to this work.
This page intentionally left blank