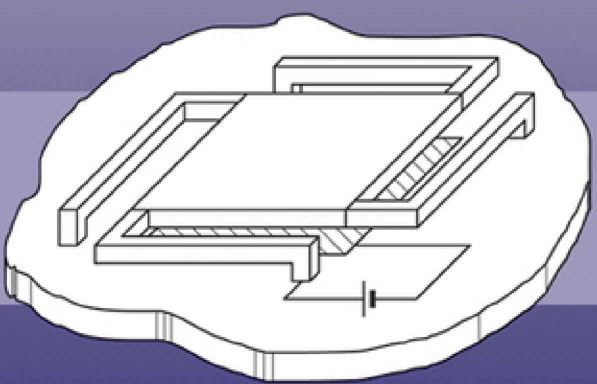
Principles of Microelectromechanical Systems



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To
my parents, Won Eui Lee and Cha Won Cho
parents-in-law, Jong Hoon Suh and Soon Ja Chung
wife, Jeong Soo Seo
and children, Jong Eun Lee, Jong Min Lee, and Jong Hyeon Lee

CONTENTS

PREFACE

This book aims to describe the principles of microelectromechanical systems (MEMS) via *a unified approach and closed-form solutions* that I have newly developed. The book is designed for senior (or graduate) students in universities and engineers from industry who want to design MEMS without numerical simulation. Since readers will have diverse backgrounds (e.g., electrical engineering, mechanical engineering, physics, chemistry), the book is organized so that it will be easy to understand the theory and closed-form solutions presented, which are used to design micro- and even nanosystems.

Through eighteen years of professional experience on MEMS, I have come to realize that most systems are so complicated that their static and dynamic behavior is too difficult to be expressed in closed forms. MEMS are systems whose mechanical behavior is coupled with electrical or other behavior, such as optical behavior. As a result, the governing equations for many MEMS become complicated or nonlinear, and their solutions have generally been obtained using numerical methods. For example, parallel plates connected to a voltage source show highly nonlinear behavior, including the pull-in phenomenon (jumping at a critical voltage), and it was too difficult to obtain simple and exact solutions for the interplate gap, resonant frequency, and capacitance, and their sensitivity at arbitrary voltages. In order to develop theory on linear and nonlinear MEMS, I spent four years, and then another year, in writing this book, which is a compilation of my research notebooks. Some of these theories and solutions have been published in journals and others are published in this book for the first time. I believe that the theory and closed-form solutions will help readers to understand coupled or nonlinear behavior of micro- and nanosystems and to easily design systems using micro- and nanoelements.

The book consists of ten chapters and presents building blocks for MEMS design in each chapter. MEMS behavior can be understood by combining the building blocks so that the reader can design a MEMS device for a specific purpose. Chapter 1 is an introduction, in which MEMS are defined and the knowledge required to understand them is examined briefly. Also presented is dimensional analysis, which provides basic dimensionless parameters existing in large- and small-scale worlds which we can use to understand the forces and other parameters in the small-scale world. Chapter 2 covers basic microfabrication, which presents knowledge on common fabrication processes employed in designing realistic MEMS. Chapter 3 is focused on statics, which is concerned with forces and moments acting on mechanical structures in static equilibrium. The static response of such mechanical elements as a prismatic beam is described using the governing equations obtained from force and moment equilibria. Chapter 4 treats the static behavior of structures consisting of mechanical elements. Chapter 5 deals with dynamic responses of mechanical structures by solving linear as well as nonlinear governing equations. Chapter 6 presents fluid flow in MEMS and evaluates the damping force acting on moving structures. Chapter 7 covers basic equations of electromagnetics, which govern the electrical behavior of MEMS and by which movable elements are actuated or sensing elements work.

Subsequent chapters are devoted to combing MEMS building blocks to form actuators and sensors for specific purposes. In Chapter 8 I set up models for actuators using piezoelectric material and thermal expansion and find closed-form solutions from the models. Chapter 9 deals with electrostatic and electromagnetic actuators whose mechanical and electrical behavior is coupled. Chapter 10 provides sensors that employ mechanical and electrical elements and the actuators covered in Chapters 8 and 9. In each chapter, theoretical models are presented for the elements and systems, and solutions for the static and dynamic responses are obtained in closed form. For easy understanding of complex MEMS, I use analogies and illustrations.

I would like to acknowledge my parents, Won Eui Lee and Cha Won Cho, and parents-in-law, Jong Hoon Suh and Soon Ja Chung, who live in South Korea and have supported me through thick and thin. My lovely wife, Jeong Soo Seo, gave me the confidence to write. Finally, I would like to thank George J. Telecki, Lucy Hitz, Angioline Loredo, and many other staff members of John Wiley & Sons, Inc. for their support. The book would not exist had it not been for their continuous encouragement and patience. If I become a professor at a university, I will use this book as a textbook.

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CHAPTER 1

INTRODUCTION

1.1 MICROELECTROMECHANICAL SYSTEMS

MEMS, *microelectromechanical systems*, are systems that consist of small-scale electrical and mechanical components for specific purposes. MEMS were translated into systems with electrical and mechanical components but have extended their boundaries to include optical, radio-frequency, and nano devices. As a result, depending on the components included and applications desired, MEMS have different names: for example, MOEMS (micro-optoelectromechanical systems) for optical applications, RF MEMS (radio-frequency MEMS) to refer to radio-frequency components and applications, and NEMS (nanoelectromechanical systems) if the systems include at least one component whose dimension is less than 1 μm. When MEMS use bio-related material (e.g., strands of DNA) to detect desired targets or to manipulate cells, the corresponding MEM system is currently called bioMEMS. Different names may refer to MEMS: *microsystems technology* (MST) in Europe and *micromachines* in Japan. Throughout this book, MEMS will be referred to as systems that include at least one set of electrical and mechanical components for a specific purpose. Depending on the specific purpose, more components, such as a reflective surface for a micromirror, can be added to a MEMS device. A typical dimension of a component of MEMS varies from 1 μm to a few hundred micrometers, and the overall size is approximately less than 1 mm. In this book we describe MEMS principles via a unified approach

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and newly developed closed-form solutions. Readers are assumed to be familiar with mathematical background at the third-year college and university level.

1.2 COUPLED SYSTEMS

MEMS are coupled systems since they consist of electrical and mechanical components; the mechanical behavior of MEMS are in general coupled with the electrical behavior. For example, let us consider the first electrostatic MEMS device (Fig. 1.1), presented by Nathanson et al. in the 1960s to filter or amplify electrical signals using the resonance of an electroplated cantilever. When an input signal (electrical signal) is applied across the end of the cantilever and the actuation electrode on a substrate, the electrical attractive force, given by Coulomb's law, actuates the cantilever, and a detection circuit formed under the cantilever detects the filtered or amplified electrical signal that is generated by the mechanical vibration of the cantilever.

Since the development of the first MEMS device, many other MEMS have been developed. For example, as one of the important components of MEMS, the parallel plate shown in Fig. 1.2 (similar to the cantilever of Fig. 1.1) is widely used in many microdevices that employ electrostatic forces for actuation of a microstructure or detection of a physical quantity. The typical parallel plate shown in Fig. 1.2 illustrates the basic knowledge that is required to understand MEMS behavior. The parallel plate consists of a movable plate suspended by flexures, a stationary plate, and a voltage source to supply voltage or electrical charge to the movable and stationary plates. The flexures are used to support the movable plate and act as a spring. The gap between plates can be adjusted when a force (e.g., electrostatic force or inertial force) acts on the plate.

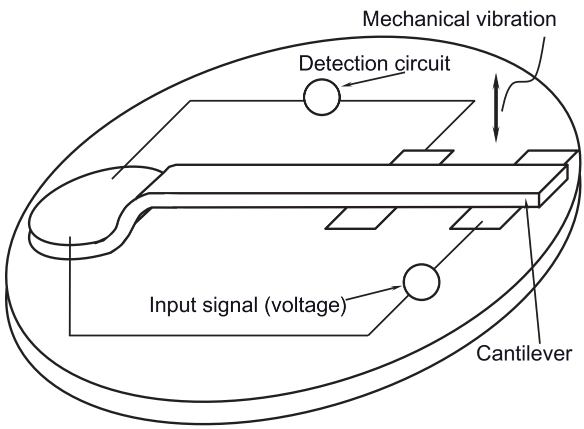


Figure 1.1 Resonant gate transistor.

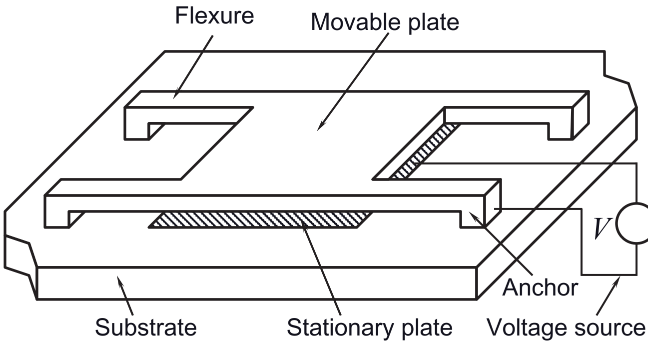


Figure 1.2 Parallel plate.

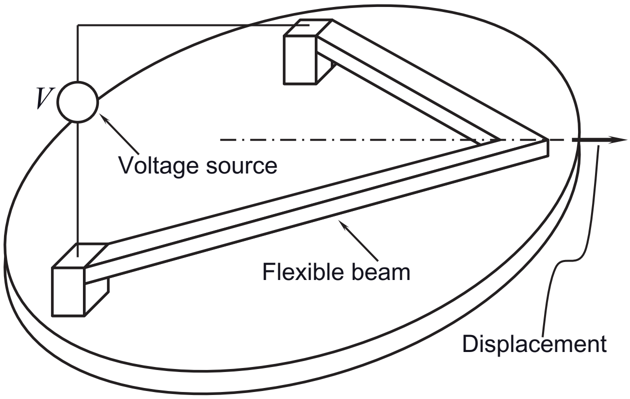


Figure 1.3 Electrothermal actuator.

Let us suppose that we apply a voltage across the movable and stationary plates. Upon applying the voltage, positive charges (or negative charges, depending on the electrical connection) are accumulated on the movable plate while opposite charges are accumulated on the stationary plate. As a result, the positive and negative charges on the plates generate an attractive force, the electrostatic force, which can push down the movable plate. The movable plate is displaced until the spring force (restoring force) due to the flexures balances the electrostatic force; that is, the displaced movable plate is in equilibrium while the voltage is applied. However, when the voltage is greater than a critical voltage called the *pull-in voltage*, the movable plate collapses into the lower plate.

A thermal actuator (Fig. 1.3) utilizes the thermal expansion due to Joule heating. As a voltage source supplies electrical current through the flexible beam that acts as a heater, heat is generated in the heater. The thermal

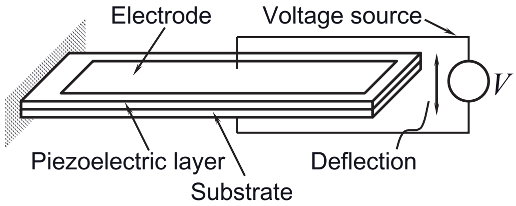


Figure 1.4 Piezoelectric actuator.

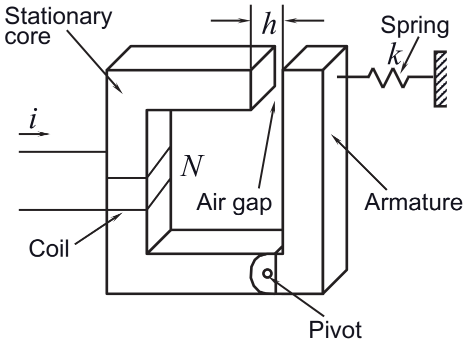


Figure 1.5 Electromagnetic relay.

expansion of the beam provides the displacement shown in the figure. The displacement depends on the voltage applied, the resistance of the beam, and the stiffness. Therefore, the mechanical behavior (e.g., displacement) of thermal actuators is coupled with the electrical and thermal behavior.

A piezoelectric actuator (Fig. 1.4) utilizes a piezoelectric material whose shape is deformed when exposed to an electric field. In Fig. 1.4 a piezoelectric layer is glued or deposited on a substrate. A thin conductive electrode is placed or deposited on the piezoelectric layer so that the layer is exposed to an electric field when a voltage source applies a voltage across the layer. In this situation, the layer expands or contracts, depending on the polarity of the voltage. For example, if the piezoelectric layer expands in the longitudinal direction, the right end of the actuator moves downward. The end of the actuator moves upward when the polarity of the voltage is reversed. The mechanical behavior of the piezoelectric actuator is then coupled with the piezoelectric constants that relate the voltage to the deformation of the piezoelectric layer, the mechanical properties (e.g., Young's modulus), and the layer geometry.

Electromagnetic force is also used to actuate microstructures. Figure 1.5 shows a model of an electromagnetic relay, one type of electromagnetic actuator. The relay consists of a movable bar (called an *armature*), a stationary core connected to the movable bar, a coil to generate magnetic field in the movable bar and stationary core, and a spring to provide the movable bar with a restoring force. When an electric current is applied to the coil, the relay is magnetized to generate an attractive force between the movable bar and the stationary core, and the movable bar is then attached to the stationary core. If the current is removed, the movable bar returns to its initial position under the restoring force of the spring. Thus, the mechanical behavior of electromagnetic actuators depends on the applied current, the magnetic and mechanical properties of the material used, the geometry of the actuator, and the stiffness of the spring.

As briefly discussed above, actuators use electricity to generate mechanical motion such as displacement, and the resulting mechanical behaviors are then coupled with electrical behavior, material properties, geometry, and so on. As a result of the coupling, the mechanical behavior is, in general, related nonlinearly to electric input (e.g., applied voltage) except in a few cases, or are expressed as complicated functions of electric input. To understand these nonlinear actuators and sensors, numerical analyses have been widely used. For example, to obtain the sensitivity to voltage of the capacitance of a parallel plate (Fig. 1.2), numerical analyses have been used to solve the equilibrium equation that governs the equilibrium position of the movable plate. Therefore, researchers, designers, and students have required commercial software to solve a problem or the skill to develop codes or programs that obtain the solution numerically. This book is designed to provide analytical closed-form solutions of both linear and nonlinear actuators in which mechanical behavior and electrical behavior are coupled. Since most MEMS-based sensors use actuators to measure physical quantities, this book can be used to design and analyze sensors.

1.3 KNOWLEDGE REQUIRED

As discussed in the foregoing section, MEMS are systems that consist of mechanical and electrical components and that may also involve other components, such as a reflective layer for a micromirror, depending on the purpose. Since the mechanical behavior of MEMS are coupled with other behavior, we should study interdisciplinary subjects in the fields of science and engineering to understand the coupled behaviors. Figure 1.6 shows an overview of the knowledge required for the research and development of MEMS and their derivatives. Because the most convenient and controllable energy is the electrical energy, electrical and electronic engineering covering electromagnetics, circuit theory, or signal processing is required to control phenomena associated with electric charge (i.e., electron, current). For example, from the point of view of electrical engineering, the parallel plate of Fig. 1.2 may be considered to be a capacitor consisting of movable and stationary plates, so knowledge of electrical engineering is necessary to calculate the capacitance of the

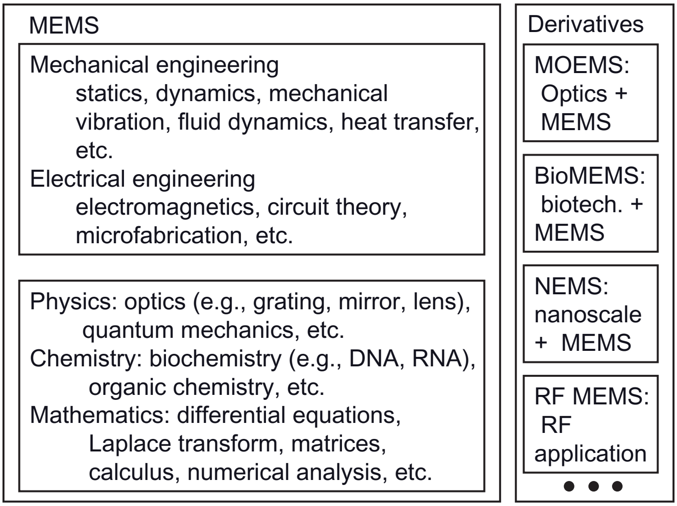


Figure 1.6 Knowledge required to understand MEMS.

parallel plate and to obtain the electrostatic force acting on the movable plate as a function of the interplate gap and the applied voltage. Similarly, since the magnetic relay of Fig. 1.5 is an electromagnet with a variable air gap, we use the magnetic energy that is stored in the electromagnet and calculate the magnetic force pulling the movable bar into the stationary core.

Physically, MEMS are mechanical structures that are designed for specific purposes. For desired functions, components of MEMS must be mechanically stable, vibrate if the mechanical resonance is utilized, and be deformed if deformation or displacement is needed. For the design and analysis of mechanical components, we need statics for the mechanical structure design, dynamics and vibration for resonance and mechanical vibration, heat transfer for thermal actuation, and fluid dynamics for the evaluation of damping due to the movement of microstructures. Let us consider the parallel plate in Fig. 1.2 as an example of a mechanical structure. Since the four flexures support the movable plate under the electrostatic force, we need statics to determine the flexure dimensions: the length, width, and thickness of the flexures. If the movable plate operates at resonance for a mechanical filter, we should use our knowledge of dynamics or mechanical vibration to design the resonant frequency desired. If we wish to set up a mechanical quality factor that affects the bandwidth of a mechanical filter, we should evaluate the damping force or damping coefficient that can be provided by fluid dynamics. If we wish to design an accelerometer or acceleration switch using the parallel plate shown in Fig. 1.2, we need to know the dynamics and mechanical vibration.

All the above-mentioned knowledge is coupled, so the design of a parallel plate for a specific application is very complicated even though the parallel plate in Fig. 1.2 looks simple. In addition to these complexities, we may need physics, chemistry, mathematics, and other subject areas, and MEMS may have different names, as described in Fig. 1.6: MOEMS if a MEMS device involves at least one optical component; RF MEMS if a MEMS device is designed for a radio-frequency application such as an RF filter; bioMEMS if a MEMS device is used for biological applications such as the detection of DNA strands; NEMS if at least one dimension of the mechanical structure is less than 1 μm; and perhaps other names in future applications if mechanical structures with electrical components are used for a specific purpose.

1.4 DIMENSIONAL ANALYSIS

Dimensional analysis and dimensionless numbers allow us to investigate complicated or coupled systems such as the MEMS described in Section 1.3. Using dimensional analysis and experimental results (or numerical simulation), we can find relationships between variables that are involved in a problem or system. If we apply dimensional analysis to a governing equation that describes a physical phenomenon and cannot be solved due to its nonlinearity, we can obtain useful dimensionless numbers that play crucial roles in describing the phenomenon. We begin with easy dimensionless numbers with which we are familiar.

Let us begin by considering the ratio of the circumference of a circle to its diameter, the well-known constant. Figure 1.7a, b, and c show a circular column, a rectangular column, and an arbitrarily shaped body, respectively. As the radius and height of the circular column (Fig. 1.7a) are represented by r and t, respectively, the perimeter l of the top view, the top-view area A, and the volume V are given by

$$l = 2\pi r = \pi d$$

$$A = \pi r^2 = \frac{\pi}{4} d^2$$

$$V = \pi r^2 t = \frac{\pi}{4} d^2 t$$

where π denotes the ratio of the circumference of a circle to its diameter and d represents the diameter of the column. It is worth noting that the perimeter, area, and volume are proportional to the diameter, the square of the diameter, and the product of the area and thickness, respectively. It is also noted that if the circular column become n times larger than its original dimensions d and t, the corresponding length, area, and volume will be nl, n^2A, and n^3V, respectively. Manipulating the equations above gives dimensionless forms as follows:

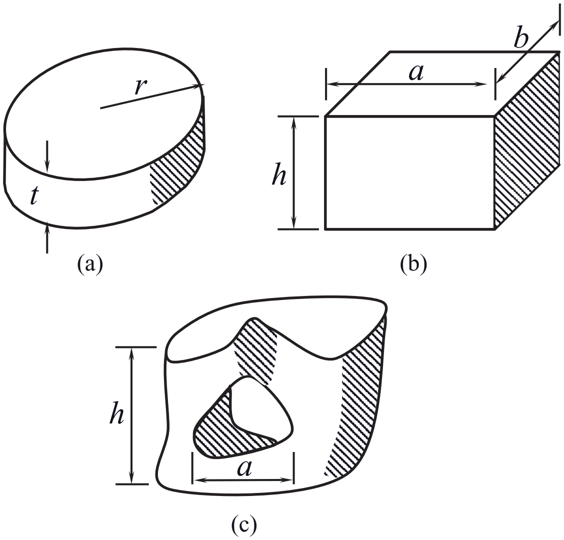


Figure 1.7 Various three-dimensional objects.

$$\frac{l}{d} = \pi$$

$$\frac{A}{r^2} = \pi \quad \text{or} \quad \frac{A}{d^2} = \frac{\pi}{4}$$

$$\frac{V}{d^2 t} = \frac{\pi}{4} \quad \text{or} \quad \frac{V}{d^3} = \frac{\pi}{4}\frac{t}{d}$$

In the preceding equations, the diameter d may be considered a characteristic length that represents the dimension of the circular column. The first two equations above give the constants (numbers) on their right-hand sides, and the third equation also yields a constant if the t/d remains unchanged. In this case, the dimensionless numbers l/d and A/d^2 remain unchanged even though the diameter becomes larger or smaller. However, the dimensionless number V/d^3 is proportional to the dimensionless number t/d. If the diameter and thickness become n times the original dimensions, the resulting length and area are, respectively, nl and n^2A, and the volume becomes n^3V since t/d does not change for a uniform transform (i.e., $nt/nd = t/d$). The preceding equations may be expressed in more general dimensionless forms as

$$f_1\left(\frac{l}{d}\right) = \frac{l}{d} - \pi = 0$$

$$f_2\left(\frac{A}{d^2}\right)=\frac{A}{d^2}-\frac{\pi}{4}=0$$

$$f_3=\left(\frac{V}{d^3},\frac{t}{d}\right)=\frac{V}{d^3}-\frac{\pi}{4}\frac{t}{d}=0$$

Similarly, for the rectangular column of Fig. 1.7b, the perimeter l of the top view, the top-view area A, and the volume V are given by

$$l=2(a+b)$$

$$A=ab$$

$$V=abh$$

We transform the equations above into dimensionless equations as follows:

$$f_4\left(\frac{l}{a},\frac{b}{a}\right)=\frac{l}{a}-2\left(1+\frac{b}{a}\right)=0$$

$$f_5\left(\frac{A}{a^2},\frac{b}{a}\right)=\frac{A}{a^2}-\frac{b}{a}=0$$

$$f_6\left(\frac{V}{a^3},\frac{b}{a},\frac{h}{a}\right)=\frac{V}{a^3}-\frac{b}{a}\frac{h}{a}=0$$

The dimensionless equations f_4, f_5, and f_6 represent functions for the perimeter and area of the top view of the rectangular column and the volume, respectively. It should be noted that the dimensionless length l/a, area A/a^2, and volume V/a^3 are expressed as functions of dimensionless variables b/a and h/a. This concept may be extended into more general cases.

Let us consider the complex three-dimensional structure shown in Fig. 1.7c. We wish to obtain the dimension a, the area, and the volume of the structure as functions of a characteristic length. The relations may be used to build a miniature or larger structure. Let l (not shown in Fig. 1.7c), h, A, and V represent a length, the height, the area, and the volume of a structure, respectively. The following equations can be written for a dimensional analysis:

$$f_7(a,l,h)=0$$

$$f_8(A,a,l,h)=0$$

$$f_9(V,a,l,h)=0$$

Let l be a characteristic length of a structure. Since a, l, and h have the dimensions of length and A and V have the dimensions of the square and cube of length, respectively, the dimensionless equations are given by

$$f_{10}\left(\frac{a}{l},\frac{h}{l}\right)=0$$

$$f_{11}\left(\frac{A}{l^2},\frac{a}{l},\frac{h}{l}\right)=0$$

$$f_{12}\left(\frac{V}{l^3},\frac{a}{l},\frac{h}{l}\right)=0$$

When the preceding equations are set up by dividing the arguments of the function by l, l_2, or l_3, the number of arguments is reduced by one in each equation. Rearranging the preceding equations yields the length a, the area A, and the volume V in dimensionless forms:

$$\frac{a}{l}=f_{13}\left(\frac{h}{l}\right)$$

$$\frac{A}{l^2}=f_{14}\left(\frac{a}{l},\frac{h}{l}\right)$$

$$\frac{V}{l^3}=f_{15}\left(\frac{a}{l},\frac{h}{l}\right)$$

For the enlargement or contraction of the structure, the ratio of linear dimensions, h/l, remains constant, and then the ratio a/l also becomes constant. The first equation above becomes $a/l = c_1$, where c_1 is a constant. Consequently, the first equation above gives a linear relation between a and l:

$$a=c_1 l \tag{1.1}$$

Similarly, the equations for the area A and the volume can be expressed as

$$A=c_2 l^2 \tag{1.2}$$

$$V=c_3 l^3 \tag{1.3}$$

where c_2 and c_3 denote constants. These three equations state that for any structures in three-dimensional space, if the shape of the structure remains unchanged for enlargement or contraction, the length from one point to another, the area of any portion of the structure, and the volume of the structure are proportional to a characteristic length and to the square and cube of the characteristic length, respectively. During derivation of equations (1.1) to (1.3), the characteristic length can be taken to be any dimension: for example, the width or the height. If the height h is selected as the characteristic length, the foregoing equations may be expressed as follows: $a = d_1 h$, $A = d_2 h^2$, *and* $V = d_3 h^3$, where d_1, d_2, and d_3 represent constants.

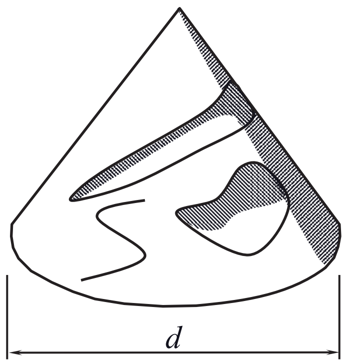


Figure 1.8 A piece of chocolate.

Example 1.1 A chocolate company decides to build an enlarged model of a piece of chocolate for an advertisement. The piece of chocolate shown in Fig. 1.8 will be enlarged n times. In other words, the dimension d will be nd in the model. In order to build the model and paint the outside, the company must calculate the length of the company logo S shown in Fig. 1.8 and the area and volume of the model. If the logo length is l_p, the outside area A_p, and the volume V_p, find the logo length of the model, the required volume of the piece of chocolate, and the outside area. If the company paints the outside to a thickness of t (wet paint), determine the volume of paint required.

As discussed above, the dimension, area, and volume of the model are proportional to a characteristic length and to the characteristic length squared and cubed, respectively. Let the logo length of model be l_m, the outside area A_m, and the volume V_m. For the length, area, and volume of the prototype (original) piece of chocolate, (1.1) to (1.3) give

$$l_p = c_1 d \tag{a}$$

$$A_p = c_2 d^2 \tag{b}$$

$$V_p = c_3 d^3 \tag{c}$$

where c_1, c_2, and c_3 denote constants for the relationships between the prototype and the model. The preceding equations also hold for the model as follows:

$$l_m = c_1 nd \tag{d}$$

$$A_m = c_2 (nd)^2 \tag{e}$$

$$V_m = c_3 (nd)^3 \tag{f}$$

From the preceding equations, we thus find relationships that will hold for both the prototype and the model:

$$l_m = nl_p \tag{g}$$

$$A_m = n^2 A_p \tag{h}$$

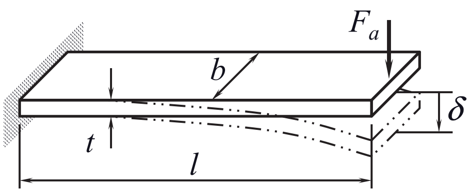
$$V_m = n^3 V_p \tag{i}$$

The volume of paint required for the model is given by $tA_m = n^2 tA_p$. According to (g) to (i), if the dimensions of a model are n times those of a prototype (i.e., geometrically similar), the linear dimension, area, and volume of the model are increased to n, n^2, and n^3 times those of the prototype, respectively. For example, if a structure is magnified by a factor of 10, any length, area, and volume of the magnified structure become, respectively, 10, 10^2, and 10^3 times those of the original structure.

In the foregoing discussion and Example 1.1, the dimensional analysis has been described for similar structures. These concepts may be extended to involve more general cases that are related to force, stress, energy, or any other physical quantities. As a physical quantity, force such as the weight of a structure is measured in newtons (N) if we use SI units (an abbreviated form of the French term corresponding to "international system of units"). Weight may be measured in other units: for example, lb_f (pound-force). To avoid any confusion associated with force units such as N and lb_f, F is used to represent the force dimension in dimensional analysis. Similarly, L and T represent the dimensions of length and time, respectively. In many cases, force (F), length (L), and time (T) are used as the fundamental units if physical quantities involved in a problem are expressed using force, length, and time. If force, length, and time are used as the fundamental units, the system of units is called the *F–L–T system*. Other dimensions that can be derived from the fundamental units are known as *derived units*. Derived units can be derived easily from basic equations. For example, the mass of a structure is a derived unit that is defined as $FL^{-1}T^2$, since the mass may be expressed as $m = F/a$, where F and a denote the force acting on the mass and the acceleration of the structure, respectively. If physical quantities under consideration cannot be derived from the preceding fundamental units (F–L–T units), the physical unit may be added to the list of fundamental units. For example, the temperature for thermal study and the electric charge for electric phenomena can be considered as additional units. As the fundamental units, the temperature and electric charge are represented by θ and Q, respectively. Fundamental and derived units that are widely used in scientific and engineering problems are shown in Table 1.1. In physics, researchers may use mass as a fundamental unit instead of force. In this case, the dimensions in Table 1.1 can be converted into an M–L–T system as MLT^{-2} is substituted for force F. For example, FL of the F–L–T system, representing energy, is converted to ML^2T^{-2} in the M–L–T system. The F–L–T and M–L–T systems yield the same results for the dimensional analysis of a physical problem.

TABLE 1.1 Dimensions of Physical Quantities

Physical Quantities	Dimensions	Physical Quantities	Dimensions
Force	F	Thermal conductivity	$FT^{-1}\theta^{-1}$
Length, displacement	L	Electric charge	Q
Time	T	Current	QT^{-1}
Mass	$FL^{-1}T^{2}$	Voltage	FLQ^{-1}
Density	$FL^{-4}T^{2}$	Resistance	$FLTQ^{-2}$
Velocity	LT^{-1}	Permittivity	$F^{-1}L^{-2}Q^{2}$
Acceleration	LT^{-2}	Capacitance	$F^{-1}L^{-1}Q^{2}$
Energy	FL	Magnetic field strength	$L^{-1}T^{-1}Q$
Stress, pressure	FL^{-2}	Magnetic flux density	$FL^{-1}T\,Q^{-1}$
Viscosity	$FL^{-2}T$	Permeability	$FT^{2}Q^{-2}$
Angle	dimensionless	Inductance	$FLT^{2}Q^{-2}$
Temperature	θ		



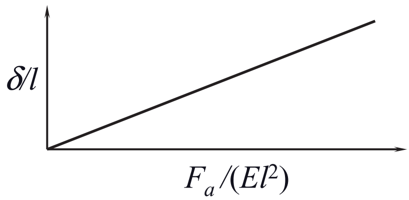


Figure 1.9 Cantilever subjected to a force.

Using the fundamental and derived units listed in Table 1.1, let us perform dimensional analysis that may be used to obtain a relation among many variables. Consider the cantilever beam of Young's modulus E (dimensions: N/m^2; a modulus relating stress to strain), length l, width b, and thickness t, as shown in Fig. 1.9a. For this simple problem we know from statics (that will be dealt with in Chapter 3) that a solution for the deflection δ at the end of a cantilever under an applied force F_a is given by

$$\delta = \frac{1}{3}\frac{F_a l^3}{EI} \tag{1.4}$$

where I denotes the moment of inertia of the cantilever, defined as $bt^3/12$. However, in order to know how to conduct the dimensional analysis of this problem, it is assumed that at this stage we don't know the solution. From Fig. 1.9a we know that F_a, E, δ, b, t, and l enter into the problem. The problem may be expressed as

$$f(F_a, E, \delta, b, t, l) = 0 \tag{1.5}$$

From Table 1.1 we know the dimensions of the variables as follows (E has the dimension of stress):

$$\begin{aligned}
F_a &= [F] \\
E &= [F/L^2] \\
\delta &= [L], \quad b = [L], \quad t = [L], \quad l = [L]
\end{aligned}$$

These variables involve the two fundamental dimensions F and L. If we divide the first equation by the second to eliminate the fundamental unit F, we have

$$\frac{F_a}{E} = \frac{[F]}{[F/L^2]} = [L^2]$$

which involves only the fundamental dimension L. We have δ, b, t, and l for the fundamental dimension of L, and any of them may be selected to represent L. Using l for a characteristic length representing the fundamental dimension L, the preceding equation can be converted into the following dimensionless number:

$$\frac{F_a}{El^2} = \frac{[L^2]}{[L^2]} = [0]$$

where [0] states that the number Fa/El^2 is dimensionless. Similarly, the other variables, δ, b, and t, can be converted into corresponding dimensionless numbers by dividing the variables by the characteristic length l, and the original problem can be expressed as

$$f\left(\frac{F_a}{El^2}, \frac{\delta}{l}, \frac{b}{l}, \frac{t}{l}\right) = 0$$

Since we wish to obtain an expression for the beam deflection δ, the foregoing equation may be expressed as follows:

$$\frac{\delta}{l} = f_1\left(\frac{F_a}{El^2}, \frac{b}{l}, \frac{t}{l}\right) \tag{1.6}$$

Equation (1.6) states that the deflection of the cantilever in Fig. 1.9a is expressed as the four dimensionless numbers Fa/El^2, δ/l, b/l, and t/l. Equation (1.6) is all that we can obtain from dimensional analysis. However, if we obtain more information from experiments or numerical simulations, the relationship (1.6) can be expressed in a more accurate expression. From an experiment or numerical analysis, we can find more relationships:

$$\frac{\delta}{l} = c_1 \frac{F_a}{El^2} \qquad \frac{\delta}{l} = c_2 \left(\frac{b}{l}\right)^{-1} \qquad \frac{\delta}{l} = c_3 \left(\frac{t}{l}\right)^{-3} \tag{1.7}$$

These equations are substituted into equation (1.6) and we then have

$$\frac{\delta}{l} = f_1\left(\frac{F_a}{El^2}, \frac{b}{l}, \frac{t}{l}\right) = c\frac{F_a}{El^2}\left(\frac{b}{l}\right)^{-1}\left(\frac{t}{l}\right)^{-3} = c\frac{F_a l^2}{Ebt^3}$$

or

$$\delta = c\frac{F_a l^3}{Ebt^3}$$

where c is a constant that represents the product of c_1, c_2, and c_3 and can be obtained from the experiment or numerical analysis (e.g., Fig. 1.9b) and obtained easily from a graph of $F_a l^2/(Ebt^3)$ against δ/l. We know that the constant will be 4 when the equation above is compared with the analytical solution, (1.4).

In the procedure used to obtain the dimensionless equation above, note that after selecting variables as characteristic variables (E and l), the other variables were divided by the variables selected. Note also that the number of resulting dimensionless variables is reduced by the number of fundamental units. In the cantilever problem in Fig. 1.9a, the number of dimensional variables of (1.5) was 6, but in the dimensionless form, the number of dimensionless variables was reduced to 4 (6 – 2, where 2 is the number of fundamental units, F and L in this case). This procedure is generalized by *Buckingham's π-theorem* (Buckingham, 1914), which may be stated as follows:

If n variables (υ_1, υ_2, … , υ_n), which can be expressed by N fundamental units, are involved in a problem, the dimensional equation for the problem may be expressed as

$$f(\upsilon_1, \upsilon_2, \upsilon_3, \ldots, \upsilon_n) = 0 \tag{1.8}$$

and the corresponding dimensionless equation can be a function of the $n - N$ dimensionless variables ($\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-N}$), as follows:

$$g(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-N}) = 0 \tag{1.9}$$

As discussed in the foregoing problem (Fig. 1.9a), equation (1.9) can give useful information and becomes more accurate if more information, such as data from an experiment or numerical analysis, is available.

The problem shown in Fig. 1.9a can also be solved using the π-theorem. In Fig. 1.9a we have six dimensional variables (F_a, E, δ, b, t, and l), and the problem of obtaining the deflection at the end of the cantilever can be expressed as

$$f(F_a, E, \delta, b, t, l) = 0$$

Since all the dimensional variables are expressed by the fundamental units of F and L, we expect that four dimensionless variables (6 – 2 = 4) will appear in a dimensionless equation:

$$g(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \tag{1.10}$$

δ, b, t, and l have the dimension of L, and then we take the first three dimensionless variables as $\pi_1 = \delta/l$, $\pi_2 = b/l$, and $\pi_3 = t/l$. The fourth variable, π_4, may be expressed in the form

$$\pi_4 = F_a E^a l^b = [F][F/L^2]^a [L]^b = [F^{1+a} L^{-2a+b}]$$

for π_4 to be a dimensionless number (i.e., [0]), we have two equations,

$$1 + a = 0 \quad \text{and} \quad -2a + b = 0$$

From these equations, $a = -1$ and $b = 2a = -2$ are obtained. Substituting a and b into π_4 above gives

$$\pi_4 = F_a E^{-1} l^{-2} = \frac{F_a}{El^2}$$

The dimensionless variables (π_1, π_2, π_3 and π_4) are substituted into (1.10) to yield

$$g\left(\frac{\delta}{l}, \frac{b}{l}, \frac{t}{l}, \frac{F_a}{El^2}\right) = 0$$

The preceding equation is the same as (1.6). For more general applications of dimensional analysis, more examples are presented below.

Example 1.2 Parallel plates are used widely in MEMS to actuate microstructures and to sense physical quantities. As illustrated in Fig. 1.10, a parallel plate consists of an upper plate on a lower plate, and a voltage V may be applied across the plates, which are separated by a gap h. The length and width of the upper plate are l_1 and l_2 (Fig. 1.10) and the thickness and fringing field effect