An Introduction to Advanced Quantum Physics

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Preface

Over the years, material that used to be taught in graduate school made its way into the undergraduate curriculum to create room in graduate courses for new material that must be included because of new developments in various fields of physics. Advanced Quantum Physics and its relativistic extension to Quantum Field Theory are a case in point. This book is intended to facilitate this process. It is written to support a second course in Quantum Physics and attempts to present the material in such a way that it is accessible to advanced undergraduates and starting graduate students in Physics or Electrical Engineering.

This book consists of two parts.

Part 1, comprising Chapters 1 through 5, contains the material for a second course in Quantum Physics. This is where concepts from classical mechanics, electricity and magnetism, statistical physics, and quantum physics are pulled together in a discussion of the interaction of radiation and matter, selection rules, symmetries and conservation laws, scattering, relativistic quantum physics, questions related to the validity of quantum physics, and more. This is material that is suitable to be taught as part of an undergraduate quantum physics course for physics and electrical engineering majors. Surprisingly, there is no undergraduate textbook that treats this material at the undergraduate level, although it is (or ought to be) taught at many institutions.

In Part 2, comprising Chapters 6 through 8, we present elementary Quantum Field Theory. That this material should be studied by undergraduates is controversial but I expect it will become accepted practice in the future. This material is intended for undergraduates that are interested in the topics discussed and need it, for example, in a course on elementary particle physics or condensed matter. Traditionally such a course is taught in the beginning of graduate school. When teaching particle physics to advanced undergraduate students I felt that the time was ripe for an elementary introduction to quantum field theory, concentrating on only those topics that have an
application in particle physics at that level. I have also taught the material in Chapters 6 through 8 to undergraduates and have found that they had no problem in understanding the material and doing the homework.

It is hoped that the presentation of the material is such that any good undergraduate student in physics or electrical engineering can follow it, and that such a student will be motivated to continue the study of quantum field theory beyond its present scope. Additionally, beginning graduate students may also find it of use.

Please communicate suggestions, criticisms and errors to the author at hpaar@ucsd.edu.

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UNITS AND METRIC

It is customary in advanced quantum physics to use natural units. These are cgs units with Planck’s constant divided by $2\pi$ and the velocity of light set to unity. Thus we set $\hbar = c = 1$.

In natural units we have for example that the Bohr radius of the hydrogen atom is $1/\alpha m$ with $\alpha = e^2/(4\pi)$ the fine-structure constant and $m$ the mass of the electron. Likewise we have that the energy of the hydrogen atom is $\alpha m/(2n^2)$ and the classical radius of the electron is $\alpha/m$.

When results of calculations have to be compared with experiment, we must introduce powers of $\hbar$ and $c$. This can be done easily with dimensional analysis using, for example, that the product $\hbar c = 0.197$ GeVfm and $(\hbar c)^2 = 0.389$ GeV$^2$mb.

In summations I use the Einstein convention that requires one to sum over repeated indices, 1 to 3 for Latin and 1 to 4 for Greek letter indices.

Three-vectors are written bold faced such as $\mathbf{x}$ for the coordinate vector. We use the ‘East-coast metric’, introduced by Minkovski and made popular by Pauli in special relativity. Its name is obviously US-centric. In it, four-vectors have an imaginary fourth component. The alternative is to use a metric tensor as in General Relativity. This is called the ‘West-coast metric’, made popular by the textbook on Field Theory by Bjorken and Drell, but is overkill for our purposes. Thus $x_\mu = (x, it)$, $p_\mu = (p, iE)$, and $k_\mu = (k, i\omega)$ with $k$ the wave vector and $\omega$ the angular frequency. A traveling wave can be written as $\exp i k \mathbf{x} = \exp i(k_x x + k_y y + k_z z - \omega t)$. Squaring four-vectors, we get for example $p^2 = p_\mu p_\mu = p^2 - E^2$ which is $-m^2$ (unfortunately negative) and $k^2 = 0$ for electromagnetic radiation. When integrating in four-dimensional space we use $d^4 x$ which one might think is equal to $d^3 x dt$ but this is not so. By $d^4 x$ we mean $d^3 x dt$ (the West-coast metric does not have this inconsistency, sorry). With this convention the gamma matrices in the Dirac equation are all Hermitian.

We also find that we can write the commutation relations of $\mathbf{p}$ and $\mathbf{x}$ on the one hand and $E$ and $t$ on the other in relativistically covariant form as $[p_\mu, x_\nu] = \delta_{\mu\nu}/i$ where $\delta_{\mu\nu}$ is the Kronecker delta. Thus it is seen that the (at first sight odd) difference in sign of the two original commutation relations is required by relativistic invariance.

The partial derivatives $\partial/\partial x_\mu$ will often be abbreviated to $\partial_\mu$. For example, the Lorentz condition $\nabla \mathbf{A} + \partial \phi/\partial t = 0$ can be written as $\partial_\mu A_\mu = 0$, showing that the Lorentz condition is relativistically covariant. Furthermore we have that $\partial^2 \phi = \partial_\mu \partial_\mu \phi = (\nabla^2 - \partial^2/\partial t^2)\phi$, an expression that is useful in writing down a Lorentz invariant wave equation for the function $\phi$. 
In some equations the notation $\text{h.c.}$ appears. This differs from the usual h.c. in that $\text{h.c.}$ preserves the order of operators to which it is applied. So the h.c. of $AB$ is $B^\dagger A^\dagger$ while the h.c. of $AB$ is $A^\dagger B^\dagger$.

We do not follow the convention of some textbooks in which $e$ stands for the absolute value of the electron charge; we use $e = -1.602 \times 10^{-19}$ C.
Part I

Relativistic Quantum Physics
1

Electromagnetic Radiation
and Matter

1.1 HAMILTONIAN AND VECTOR POTENTIAL

The classical Hamiltonian describing the interaction of a particle with mass $m$ and charge $e$ with an electromagnetic field with vector potential $A$ and scalar potential $\phi$ is

$$H = \frac{1}{2m}(p - eA)^2 + e\phi$$

(1.1)

where $p$ is the momentum of the particle. For example, $A$ could be the vector potential of an external magnetic field while $\phi$ could be the Coulomb potential due to the presence of another charge. We do not follow the convention where $e$ stands for the absolute value of the electron charge; in our case $e = -1.602 \times 10^{-19} \text{C}$ for an electron. This Hamiltonian is derived by casting the Lorentz force in the Hamiltonian formalism. It can also be obtained from the Hamiltonian of a free particle

$$H = \frac{p^2}{2m}$$

(1.2)

with the substitutions

$$p \to p - eA \quad H \to H - e\phi$$

(1.3)

This is called the ‘Minimal Substitution’. The substitutions (1.3) can be written in covariant form as

$$p_\mu \to p_\mu - eA_\mu$$

(1.4)
with $p_\mu = (p, iE)$ and $A_\mu = (A, i\phi)$, $c = 1$. The word ‘minimal’ indicates that no additional terms that would in principle be allowed are included (experiment is the arbiter). An example would be a term that accounts for the intrinsic magnetic moment of the particle. As is known from the study of an atom in an external static magnetic field, the Hamiltonian Equation (1.1) already accounts for a magnetic moment associated with orbital angular momentum. To obtain the Hamiltonian of the quantized system we use as always the replacements

$$ p \rightarrow \frac{1}{i} \nabla \quad H \text{(or } E) \rightarrow -\frac{1}{i} \frac{\partial}{\partial t} $$ \hspace{1cm} (1.5)

or in covariant form

$$ p_\mu \rightarrow \frac{1}{i} \frac{\partial}{\partial x_\mu} = \frac{1}{i} \partial_\mu $$ \hspace{1cm} (1.6)

The requirement that the substitutions in Equation (1.5) be covariant, that is, that they can be written in the form of Equation (1.6), explains the minus sign in Equation (1.5). The substitutions are a manifestation of ‘First Quantization’ in which momenta and energies, and functions dependent upon these, become operators. They lead to the commutation relations

$$ [p_\mu, x_\nu] = \frac{1}{i} \delta_{\mu\nu} $$ \hspace{1cm} (1.7)

where $\delta_{\mu\nu}$ is the Kronecker delta.

The question arises as to how to quantize the electromagnetic field. We know from the hypothesis of Planck and its extension by Einstein in the treatment of black body radiation and the photo-electric effect respectively that electromagnetic energy is quantized with quanta equal $\hbar \omega$. The replacements Equation (1.6) are of no use for an explanation. We will address this issue fully in the next subsection when we introduce ‘Second Quantization’ in which the vector and scalar potentials and thus the electric and magnetic fields become operators. This is a prototype of relativistic quantum field theory.

In preparation for quantization of the electromagnetic field, we will briefly review the arguments that lead to the wave equation for $A$ with $\phi = 0$ (the Coulomb Gauge). Recall from classical electromagnetism that when the vector and scalar potentials are transformed into new ones by the Gauge transformation

$$ A \rightarrow A' = A + \nabla \chi \quad \phi \rightarrow \phi' = \phi - \frac{\partial \chi}{\partial t} $$ \hspace{1cm} (1.8)
that the values of electric and magnetic fields $E$ and $B$ do not change. This is so because under the Gauge transformation in Equation (1.8) we have

$$E = -\nabla \phi - \frac{\partial A}{\partial t} \rightarrow E' = -\nabla \phi' - \frac{\partial A'}{\partial t} = E$$

(1.9)

and

$$B = \nabla \times A \rightarrow B' = \nabla \times A' = A$$

(1.10)

where we used in Equation (1.9) that the order of $\nabla$ and $\partial/\partial t$ can be exchanged while we used in Equation (1.10) that $\nabla \times \nabla \chi = 0$ for all $\chi$. The function $\chi$ is arbitrary (unconstrained). The minus sign in Equation (1.8) is necessary for $E$ to remain unchanged under the Gauge transformation. The minus sign also follows if we require that the relations Equation (1.8) can be written in a covariant form

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \chi$$

(1.11)

Writing out the fourth component of $A_\mu = (A, i\phi)$ in Equation (1.11), one finds the second relation of Equation (1.8). One can make a Gauge transformation to find new $A'$ and $\phi'$ such that they satisfy the Lorentz condition

$$\nabla \cdot A' + \frac{\partial \phi'}{\partial t} = \partial_\mu A'_\mu = 0$$

(1.12)

as follows. If $\nabla \cdot A + \partial \phi / \partial t = f(x, t) \neq 0$ then the new $A'$ and $\phi'$ will satisfy the Lorentz condition in Equation (1.12) if $\chi$ is required to satisfy the inhomogeneous wave equation

$$\nabla^2 \chi - \frac{\partial^2 \chi}{\partial t^2} = -f(x, t)$$

(1.13)

The Lorentz condition is seen to be covariant as well. The Lorentz condition simplifies the differential equations for the vector and scalar potentials to

$$\nabla^2 A - \frac{\partial^2 A}{\partial t^2} = -4\pi j \quad \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho$$

(1.14)

or

$$\partial_\mu \partial_\mu A_\alpha = \partial_\alpha A_\alpha = 0$$

(1.15)

showing that the wave equations for the potentials are covariant as they should be. This is the Lorentz Gauge.
There is more freedom left in the choice of $A$ and $\phi$ in that a further Gauge transformation as in Equation (1.11) can be made that results in the scalar potential $\phi$ being zero if we require that the new $\chi$ satisfies

$$
\phi = \frac{\partial \chi}{\partial t} \quad \text{and} \quad \nabla^2 \chi - \frac{\partial^2 \chi}{\partial t^2} = 0
$$

(1.16)

compared with Equation (1.13). This is the Coulomb or Radiation Gauge. The electric and magnetic fields are now given by

$$
E = -\frac{\partial A}{\partial t} \quad \text{and} \quad B = \nabla \times A
$$

(1.17)

compared with the relations for $E$ and $B$ used in Equation (1.9) and Equation (1.10). The Lorentz condition in Equation (1.12) in the Coulomb Gauge is

$$
\nabla \cdot A = 0
$$

(1.18)

We introduce the wave vector $k$ and the corresponding angular frequency $\omega$ with $k^2 = \omega^2$. It follows from Equation (1.18) for a traveling wave of the form

$$
A(x, t) = A_0 \exp(ik \cdot x - \omega t)
$$

so $k$ and $A$ are perpendicular to each other. From the first relation in Equation (1.17) we find that $E = i\omega A$, so $E$ and $A$ are parallel and thus $k$ and $E$ are also perpendicular. The second relation in Equation (1.17) gives $B = ik \times A = k \times E/\omega$. This relation shows that $B$ is in phase with $E$ and is perpendicular to both $k$ and $E$, so all three vectors $E$, $B$, $k$ are mutually perpendicular. They form a right-handed triplet in that order because $E \times B = E \times (k \times E)/\omega = E^2 k/\omega$ where we used that $E \cdot k = 0$. We define the polarization of the electromagnetic field as the direction of the electric field. This is so because the effects of the electric field dominate those of the magnetic field, for example in the exposure of photographic film. Because the electric field is perpendicular to its momentum we say that the electromagnetic field is transversely polarized. The cross product $E \times B$ equals the Poynting vector $S$, which is in the direction of $k$ as it should be. Because Gauge transformations do not change the physical properties of the electromagnetic field, the last conclusion about the orientation of $E$, $B$ and $k$ holds as well in the Lorentz Gauge and indeed in general. Note that by transforming away $\phi$ we have not removed a Coulomb potential that might be present, we only removed the scalar potential associated with the electromagnetic field described by the coupled $E$ and $B$.

The reader is urged to review this material from the text used in classical electromagnetism. A problem about Gauge transformations and the Coulomb Gauge is provided at the end of this chapter.
We stress that the sequence of two Lorentz transformations has left the electric and magnetic fields unchanged, so the physical properties of the system have not been affected. This is in analogy with elementary classical mechanics where it is shown that the potential energy $U(r)$ is defined up to a constant because adding a constant to $U$ does not change the force $F = -\nabla U$ and does not change the physical properties of the system.

The total Hamiltonian of the system consisting of a charged particle in an electromagnetic field with no other charges and currents but the ones associated with the particle in Equation (1.1) consists of the part given in Equation (1.1) and the energy of the electromagnetic field

$$E_{em} = \frac{1}{8\pi} \int d^3x (|E|^2 + |B|^2)$$

(1.20)

Because $E$ or $B$ may be complex we use absolute values in the integrand. The homogeneous wave Equation (1.14) for $A$ becomes

$$\nabla^2 A - \frac{\partial^2 A}{\partial t^2} = 0$$

(1.21)

This equation describes ‘free’ electromagnetic fields, that is, fields in the absence of currents and charges. We now seek solutions of the homogeneous wave Equation (1.21). We introduce the four-vector $k_\mu = (k, i\omega)$ where $k$ is the wave vector and $\omega$ the corresponding angular frequency. With $x_\mu = (x, i ct)$ we find that $kx = k \cdot x - \omega t$. Therefore we can write a traveling wave $A(x, t) = A_0 \exp(ik \cdot x - \omega t)$ as $A(x) = A_0 \exp(ikx)$. Traveling waves with the vector potential of this form satisfy the wave equation provided that $k^2 - \omega^2 = k^2 = 0$. This condition can be enforced in the solution by including a factor $\delta(k^2)$ where $\delta$ is the Dirac $\delta$ function. The solution must represent a three-dimensional vector so we make use of three mutually perpendicular unit vectors $\varepsilon_\lambda(k)$ ($\lambda = 1, 2, 3$). The Lorentz condition in the Coulomb Gauge Equation (1.19) requires that $k$ and $A$ are perpendicular to each other. If we choose $\varepsilon_1$ and $\varepsilon_2$ perpendicular to $k$ (and to each other) and $\varepsilon_3$ parallel to $k$, such that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ form a right-handed set of unit vectors, terms proportional to $\varepsilon_3$ must be absent because of Equation (1.19). Because the wave Equation (1.21) is linear, its most general solution is a linear superposition of terms of the form $a_\lambda(k, \omega) \exp[i(k \cdot x - \omega t)]\varepsilon_\lambda(k)\delta(k^2)$ $\lambda = 1, 2$. Our notation shows that the pre-factor $a_\lambda(k, \omega)$ depends upon $k$ and $\omega$ and that the unit vectors $\varepsilon_\lambda(k)$ depend upon (the direction of) $k$. The most general solution can thus be written as

$$A(x, t) = \left(\frac{1}{2\pi}\right)^2 \int d^3k d\omega \sum_{\lambda=1}^{2} a_\lambda(k) e^{i(k \cdot x - \omega t)} \varepsilon_\lambda(k) \delta(k^2)$$

(1.22)
The $\varepsilon_\lambda(k)$ are called polarization vectors. One can interpret this solution as a four-dimensional version of the Fourier transform which is familiar from the solution of a wave equation in one dimension. The one-dimensional Fourier transform has a pre-factor $1/\sqrt{2\pi}$, hence the $(1/2\pi)^2$ in Equation (1.22). This is the integral version of the Fourier transform. Substitution of Equation (1.22) in the wave Equation (1.21) shows that Equation (1.22) is indeed a solution if the condition $k^2 \delta(k^2) = 0$ is satisfied for each term separately. This condition is satisfied for the Dirac-$\delta$ function as in general $x \delta(x) = 0$ because the Dirac-$\delta$(x) function is even in x and it is multiplied by the odd function $x$. The general solution in Equation (1.22) satisfies the condition in Equation (1.19) by construction. We know that $A$ and $E$ are parallel, see Equation (1.17), so the direction of $A$ as specified by $a_1(k, \omega) \varepsilon_1(k)$ and $a_2(k, \omega) \varepsilon_2(k)$ is the direction of polarization.

We can simplify Equation (1.22) by integrating over $\omega$ using the constraint provided by the factor $\delta(k^2) = 0$. The condition $k^2 = k^2 - \omega^2 = 0$ lead to the requirement that $\omega = \pm |k|$. This is also expressed by the property of the Dirac-$\delta$ function

$$\delta(k^2) = \delta(k^2 - \omega^2) = \frac{1}{2\omega_k} \left[ \delta(|k| + \omega) + \delta(|k| - \omega) \right]$$  (1.23)

We introduce $\omega_k = |k|$ and obtain

$$A(x, t) = \left( \frac{1}{2\pi} \right)^2 \int \frac{d^3k d\omega}{2\omega_k} \sum_{\lambda=1}^2 a_\lambda(k, \omega) e^{i(k \cdot x - \omega t)} [\delta(\omega_k + \omega) + \delta(\omega_k - \omega)] \varepsilon_\lambda(k)$$

$$= \left( \frac{1}{2\pi} \right)^2 \int \frac{d^3k}{2\omega_k} \sum_{\lambda=1}^2 [a_\lambda(k, -\omega_k) e^{i(k \cdot x + \omega_k t)} + a_\lambda(k, \omega_k) e^{i(k \cdot x - \omega_k t)}] \varepsilon_\lambda(k)$$  (1.24)

We call the first term in Equation (1.24) with the negative value of $\omega$ the negative energy solution, while we call the second term in Equation (1.24) with the positive value of $\omega$ the positive energy solution. It is conventional to remove a factor $1/\sqrt{2\pi}$ and a factor $1/\sqrt{2\omega_k}$ and to add a factor $\sqrt{4\pi}$ in Equation (1.24). This merely redefines the coefficients $a_\lambda$. The reason for the first change is that the Fourier integral over $d^3k$ ought to be accompanied by a pre-factor $1/\sqrt{2\pi}$ for each integration variable. The reason for the other two changes will become clear in the next subsection.

We would like to reinstate the compact notation $\exp(i k x)$ in Equation (1.24). The positive energy term is already of this form but the negative energy term is not because the terms $k \cdot x$ and $\omega_k t$ in the exponential do not have opposite signs. We can arrange for that by introducing a new integration variable $k' = -k$ in the negative energy term only. This gives
three minus signs from $d^3k' = -d^3k$ and changes the signs of the integration limits in each of the three integrals. Exchanging the new upper and lower integration limits in each of the three integrals gives another three minus signs. The net result is that all minus signs cancel. We drop the prime in $k'$. The argument of $a_\lambda$ in the negative energy term will now be $-k$. Another way to see this is to consider that we are looking for solutions of the wave equation in a large box with periodic boundary conditions at its surface. Solutions with $+k$ and $-k$ have the same physical properties.

The expression for $A$ becomes

$$A(x) = \left(\frac{1}{2\pi}\right)^\frac{3}{2} \sqrt{4\pi} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \left[ a_\lambda(-k, -\omega_k) e^{-ikx} \epsilon_\lambda(-k) + a_\lambda^*(k, \omega_k) e^{ikx} \epsilon_\lambda^*(k) \right]$$

(1.25)

$A$ is used to calculate $E$ and $B$ according to Equation (1.17). In the next section $A$, $E$ and $B$ become operators. The eigenvalues of $E$ and $B$ are observable, so the operators $E$ and $B$ and therefore $A$ must be Hermitian operators. This means that $A$ must be real quantity and we require that $A = A^*$. We find that

$$A^*(x) = \left(\frac{1}{2\pi}\right)^\frac{3}{2} \sqrt{4\pi} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \left[ a_\lambda^*(-k, -\omega_k) e^{ikx} \epsilon_\lambda^*(-k) + a_\lambda^*(k, \omega_k) e^{-ikx} \epsilon_\lambda^*(k) \right]$$

(1.26)

Comparing Equation (1.26) with Equation (1.25) and using that the exponentials $\exp(\pm i kx)$ are orthogonal to each other, we find that

$$\sum_{\lambda} a_\lambda(-k, -\omega_k) \epsilon_\lambda(-k) = \sum_{\lambda} a_\lambda^*(k, \omega_k) \epsilon_\lambda^*(k)$$

(1.27)

$$\sum_{\lambda} a_\lambda(k, \omega_k) \epsilon_\lambda(k) = \sum_{\lambda} a_\lambda^*(-k, -\omega_k) \epsilon_\lambda^*(-k)$$

(1.28)

The two relations are each other’s complex conjugate, so there is just one condition on $a_\lambda$ and its complex conjugate. We will not attempt to formulate relations between individual terms in the sums over $\lambda$ because we do not need those, and if we try it would lead to a left-handed set of three unit vectors $\epsilon$ or the appearance of minus signs in nasty places. Using Equation (1.27) in Equation (1.25) we have the result

$$A(x) = \left(\frac{1}{2\pi}\right)^\frac{3}{2} \sqrt{4\pi} \int \frac{d^3k}{\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \left[ a_\lambda(k) e^{ikx} \epsilon_\lambda(k) + a_\lambda^*(k) e^{-ikx} \epsilon_\lambda^*(k) \right]$$

(1.29)