

---

# DYNAMICS AND RELATIVITY

---

**Jeffrey R. Forshaw and A. Gavin Smith**

*School of Physics and Astronomy,  
The University of Manchester, Manchester, U.K.*



A John Wiley and Sons, Ltd., Publication



# **Dynamics and Relativity**

---

# The Manchester Physics Series

*General Editors*

**F.K. LOEBINGER: F. MANDL: D.J. SANDIFORD:**

*School of Physics and Astronomy,  
The University of Manchester*

---

<b>Properties of Matter:</b>	B. H. Flowers and E. Mendoza
<b>Statistical Physics:</b> <i>Second Edition</i>	F. Mandl
<b>Electromagnetism:</b> <i>Second Edition</i>	I. S. Grant and W. R. Phillips
<b>Statistics:</b>	R. J. Barlow
<b>Solid State Physics:</b> <i>Second Edition</i>	J. R. Hook and H. E. Hall
<b>Quantum Mechanics:</b>	F. Mandl
<b>Computing for Scientists:</b>	R. J. Barlow and A. R. Barnett
<b>The Physics of Stars:</b> <i>Second Edition</i>	A. C. Phillips
<b>Nuclear Physics:</b>	J. S. Lilley
<b>Introduction to Quantum Mechanics:</b>	A. C. Phillips
<b>Particle Physics:</b> <i>Third Edition</i>	B. R. Martin and G. Shaw
<b>Dynamics and Relativity:</b>	J. R. Forshaw and A. G. Smith
<b>Vibrations and Waves:</b>	G.C. King

---

# DYNAMICS AND RELATIVITY

---

**Jeffrey R. Forshaw and A. Gavin Smith**

*School of Physics and Astronomy,  
The University of Manchester, Manchester, U.K.*



A John Wiley and Sons, Ltd., Publication

This edition first published 2009  
© 2009 John Wiley & Sons Ltd

*Registered office*

John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

For details of our global editorial offices, for customer services and for information about how to apply for permission to reuse the copyright material in this book please see our website at [www.wiley.com](http://www.wiley.com).

The right of the author to be identified as the author of this work has been asserted in accordance with the Copyright, Designs and Patents Act 1988.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, except as permitted by the UK Copyright, Designs and Patents Act 1988, without the prior permission of the publisher.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic books.

Designations used by companies to distinguish their products are often claimed as trademarks. All brand names and product names used in this book are trade names, service marks, trademarks or registered trademarks of their respective owners. The publisher is not associated with any product or vendor mentioned in this book. This publication is designed to provide accurate and authoritative information in regard to the subject matter covered. It is sold on the understanding that the publisher is not engaged in rendering professional services. If professional advice or other expert assistance is required, the services of a competent professional should be sought.

The Publisher and the Author make no representations or warranties with respect to the accuracy or completeness of the contents of this work and specifically disclaim all warranties, including without limitation any implied warranties of fitness for a particular purpose. This work is sold with the understanding that the Publisher is not engaged in rendering professional services. The advice and strategies contained herein may not be suitable for every situation. In view of ongoing research, equipment modifications, changes in governmental regulations, and the constant flow of information relating to the use of experimental reagents, equipment, and devices, the reader is urged to review and evaluate the information provided in the package insert or instructions for each chemical, piece of equipment, reagent, or device for, among other things, any changes in the instructions or indication of usage and for added warnings and precautions. The fact that an organization or Website is referred to in this work as a citation and/or a potential source of further information does not mean that the Author or the Publisher endorses the information the organization or Website may provide or recommendations it may make. Further, readers should be aware that Internet Websites listed in this work may have changed or disappeared between when this work was written and when it is read. No warranty may be created or extended by any promotional statements for this work. Neither the Publisher nor the Author shall be liable for any damages arising herefrom.

***Library of Congress Cataloging-in-Publication Data***

Forshaw, J. R. (Jeffrey Robert), 1968 –

Dynamics and relativity/Jeffrey R. Forshaw and A. Gavin Smith.

p. cm.

Includes bibliographical references and index.

ISBN 978-0-470-01459-2 (cloth : alk. paper) – ISBN 978-0-470-01460-8 (pbk. : alk. paper)

1. Special relativity (Physics) 2. Dynamics. I. Smith, A. Gavin. II. Title.

QC173.65.F67 2009

530.11 – dc22

2008053366

A catalogue record for this book is available from the British Library.

ISBN 978-0-470-01459-2 (HB)

ISBN 978-0-470-01460-8 (PB)

Typeset in 10/12 Times by Laserwords Private Limited, Chennai, India

Printed and bound in Great Britain by CPI Antony Rowe, Chippenham, Wiltshire

Dedicated to the memory of Howard North and Edward Swallow.





# Contents

Editors' Preface to the Manchester Physics Series	xi
Author's Preface	xiii
<b>I INTRODUCTORY DYNAMICS</b>	<b>1</b>
<b>1 SPACE, TIME AND MOTION</b>	<b>3</b>
1.1 Defining Space and Time	3
1.1.1 Space and the classical particle	4
1.1.2 Unit vectors	6
1.1.3 Addition and subtraction of vectors	6
1.1.4 Multiplication of vectors	7
1.1.5 Time	8
1.1.6 Absolute space and space-time	10
1.2 Vectors and Co-ordinate Systems	11
1.3 Velocity and Acceleration	14
1.3.1 Frames of reference	16
1.3.2 Relative motion	16
1.3.3 Uniform acceleration	18
1.3.4 Velocity and acceleration in plane-polar co-ordinates: uniform circular motion	20
1.4 Standards and Units	21
<b>2 FORCE, MOMENTUM AND NEWTON'S LAWS</b>	<b>25</b>
2.1 Force and Static Equilibrium	25
2.2 Force and Motion	31
2.2.1 Newton's Third Law	35
2.2.2 Newton's bucket and Mach's principle	39
2.3 Applications of Newton's Laws	41
2.3.1 Free body diagrams	41
2.3.2 Three worked examples	42
2.3.3 Normal forces and friction	46
2.3.4 Momentum conservation	49

2.3.5 Impulse	51
2.3.6 Motion in fluids	51
<b>3 ENERGY</b>	<b>55</b>
3.1 Work, Power and Kinetic Energy	56
3.2 Potential Energy	61
3.2.1 The stability of mechanical systems	64
3.2.2 The harmonic oscillator	65
3.2.3 Motion about a point of stable equilibrium	67
3.3 Collisions	68
3.3.1 Zero-momentum frames	68
3.3.2 Elastic and inelastic collisions	71
3.4 Energy Conservation in Complex Systems	75
<b>4 ANGULAR MOMENTUM</b>	<b>81</b>
4.1 Angular Momentum of a Particle	81
4.2 Conservation of Angular Momentum in Systems of Particles	83
4.3 Angular Momentum and Rotation About a Fixed Axis	86
4.3.1 The parallel-axis theorem	94
4.4 Sliding and Rolling	95
4.5 Angular Impulse and the Centre of Percussion	97
4.6 Kinetic Energy of Rotation	99
<b>II INTRODUCTORY SPECIAL RELATIVITY</b>	<b>103</b>
<b>5 THE NEED FOR A NEW THEORY OF SPACE AND TIME</b>	<b>105</b>
5.1 Space and Time Revisited	105
5.2 Experimental Evidence	108
5.2.1 The Michelson-Morley experiment	108
5.2.2 Stellar aberration	110
5.3 Einstein's Postulates	113
<b>6 RELATIVISTIC KINEMATICS</b>	<b>115</b>
6.1 Time Dilation, Length Contraction and Simultaneity	115
6.1.1 Time dilation and the Doppler effect	116
6.1.2 Length contraction	121
6.1.3 Simultaneity	123
6.2 Lorentz Transformations	124
6.3 Velocity Transformations	129
6.3.1 Addition of velocities	129
6.3.2 Stellar aberration revisited	130
<b>7 RELATIVISTIC ENERGY AND MOMENTUM</b>	<b>135</b>
7.1 Momentum and Energy	135

---

7.1.1 The equivalence of mass and energy	142
7.1.2 The hint of an underlying symmetry	144
7.2 Applications in Particle Physics	145
7.2.1 When is relativity important?	146
7.2.2 Two useful relations and massless particles	149
7.2.3 Compton scattering	152
<b>III ADVANCED DYNAMICS</b>	<b>157</b>
<b>8 NON-INERTIAL FRAMES</b>	<b>159</b>
8.1 Linearly Accelerating Frames	159
8.2 Rotating Frames	161
8.2.1 Motion on the earth	165
<b>9 GRAVITATION</b>	<b>173</b>
9.1 Newton's Law of Gravity	174
9.2 The Gravitational Potential	177
9.3 Reduced Mass	182
9.4 Motion in a Central Force	184
9.5 Orbits	186
<b>10 RIGID BODY MOTION</b>	<b>197</b>
10.1 The Angular Momentum of a Rigid Body	198
10.2 The Moment of Inertia Tensor	200
10.2.1 Calculating the moment of inertia tensor	203
10.3 Principal Axes	207
10.4 Fixed-axis Rotation in the Lab Frame	212
10.5 Euler's Equations	214
10.6 The Free Rotation of a Symmetric Top	216
10.6.1 The body-fixed frame	216
10.6.2 The lab frame	218
10.6.3 The wobbling earth	223
10.7 The Stability of Free Rotation	224
10.8 Gyroscopes	226
10.8.1 Gyroscopic precession	226
10.8.2 Nutation of a gyroscope	232
<b>IV ADVANCED SPECIAL RELATIVITY</b>	<b>237</b>
<b>11 THE SYMMETRIES OF SPACE AND TIME</b>	<b>239</b>
11.1 Symmetry in Physics	239
11.1.1 Rotations and translations	240
11.1.2 Translational symmetry	245

---

11.1.3 Galilean symmetry	246
11.2 Lorentz Symmetry	247
<b>12 FOUR-VECTORS AND LORENTZ INVARIANTS</b>	<b>253</b>
12.1 The Velocity Four-vector	254
12.2 The Wave Four-vector	255
12.3 The Energy-momentum Four-vector	258
12.3.1 Further examples in relativistic kinematics	259
12.4 Electric and Magnetic Fields	262
<b>13 SPACE-TIME DIAGRAM AND CAUSALITY</b>	<b>267</b>
13.1 Relativity Preserves Causality	270
13.2 An Alternative Approach	272
<b>14 ACCELERATION AND GENERAL RELATIVITY</b>	<b>279</b>
14.1 Acceleration in Special Relativity	279
14.1.1 Twins paradox	280
14.1.2 Accelerating frames of reference	282
14.2 A Glimpse of General Relativity	288
14.2.1 Gravitational fields	290
<b>A DERIVING THE GEODESIC EQUATION</b>	<b>295</b>
<b>B SOLUTIONS TO PROBLEMS</b>	<b>297</b>

# Editors' Preface to the Manchester Physics Series

The Manchester Physics Series is a series of textbooks at first degree level. It grew out of our experience at the University of Manchester, widely shared elsewhere, that many textbooks contain much more material than can be accommodated in a typical undergraduate course; and that this material is only rarely so arranged as to allow the definition of a short self-contained course. In planning these books we have had two objectives. One was to produce short books so that lecturers would find them attractive for undergraduate courses, and so that students would not be frightened off by their encyclopaedic size or price. To achieve this, we have been very selective in the choice of topics, with the emphasis on the basic physics together with some instructive, stimulating and useful applications. Our second objective was to produce books which allow courses of different lengths and difficulty to be selected with emphasis on different applications. To achieve such flexibility we have encouraged authors to use flow diagrams showing the logical connections between different chapters and to put some topics in starred sections. These cover more advanced and alternative material which is not required for the understanding of latter parts of each volume.

Although these books were conceived as a series, each of them is self-contained and can be used independently of the others. Several of them are suitable for wider use in other sciences. Each Author's Preface gives details about the level, prerequisites, etc., of that volume.

The Manchester Physics Series has been very successful since its inception 40 years ago, with total sales of more than a quarter of a million copies. We are extremely grateful to the many students and colleagues, at Manchester and elsewhere, for helpful criticisms and stimulating comments. Our particular thanks go to the authors for all the work they have done, for the many new ideas they have contributed, and for discussing patiently, and often accepting, the suggestions of the editors.

Finally we would like to thank our publishers, John Wiley & Sons, Ltd., for their enthusiastic and continued commitment to the Manchester Physics Series.

F. K. Loebinger

F. Mandl

D. J. Sandiford

*August 2008*

# Authors' Preface

In writing this book, our goal is to help the student develop a good understanding of classical dynamics and special relativity. We have tried to start out gently: the first part of the book aims to provide the solid foundations upon which the second half builds. In the end, we are able, in the final chapter, to cover some quite advanced material for a book at this level (when we venture into the terrain of Einstein's General Theory of Relativity) and it is our hope that our pedagogical style will lead the keen student all the way to the denouement. That said, we do not assume too much prior knowledge. A little calculus, trigonometry and some exposure to vectors would help but not much more than that is needed in order to get going. We have in mind that the first half of the book covers material core to a typical first year of undergraduate studies in physics, whilst the second half covers material that might appear in more advanced first or second year courses (e.g. material such as the general rotation of rigid bodies and the role of four-vectors in special relativity).

The classical mechanics of Newton and the theory of relativity, developed by Einstein, both make assumptions as to the structure of space and time. For Newton time is an absolute, something to be agreed upon by everyone, whilst for Einstein time is more subjective and clocks tick at different rates depending upon where they are and how they are moving. Such different views lead to different physics and by presenting Newtonian mechanics alongside relativity, as we do in this book, it becomes possible to compare and contrast the two. Of course, we shall see how Newtonian physics provides a very good approximation to that of Einstein for most everyday phenomena, but that it fails totally when things whizz around at speeds approaching the speed of light.

In this era of electronic communications and online resources that can be researched at the push of a button, it might seem that the need for textbooks is diminished. Perhaps not surprisingly we don't think that is the case. Quiet time spent with a textbook, some paper and a pen, reading and solving problems, is probably still the best way to do physics. Just as one cannot claim to be a pianist without playing a piano, one cannot claim to be a physicist without solving physics problems. It is a point much laboured, but it is true nonetheless. The problems that really help develop understanding are usually those that take time to crack. The painful process of failing to solve a problem is familiar to every successful physicist, as is the excitement of figuring out the way forward. Our advice when solving the problems in this book is to persevere for as long as

possible before peeking at the solution, to try and enjoy the process and not to panic if you cannot see how to start a problem.

We have deliberately tried to keep the figures as simple as possible. A good drawing can often be an important step to solving a physics problem, and we encourage you to make them at every opportunity. For that reason, we have illustrated the book with the sorts of drawings that we would normally use in lectures or tutorials and have deliberately avoided the sort of embellishments that would undoubtedly make the book look prettier. Our aim is to present diagrams that are easy to reproduce.

A comment is in order on our usage of the word “classical”. For us “classical” refers to physics pre-Einstein but not everyone uses that terminology. Sometimes, classical is used to refer to the laws of physics in the absence of quantum mechanics and in that sense, special relativity could be said to be a classical theory. We have nothing to say about the quantum theory in this book, except that quantum theories that are also consistent with relativity lie at the very heart of modern physics. Hopefully this book will help whet the appetite for further studies in that direction.

We should like to express our gratitude to all those who have read the manuscript and provided helpful suggestions. In particular we thank Rob Appleby, Richard Battye, Mike Birse, Brian Cox, Joe Dare, Fred Loebinger, Nicola Lumley, Franz Mandl, Edward Reeves, David Sandiford and Martin Yates.

Finally, we would like to express particular gratitude to our parents, Thomas & Sylvia Forshaw and Roy & Marion Smith, for their constant support. For their love and understanding, our heartfelt thanks go to Naomi, Isabel, Jo, Ellie, Matt and Josh.

Jeffrey R. Forshaw  
A. Gavin Smith  
October 2008



# **Part I**

## **Introductory Dynamics**



# 1

## Space, Time and Motion

### 1.1 DEFINING SPACE AND TIME

If there is one part of physics that underpins all others, it is the study of motion. The accurate description of the paths of celestial objects, of planets and moons, is historically the most celebrated success of a classical mechanics underpinned by Newton's laws<sup>1</sup>. The range of applicability of these laws is vast, encompassing a scale that extends from the astronomical to the microscopic. We have come to understand that many phenomena not previously associated with motion are in fact linked to the movement of microscopic objects. The absorption and emission spectra of atoms and molecules arise as a result of transitions made by their constituent electrons, and the random motion of ensembles of atoms and molecules forms the basis for the modern statistical description of thermodynamics. Although atomic and subatomic objects are properly described using quantum mechanics, an understanding of the principles of classical mechanics is essential in making the conceptual leap from continuous classical systems with which we are most familiar, to the discretised quantum mechanical systems, which often behave in a manner at odds with our intuition. Indeed, the calculational techniques that are routinely used in quantum mechanics have their roots in the classical mechanics of particles and waves; a close familiarity with their use in classical systems is an asset when facing problems of an inherently quantum mechanical nature.

As we shall see in the second part of this book, when objects move at speeds approaching the speed of light classical notions about the nature of space and time fail us. As a result, the classical mechanics of Newton should be viewed as a low-velocity approximation to the more accurate relativistic theory of Einstein<sup>2</sup>. To look carefully at the differences between relativistic and non-relativistic theories

<sup>1</sup> After Isaac Newton (1643–1727).

<sup>2</sup> Albert Einstein (1879–1955).

forces us to recognise that our intuitive ideas about how things move are often incorrect. At the most fundamental level, mechanics of either the classical or the quantum kind, in either the relativistic or non-relativistic limit, is a study of motion and to study motion is to ask some fundamental questions about the nature of space and time. In this book we will draw out explicitly the different underlying structures of space and time used in the approaches of Newton and Einstein.

### 1.1.1 Space and the classical particle

We all have strong intuitive ideas about space, time and motion and it is precisely because of this familiarity that we must take special care in our attempts to define these fundamental concepts, so as not to carry too many unrecognised assumptions along with us as we develop the physics. So let us start by picking apart what we mean by position. We can usually agree what it means for London to be further away than Inverness and we all know that in order to go to London from Inverness we must also know the direction in which to travel. It may also seem to be fairly uncontentious that an object, such as London, has a position that can be specified, i.e. it is assumed that given enough information there will be no ambiguity about where it is. Although this seems reasonable, there is immediately a problem: day-to-day objects such as tennis balls and cities have finite size; there are a number of ‘positions’ for a given object that describe different parts of the object. Having directions to London may not be enough to find Kings Cross station, and having directions to Kings Cross station may not be enough to find platform number nine. To unambiguously give the position of an object is therefore only possible if the object is very small – vanishingly small, in fact. This hypothetical, vanishingly small object is called a particle. It might be suggested that with the discovery of the substructure of the atom, true particles, with mass but no spatial extent, have been identified. However, at this level, the situation becomes complicated by quantum uncertainty which makes the simultaneous specification of position and momentum impossible. The classical particle is therefore an idealisation, a limit in which the size of an object tends to zero but in which we ignore quantum phenomena. Later we shall see that it is possible to define a point called the centre of mass of an extended object and that this point behaves much like a classical particle. The collection of all possible positions for a particle forms what we call space.

The mathematical object possessing the properties we require for the description of position is called the vector. A vector has both magnitude and direction and we must be careful to distinguish it from a pure number which has a magnitude, but no directional properties. The paradigm for the vector comes from the displacement of a particle from point  $A$  to point  $B$  as shown in Figure 1.1. The displacement from  $A$  to  $B$  is represented by the directed-line-segment  $AB$ . We can imagine specifying the displacement as, for example, “start at  $A$  and move 3 km to the northeast” or “start at  $A$  and go 1 parsec in the direction of Alpha Centuri”. Once we have specified a displacement between the two points  $A$  and  $B$  we can imagine sliding each end of the line segment in space until it connects another two points  $C$  and  $D$ . To do this, we move each end through the same distance and in the same direction,

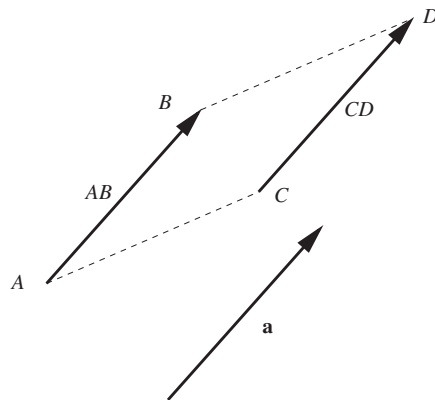


Figure 1.1 Displacement of a particle from point  $A$  to point  $B$  is illustrated by the directed line segment  $AB$ . Parallel transport of this line gives the displacement from point  $C$  to point  $D$ . The displacement vector  $\mathbf{a}$  is not associated with any particular starting point.

an operation that is known as parallel transport. Now the displacement is denoted  $CD$  but its direction and magnitude are the same. It should be clear that there is an infinity of such displacements that may be obtained by parallel transport of the directed line segment. The displacement vector  $\mathbf{a}$  has the magnitude and direction common to this infinite set of displacements but is not associated with a particular position in space. This is an important point which sometimes causes confusion since vectors are illustrated as directed line segments, which appear to have a well defined beginning and an end in space: A vector has magnitude and direction but not location. The position of a particle in space may be given generally by a position vector  $\mathbf{r}$  only in conjunction with a fixed point of origin.

Now, all of this assumes that we understand what it means for lines to be parallel. At this point we assume that we are working in Euclidean space, which means that parallel lines remain equidistant everywhere, i.e. they never intersect. In non-Euclidean spaces, such as the two-dimensional surface of a sphere, parallel lines do intersect<sup>3</sup> and extra mathematics is required to specify how local geometries are transported to different locations in the space. For the moment, since we have no need of non-Euclidean geometry, we will rest our discussion of vectors firmly on the familiar Euclidean notion of parallel lines. Later, when we consider the space-time geometry associated with relativistic motion we will be forced to drop this deep-rooted assumption about the nature of space.

So far, we have been considering only vectors that are associated with displacements from one point to another. Their utility is far more wide ranging than that though: vectors are used to represent other interesting quantities in physics. For example the electric field strength in the vicinity of an electric charge is correctly represented by specifying both its magnitude and direction, i.e. it is a vector. Since it is important to maintain the distinction between vectors and ordinary numbers

<sup>3</sup> For example lines of longitude meet at the poles.

(called scalars) we identify vector quantities in this book by the use of bold font. When writing vectors by hand it is usual to either underline the vector, or to put an arrow over the top. Thus

$$\underline{a} \equiv \vec{a} \equiv \mathbf{a}.$$

Use the notation that you find most convenient, but always maintain the distinction between vector and scalar quantities. In this book both upper case ( $\mathbf{A}$ ) and lower case ( $\mathbf{a}$ ) notation will be used for vectors where  $\mathbf{A}$  is in general a different vector from  $\mathbf{a}$ . When a vector has zero magnitude it is impossible to define its direction; we call such a vector the null vector  $\mathbf{0}$ .

### 1.1.2 Unit vectors

The length of a vector  $\mathbf{a}$  is known as its magnitude, often denoted  $|\mathbf{a}|$ . To simplify the notation we shall adopt the convention that vectors are printed in bold and their magnitudes are indicated by dropping the bold font, thus  $a \equiv |\mathbf{a}|$ . Often we will separate the magnitude and direction of a vector, writing

$$\mathbf{a} = a\hat{\mathbf{a}},$$

where  $\hat{\mathbf{a}}$  is the vector of unit magnitude with the same direction as  $\mathbf{a}$ . Unit vectors, of which  $\hat{\mathbf{a}}$  is an example, are often used to specify directions such as the directions of the axes of a co-ordinate system (see below).

### 1.1.3 Addition and subtraction of vectors

The geometrical rules for adding and subtracting vectors are illustrated in Figure 1.2. Addition of the vectors  $\mathbf{A}$  and  $\mathbf{B}$  involves sliding the vectors until they are “head-to-tail”, so that the resultant vector connects the tail of  $\mathbf{A}$  to the head of  $\mathbf{B}$ . The vector  $-\mathbf{A}$  is defined as a vector with the same magnitude, but opposite direction to  $\mathbf{A}$ . The difference  $\mathbf{B} - \mathbf{A}$  is constructed by adding  $\mathbf{B}$  and  $-\mathbf{A}$  as shown. Subtraction of a vector from itself gives the null vector:

$$\mathbf{A} - \mathbf{A} = \mathbf{0}.$$

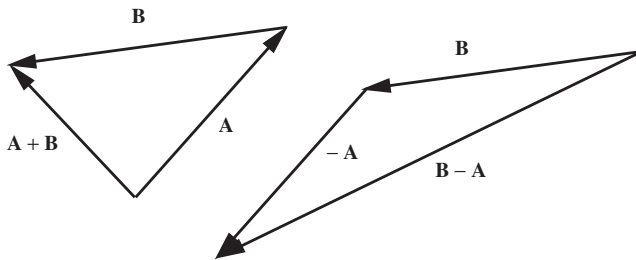


Figure 1.2 Adding and subtracting vectors.

### 1.1.4 Multiplication of vectors

There are two types of vector multiplication that are useful in classical physics. The scalar (or dot) product of two vectors  $\mathbf{A}$  and  $\mathbf{B}$  is defined to be

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta, \quad (1.1)$$

This scalar quantity (a pure number) has a simple geometrical interpretation. It is the projection of  $\mathbf{B}$  on  $\mathbf{A}$ , i.e.  $B \cos \theta$ , multiplied by the length of  $\mathbf{A}$  (see Figure 1.3). Equally, it may be thought of as the projection of  $\mathbf{A}$  on  $\mathbf{B}$ , i.e.  $A \cos \theta$ , multiplied by the length of  $\mathbf{B}$ . Clearly the scalar product is insensitive to the order of the vectors and hence  $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ . The scalar product takes its maximum value of  $AB$  when the two vectors are parallel, and it is zero when the vectors are mutually perpendicular.

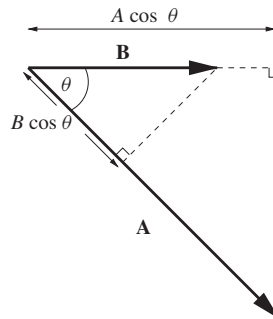


Figure 1.3 Geometry of the scalar product.  $\mathbf{A} \cdot \mathbf{B}$  is the product of the length of  $\mathbf{A}$ , and the projection of  $\mathbf{B}$  onto  $\mathbf{A}$  or alternatively the product of the length of  $\mathbf{B}$ , and the projection of  $\mathbf{A}$  onto  $\mathbf{B}$ .

The vector (or cross) product is another method of multiplying vectors that is frequently used in physics. The cross product of vectors  $\mathbf{A}$  and  $\mathbf{B}$  is defined to be

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta \hat{\mathbf{n}}, \quad (1.2)$$

where  $\theta$  is the angle between  $\mathbf{A}$  and  $\mathbf{B}$  and  $\hat{\mathbf{n}}$  is a unit vector normal to the plane containing both  $\mathbf{A}$  and  $\mathbf{B}$ . Whether  $\hat{\mathbf{n}}$  is 'up' or 'down' is determined by convention and in our case we choose to use the right-hand screw rule; turning the fingers of the right hand from  $\mathbf{A}$  to  $\mathbf{B}$  causes the thumb to point in the sense of  $\hat{\mathbf{n}}$  as is shown in Figure 1.4. Interchanging the order of the vectors in the product means that the fingers of the right hand curl in the opposite sense and the direction of the thumb is reversed. So we have

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}. \quad (1.3)$$

The magnitude of the vector product  $AB \sin \theta$  also has a simple geometrical interpretation. It is the area of the parallelogram formed by the vectors  $\mathbf{A}$  and  $\mathbf{B}$ . Alternatively it can be viewed as the magnitude of one vector times the projection

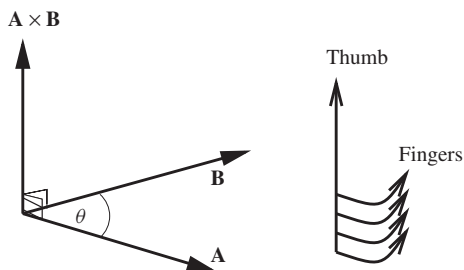


Figure 1.4 Vector product of  $\mathbf{A}$  and  $\mathbf{B}$ .

of the second on an axis which is perpendicular to the first and which lies in the plane of the two vectors. It is this second geometric interpretation that has most relevance in dynamics. As we shall see later, moments of force and momentum involve this type of perpendicular projection. In this book the vector product will find its principal application in the study of rotational dynamics.

The scalar and vector products are interesting to us precisely because they have a geometrical interpretation. That means they represent real things in space. In a sense, we can think of the scalar product as a machine that takes two vectors as input and returns a scalar as output. Similarly the vector product also takes two vectors as its input but instead returns a vector as its output. There are in fact no other significantly different<sup>4</sup> machines that are able to convert two vectors into scalar or vector quantities and as a result you will rarely see anything other than the scalar and vector products in undergraduate/college level physics. There is in fact a machine that is able to take two vectors as its input and return a new type of geometrical object that is neither scalar nor vector. We will even meet such a thing later in this book when we encounter tensors in our studies of advanced dynamics and advanced relativity.

### 1.1.5 Time

We are constantly exposed to natural phenomena that recur: the beat of a pulse; the setting of the Sun; the chirp of a cricket; the drip of a tap; the longest day of the year. Periodic phenomena such as these give us a profound sense of time and we measure time by counting periodic events. On the other hand, many aspects of the natural world do not appear to be periodic: living things die and decay without rising phoenix-like from their ashes to repeat their life-cycle; an egg dropped on the floor breaks and never spontaneously re-forms into its original state; a candle burns down but never up. There is a sense that disorder follows easily from order, that unstructured things are easily made from structured things but that the reverse is much more difficult to achieve. That is not to say that it is impossible to create order from disorder – you can do that by tidying your room – just that on average

<sup>4</sup> i.e. other than trivial changes such as would occur if we choose instead to define the scalar product to be  $\mathbf{A} \cdot \mathbf{B} = \lambda AB \cos \theta$  where  $\lambda$  is a constant. We choose  $\lambda = 1$  because it is most convenient but any other choice is allowed provided we take care to revise the geometrical interpretation accordingly.



things go the other way with the passing of time. This idea is central to the study of thermodynamics where the disorder in a system is a measurable quantity called entropy. The total entropy of the Universe appears always to increase with time. It is possible to decrease the entropy (increase the order) of a part of the Universe, but only at the expense of increasing the entropy of the rest of the Universe by a larger amount. This net disordering of the Universe is in accord with our perception that time has a direction. We cannot use natural processes to “wind the clock back” and put the Universe into the state it was in yesterday – yesterday is truly gone forever. That is not to say that the laws of physics forbid the possibility that a cup smashed on the floor will spontaneously re-assemble itself out of the pieces and leap onto the table from which it fell. They do not; it is simply that the likelihood of order forming spontaneously out of disorder like this is incredibly small. In fact, the laws of physics are, to a very good approximation, said to be “time-reversal invariant”. The exception occurs in the field of particle physics where “CP-violation” experiments indicate that time-reversal symmetry is not respected in all fundamental interactions. This is evidence for a genuine direction to time that is independent of entropy. Entropy increase is a purely statistical effect, which occurs even when fundamental interactions obey time-reversal symmetry.

Thermodynamics gives us a direction to time and periodic events allow us to measure time intervals. A clock is a device that is constructed to count the number of times some recurring event occurs. A priori there is no guarantee that two clocks will measure the same time, but it is an experimental fact that two clocks that are engineered to be the same and which are placed next to each other, will measure, at least approximately, the same time intervals. This approximate equivalence of clocks leads us to conjecture the existence of absolute time, which is the same everywhere. A real clock is thus an imperfect means of measuring absolute time and a good clock is one that measures absolute time accurately. One problem with this idea is that absolute time is an abstraction, a theoretical idea that comes from an extrapolation of the experimental observation of the similar nature of different clocks. We can only measure absolute time with real clocks and without some notion of which clocks are better than others we have no handle on absolute time. One way to identify a reliable clock is to build lots of copies of it and treat all the copies exactly the same, i.e. put them in the same place, keep them at the same temperature and atmospheric conditions etc. If it is a reliable clock the copies will deviate little from each other over long time intervals. However, a reliable clock is not necessarily a good clock; similarly constructed clocks may run down in similar ways so that, for example, the time intervals between ticks might get longer the longer a clock runs, but in such a way that the similar clocks still read the same time. We can get around this by comparing equally reliable clocks based on different mechanisms. If enough equally-reliable clocks, based on enough different physical processes, all record the same time then we can start to feel confident that there is such a thing as absolute time. It is worth pointing out that in the 17th century reliable clocks were hard to come by and Newton certainly did not come to the idea of absolute time as a result of the observation of the constancy of clocks. Newton had an innate faith in the idea of absolute time and constructed his system of mechanics on that basis.

There is no doubt that absolute time is a useful concept; in this book we shall at first examine the motion of things under the influence of forces, treating time as though it is the same for every observer, and we will get answers accurate to a high degree. However, absolute time is a flawed concept, but flawed in such a way that the cracks only begin to appear under extreme conditions. We shall see later how clocks that are moving at very high relative velocities do *not* record the same time and that time depends on the state of motion of the observer. Einstein's Special Theory of Relativity tells us how to relate the time measured by different observers although the deviations from absolute time are only important when things start to move around at speeds approaching the speed of light. In describing the motion of things that do not approach the speed of light we can ignore relativistic effects with impunity, avoiding the conceptual and computational complications that arise from a full relativistic treatment. This will allow us to focus on concepts such as force, linear and angular momentum and energy. Once the basic concepts of classical mechanics have been established we will move on to study Special Relativity in Part II. Even then we will not completely throw out the concepts that are so successful in classical mechanics. Rather, these shall be adapted into the more general ideas of energy, momentum, space and time that are valid for all speeds.

### 1.1.6 Absolute space and space-time

At a fundamental level, the natural philosophy of Aristotle and the physics of Newton differ from the physics of Galileo<sup>5</sup> and Einstein in the way that space and time are thought to be connected. One very basic question involves whether space can be thought of as absolute. Consider the corner of the room you might be sitting in. The intersection of the two walls and the ceiling of a room certainly defines a point, but will this point be at the same place a microsecond later? We might be tempted to think so, that is, until the motion of the Earth is considered; the room is hurtling through space and so is our chosen point. Clearly the corner of the room defines a 'different' point at each instant. So would it be better to define a 'fixed' point with reference to some features of the Milky Way? This might satisfy us, at least until we discover that the Milky Way is moving relative to the other galaxies, so such a point cannot really be regarded as fixed. We find it difficult to escape completely from the idea that there is some sort of fixed background framework with respect to which we can measure all motion, but there is, crucially, no experimental evidence for this structure. Such a fixed framework is known as absolute space.

The concept of absolute space, which originates with Aristotle and his contemporaries, can be represented geometrically as shown in Figure 1.5(a). Here we have time as another Cartesian axis, tacked onto the spatial axes to produce a composite space that we call space-time. Consider two things that happen at times and positions that are measured using clocks and co-ordinate axes. We call these happenings 'events' and mark them on our space-time diagram as *A* and *B*. In the picture of absolute space, if the spatial co-ordinates of events *A* and *B* are identical we say that they represent the same point in space at different times. We can construct a path shown by the dotted line that connects the same point in space for all times. Galilean

---

<sup>5</sup> Galileo Galilei (1564–1642).

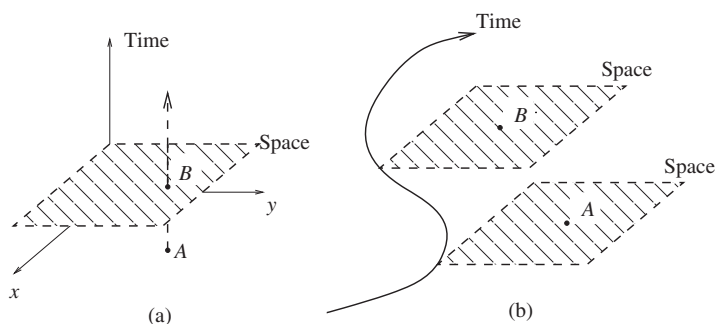


Figure 1.5 Different structures of space and time: (a) absolute space where points  $A$  and  $B$  are the same point in space; (b) a fibre-bundle structure where each moment in time has its own space.

relativity challenges this picture by rejecting the notion of absolute space and replacing it with the idea that space is defined relative to some chosen set of axes at a given instant in time. This is more like the picture in Figure 1.5(b), a structure that mathematicians call a fibre bundle. The same events  $A$  and  $B$  now lie in different spaces and the connection between them is no longer obvious. The fibre bundle is a more abstract structure to deal with than the space  $\times$  time structure of (a). Imagine, for example, trying to calculate the displacement from  $A$  to  $B$ . To do this we have to assume some additional structure of space-time that allows us to compare points  $A$  and  $B$ . It is as if space is erased and redefined at each successive instant and we have no automatic rule for saying how the ‘new’ space relates to the ‘old’ one. Notice that this view still treats time as absolute; observers at different points in the  $x - y$  plane agree on the common time  $t$ . In later chapters we will reconsider the geometry of space and time when we come to study the theory of Special Relativity, where universal time will be rejected in favour of a new space-time geometry in which observers at different positions each have their own local time.

## 1.2 VECTORS AND CO-ORDINATE SYSTEMS

As far as we can tell, space is three-dimensional, which means that three numbers are required to define a unique position. How we specify the three position-giving numbers defines what is known as the co-ordinate system. The co-ordinate system therefore introduces a sort of invisible grid or mesh that maps every point in space onto a unique ordered set of three real numbers. Figure 1.6 shows two commonly-used 3-dimensional co-ordinate systems. The Cartesian system is named after the French philosopher and mathematician René Descartes (1596–1650), who is reputed to have invented it from his bed while considering how he might specify the position of a fly that was buzzing around his room. This co-ordinate system consists of three mutually perpendicular axes, labelled  $x$ ,  $y$  and  $z$ , that intersect at the point  $O$ , called the origin. The position of a particle at  $P$  may be specified by giving the set of three distances  $(x, y, z)$ . Another frequently used co-ordinate system, the spherical-polar system, is obtained when the position of the particle is given instead by the distance from the origin  $r$  and two angles: the polar angle

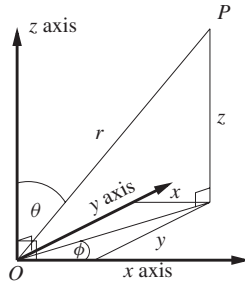


Figure 1.6 Two 3-dimensional co-ordinate systems covering the same space. The Cartesian co-ordinates consist of the set  $(x, y, z)$ . The spherical polar co-ordinates consist of the set  $(r, \theta, \phi)$ .

$\theta$  and the azimuthal angle  $\phi$ . The Cartesian and the spherical polar systems are just two possible ways of mapping the same space, and it should be clear that for any given physical problem there will be an infinite number of equally-valid co-ordinate systems. The decision as to which one to use is based on the nature of the problem, and the ease or difficulty of the calculation that results from the choice.

Choosing a co-ordinate system immediately gives us a way to represent vectors. Associated with any co-ordinate system are a set of unit vectors known as basis vectors. Each co-ordinate has an associated basis vector that points in the direction in which that co-ordinate is increasing. For example, in the 3D Cartesian system  $\mathbf{i}$  points in the direction of increasing  $x$ , i.e. along the  $x$ -axis, while  $\mathbf{j}$  and  $\mathbf{k}$  point along the  $y$ - and  $z$ -axes, respectively. Suppose that the position of a particle relative to the origin is given by the vector  $\mathbf{r}$ , known as the ‘position vector’ of the particle. Then  $\mathbf{r}$  can be written in terms of the Cartesian basis vectors as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad (1.4)$$

where the numbers  $(x, y, z)$  are the Cartesian co-ordinates of the particle. The magnitude of the position vector, which is the distance between the particle and the origin, can be calculated by Pythagoras’ Theorem and is

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{x^2 + y^2 + z^2}. \quad (1.5)$$

We have focussed upon a position vector in the Cartesian basis but we could have talked about a force, or an acceleration or a magnetic field etc. Any vector  $\mathbf{A}$  can be expressed in terms of its components  $(A_x, A_y, A_z)$  according to

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}. \quad (1.6)$$

It is not our aim here to present a full discussion of the algebraic properties of vectors. Some key results, which will prove useful later are listed in Table 1.1.

Very often, the motion of an object may be constrained to a known plane, such as in the case of a ball on a pool table, or a planet in orbit around the Sun. In such situations the full 3D co-ordinate system is not required and a two-dimensional

TABLE 1.1 Vector operations in the Cartesian basis.  $\mathbf{A}$  and  $\mathbf{B}$  are vectors,  $\lambda$  is a scalar.

Operation	Notation	Resultant
Negation	$-\mathbf{A}$	$(-A_x)\mathbf{i} + (-A_y)\mathbf{j} + (-A_z)\mathbf{k}$
Addition	$\mathbf{A} + \mathbf{B}$	$(A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$
Subtraction	$\mathbf{A} - \mathbf{B}$	$(A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} + (A_z - B_z)\mathbf{k}$
Scalar (Dot) Product	$\mathbf{A} \cdot \mathbf{B}$	$A_x B_x + A_y B_y + A_z B_z$
Vector (Cross) Product	$\mathbf{A} \times \mathbf{B}$	$(A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$
Scalar Multiplication	$\lambda \mathbf{A}$	$\lambda A_x \mathbf{i} + \lambda A_y \mathbf{j} + \lambda A_z \mathbf{k}$

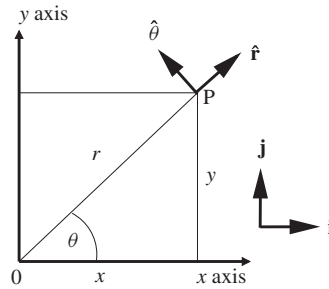


Figure 1.7 2D co-ordinate systems. The Cartesian co-ordinates consist of the set  $(x, y)$ . The plane polar co-ordinates consist of the set  $(r, \theta)$ .

system may be used. Two of these are shown in Figure 1.7. The Cartesian 2D co-ordinate system has basis vectors  $\mathbf{i}$  and  $\mathbf{j}$  and co-ordinates  $(x, y)$ . The plane-polar co-ordinates are  $(r, \theta)$ <sup>6</sup> where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{y}{x}. \tag{1.7}$$

The plane-polar system has basis vectors  $\hat{\mathbf{r}}$  and  $\hat{\boldsymbol{\theta}}$ . These may be expressed in terms of  $\mathbf{i}$  and  $\mathbf{j}$  as

$$\begin{aligned} \hat{\mathbf{r}} &= \mathbf{i} \cos \theta + \mathbf{j} \sin \theta, \\ \hat{\boldsymbol{\theta}} &= -\mathbf{i} \sin \theta + \mathbf{j} \cos \theta. \end{aligned} \tag{1.8}$$

The general position vector in the plane may therefore be written as

$$\mathbf{r} = r \hat{\mathbf{r}} = r(\mathbf{i} \cos \theta + \mathbf{j} \sin \theta) = x\mathbf{i} + y\mathbf{j}. \tag{1.9}$$

Some care is required when using polar co-ordinates to describe the motion of a particle since the basis vectors depend on the co-ordinate  $\theta$ , which may itself depend on time. This means that as the particle moves, the basis vectors change direction.

<sup>6</sup> Note the conventional use of  $\theta$  for the angle to the  $x$  axis rather than  $\phi$ , which is used for the corresponding angle in the spherical (3D) polar system.

This will lead to more complicated expressions for velocity and acceleration in polar co-ordinates than are obtained for Cartesian co-ordinates, as will be seen in the next section.

### 1.3 VELOCITY AND ACCELERATION

A particle is in motion when its position vector depends on time. The Ancient Greek philosophers had problems accepting the idea of a body being both in motion, and being ‘at a point in space’ at the same time. Zeno, in presenting his ‘runner’s paradox’, divided up the interval between the start and finish of a race to produce an infinite sum for the total distance covered. He argued that before the runner completes the full distance ( $l$ ) he must get half-way, and before he gets to the end of the second half he must get to half of that length and so on. The total distance covered can therefore be written as the infinite series

$$l \left[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right].$$

Zeno argued that it would be impossible for the runner to cover all of the sub-stretches in a finite time, and would therefore never get to the finish line. This contradiction forced him to decide that motion is impossible and that what we perceive as motion must be an illusion. We now know that the resolution of this paradox lies in an understanding of calculus. As the series continues, the steps get shorter and shorter, as do the time intervals taken for the runner to cover each step and we tend to a situation in which a vanishingly short distance is covered in a vanishingly small time.

Assuming that the position is a smooth function of time, we define the velocity as

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \lim_{\Delta t \rightarrow 0} \left( \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \right). \quad (1.10)$$

Notice that it involves a difference in the position vector at time  $t + \Delta t$  and at time  $t$ . This difference, divided by the time interval  $\Delta t$ , only becomes the velocity *in the limit* that  $\Delta t$  goes to zero. Thus the velocity is defined in terms of an infinitesimally small displacement divided by an infinitesimally small time interval. Notice that the vector nature of  $\mathbf{v}$  follows directly from the vector nature of  $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ , which differs from  $\mathbf{v}$  only by division by the scalar  $\Delta t$ . Often it is useful to refer to the magnitude of the velocity; this is known as the speed  $v$ , i.e.

$$v = |\mathbf{v}|.$$

With the notion that the ratio of two infinitesimally small quantities can be a finite number, we return to the Runner’s Paradox. Zeno’s argument does not rely on the particular choice of infinite series stated above. So we can simplify things by instead using a series made of equal-length steps. First we divide  $l$  up into