Propagation of Sound in Porous Media:

Modelling Sound Absorbing Materials, Second Edition

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WILEY

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Contents

Preface to the Second Edition

In the first edition, models initially developed to describe wave propagation in porous media saturated by heavy fluids are used to predict the acoustical performances of air saturated sound absorbing porous media. In this expanded and revised edition, we have retained, with slight modifications, most of the basic material of the first edition and expanded it by revisiting several original topics and adding new topics to integrate recent developments in the domain of wave propagation in porous media and practical numerical prediction methods that are widley used by researchers and engineers.

Chapters 1 to 3 dealing with sound propagation in solids and fluid and Chapter 9 dealing with the modelling of perforated facings were slightly modified. Chapters 4 to 6 were greatly revisited. A more detailed description of sound propagation in cylindrical pores is presented (Chapter 4), related to the more general presentation of new parameters and new models for sound propagation in rigid-framed porous media (Chapter 5). Also in Chapter 5 a short presentation of homogenization, with some results concerning double porosity media, is added. In Chapter 6, different formulations of the Biot theory for poroelastic media are given, with a simplified version for the case of media with a limp frame. In Chapter 11 we have revisited the original representation of the modelling of layered media (Chapter 7 of the first edition) and extended it to cover the systematic modelling of layered media using the Transfer Matrix Method (TMM). In particular, a step by step presentation of the numerical implementation of the method is given with several application examples.

Major additions include five new chapters. Chapter 7 discusses the acoustic field created by a point source above a rigid framed porous layer, with recent advances concerning the poles of the reflection coefficient and the reflected field at grazing incidence. Chapter 8 is concerned by the poroelastic layers excited by a point source in air or by a localized stress source on the free face of the layer, with a description of the Rayleigh waves and the resonances. Axisymmetrical poroelastic media are studied in Chapter 10. In Chapter 12, complements to the transfer matrix method are given. They concern mainly the effect of the finite lateral extend, and the excitation by point loads, of sound packages. Several examples illustrating the practical importance of these extensions are given (e.g. size effects on the random incidence absorption and transmission loss of porous media; airborne vs. structure borne insertion loss of sound packages). In Chapter 13, an introduction to the finite element modelling of poroelastic media is presented. Emphasis is put on the use of the mixed displacement-pressure formulation of the Biot theory, which appears in the Appendix of Chap. 6. Detailed description of coupling conditions between various domains including a waveguide are presented together with sections on the breakdown of the power dissipation mechanisms within a porous media and radiation conditions. Several applications are chosen to illustrate the practical use of the presented methods including modelling of double porosity materials and smart foams.

As in the first edition, the goal of the book remains to provide in a concrete manner a physical basis, as simple as possible, and the developments, analytical calculations and numerical methods, that will be useful in different fields where sound absorption and transmission and vibration damping by air saturated porous media are concerned.

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August 2009 Noureddine Atalla, Sherbrooke, Canada

1

Plane waves in isotropic fluids and solids

1.1 Introduction

The aim of this chapter is to introduce the stress–strain relations, the basic equations governing sound propagation which will be useful for the understanding of the Biot theory. The framework of the presentation is the linear theory of elasticity. Total derivatives with respect to time d/dt are systematically replaced by partial derivatives $\partial/\partial t$. The presentation is carried out with little explanation. Detailed derivation can be found in the literature (Ewing *et al*. 1957, Cagniard 1962, Miklowitz 1966, Brekhovskikh 1960, Morse and Ingard 1968, Achenbach 1973).

1.2 Notation – vector operators

A system of rectangular cartesian coordinates (x_1, x_2, x_3) will be used in the following, having unit vectors $\mathbf{i_1}$, $\mathbf{i_2}$ and $\mathbf{i_3}$. The vector operator del (or nabla) denoted by $\boldsymbol{\nabla}$ can be defined by

$$\boldsymbol{\nabla} = \mathbf{i}_1 \frac{\partial}{\partial x_1} + \mathbf{i}_2 \frac{\partial}{\partial x_2} + \mathbf{i}_3 \frac{\partial}{\partial x_3} \tag{1.1}$$

When operating on a scalar field $\varphi(x_1, x_2, x_3)$ the vector operator $\boldsymbol{\nabla}$ yields the gradient of φ

$$\mathbf{grad}\ \varphi = \nabla\varphi = \mathbf{i}_1 \frac{\partial\varphi}{\partial x_1} + \mathbf{i}_2 \frac{\partial\varphi}{\partial x_2} + \mathbf{i}_3 \frac{\partial\varphi}{\partial x_3} \tag{1.2}$$

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When operating on a vector field $\mathbf{v}$ with components $(\upsilon_1, \upsilon_2, \upsilon_3)$, the vector operator ∇ yields the divergence of $\mathbf{v}$

$$div\ \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial \upsilon_1}{\partial x_1} + \frac{\partial \upsilon_2}{\partial x_2} + \frac{\partial \upsilon_3}{\partial x_3} \tag{1.3}$$

The Laplacian of φ is:

$$\nabla \cdot \nabla\varphi = \nabla^2\varphi = \text{div}\ \textbf{grad}\ \varphi = \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2} \tag{1.4}$$

When operating on the vector $\mathbf{v}$, the Laplacian operator yields a vector field whose components are the Laplacians of υ_1, υ_2 and υ_3

$$(\nabla^2\mathbf{v})_i = \frac{\partial^2\upsilon_i}{\partial\varphi_1^2} + \frac{\partial^2\upsilon_i}{\partial\varphi_2^2} + \frac{\partial^2\upsilon_i}{\partial\varphi_3^2} \tag{1.5}$$

The gradient of the divergence of a vector $\mathbf{v}$ is a vector of components

$$(\nabla\nabla \cdot \mathbf{v})_i = \frac{\partial}{\partial x_i}\left(\frac{\partial \upsilon_1}{\partial x_1} + \frac{\partial \upsilon_2}{\partial x_2} + \frac{\partial \upsilon_3}{\partial x_3}\right) \tag{1.6}$$

The vector **curl** is denoted by

$$\textbf{curl}\ \mathbf{v} = \nabla \wedge \mathbf{v} \tag{1.7}$$

and is equal to

$$\textbf{curl}\ \mathbf{v} = \mathbf{i_1}\left(\frac{\partial \upsilon_3}{\partial x_2} - \frac{\partial \upsilon_2}{\partial x_3}\right) + \mathbf{i_2}\left(\frac{\partial \upsilon_1}{\partial x_3} - \frac{\partial \upsilon_3}{\partial x_1}\right) + \mathbf{i_3}\left(\frac{\partial \upsilon_2}{\partial x_1} - \frac{\partial \upsilon_3}{\partial x_2}\right) \tag{1.8}$$

1.3 Strain in a deformable medium

Let us consider the coordinates of the two points P and Q in a deformable medium before and after deformation. The two points P and Q are represented in Figure 1.1.

The coordinates of P are (x_1, x_2, x_3) and become $(x_1 + u_1, x_2 + u_2, x_3 + u_3)$ after deformation. The quantities (u_1, u_2, u_3) are then the components of the displacement vector $\mathbf{u}$ of P. The components of the displacement vector for the neighbouring point Q, having initial coordinates $(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3)$, are to a first-order approximation

$$\begin{aligned}
u_1' &= u_1 + \frac{\partial u_1}{\partial x_1}\Delta x_1 + \frac{\partial u_1}{\partial x_2}\Delta x_2 + \frac{\partial u_1}{\partial x_3}\Delta x_3 \\
u_2' &= u_2 + \frac{\partial u_2}{\partial x_1}\Delta x_1 + \frac{\partial u_2}{\partial x_2}\Delta x_2 + \frac{\partial u_3}{\partial x_3}\Delta x_3 \\
u_3' &= u_3 + \frac{\partial u_3}{\partial x_1}\Delta x_1 + \frac{\partial u_3}{\partial x_2}\Delta x_2 + \frac{\partial u_3}{\partial x_3}\Delta x_3
\end{aligned} \tag{1.9}$$

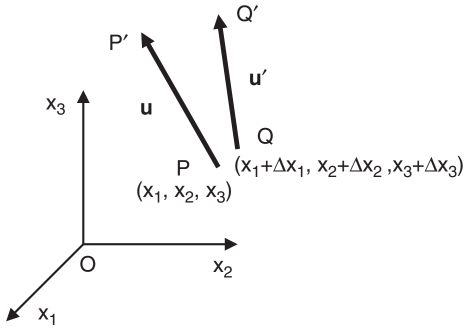

Figure 1.1 The displacement of P and Q to P' and Q' in a deformable medium.

A rotation vector $\boldsymbol{\Omega}(\Omega_1, \Omega_2, \Omega_3)$ and a 3×3 strain tensor e can be defined at P by the following equations:

$$\Omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \qquad \Omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)$$
$$\Omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{1.10}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.11}$$

The displacement components of Q can be rewritten as

$$\begin{aligned} u_1' &= u_1 + (\Omega_2 \Delta x_3 - \Omega_3 \Delta x_2) + (e_{11}\Delta x_1 + e_{12}\Delta x_2 + e_{13}\Delta x_3) \\ u_2' &= u_2 + (\Omega_3 \Delta x_1 - \Omega_1 \Delta x_3) + (e_{21}\Delta x_1 + e_{22}\Delta x_2 + e_{23}\Delta x_3) \\ u_3' &= u_3 + (\Omega_1 \Delta x_2 - \Omega_2 \Delta x_1) + (e_{31}\Delta x_1 + e_{32}\Delta x_2 + e_{33}\Delta x_3) \end{aligned} \tag{1.12}$$

The terms in the first parenthesis of each equation are associated with rotations around P, while those in the second parenthesis are related to deformations. The three components e_{11}, e_{22} and e_{33}, which are equal to

$$e_{11} = \frac{\partial u_1}{\partial x_1}, \qquad e_{22} = \frac{\partial u_2}{\partial x_2}, \qquad e_{33} = \frac{\partial u_3}{\partial x_3} \tag{1.13}$$

are an estimation of the extensions parallel to the axes.

The cubical dilatation θ is the limit of the ratio of the change in the volume to the initial volume when the dimensions of the initial volume approach zero. Hence,

$$\theta = \lim \frac{(\Delta x_1 + e_{11}\Delta x_1)(\Delta x_2 + e_{22}\Delta x_2)(\Delta x_3 + e_{33}\Delta x_3) - \Delta x_1 \Delta x_2 \Delta x_3}{\Delta x_1 \Delta x_2 \Delta x_3} \tag{1.14}$$

and is equal to the divergence of $\boldsymbol{u}$:

$$\theta = \nabla \cdot \boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = e_{11} + e_{22} + e_{33} \tag{1.15}$$

If $\mathbf{\Delta x}$ denotes the vector having components Δx_1, Δx_2 and Δx_3, after a rotation characterized by the rotation vector $\mathbf{\Omega}$, the initial vector becomes $\mathbf{\Delta x'}$ related to $\mathbf{\Delta x}$ by

$$\mathbf{\Delta x' - \Delta x = \Omega \wedge \Delta x} \tag{1.16}$$

The rotation vector $\mathbf{\Omega}$, in vector notation, is

$$\mathbf{\Omega} = \frac{1}{2}\,\mathbf{curl\ u} \tag{1.17}$$

1.4 Stress in a deformable medium

Two kinds of forces may act on a body, body forces and surface forces. Surface forces act across the surface, including its boundary. Consider a volume V in a deformable medium as represented in Figure 1.2.

Let S be the surface limiting V and ΔS an element of S around a point P that lies on S. The side of S which is outside V is called (+) while the other is called (−). The force exerted on V across ΔS is denoted by $\mathbf{\Delta F}$. A stress vector at P is defined by

$$\mathbf{T}(P) = \lim_{\Delta S \to 0} \frac{\mathbf{\Delta F}}{\Delta S} \tag{1.18}$$

The stress vector $\mathbf{T}(P)$ depends on P and on the direction of the positive outward unit normal $\mathbf{n}$ to the surface S at P. The stress vectors can be obtained from $\mathbf{T}^1(\sigma_{11}, \sigma_{12}, \sigma_{13})$, $\mathbf{T}^2(\sigma_{21}, \sigma_{22}, \sigma_{23})$, and $\mathbf{T}^3(\sigma_{31}, \sigma_{32}, \sigma_{33})$ corresponding to surfaces with normal $\mathbf{n}$ parallel to the x_1, x_2 and x_3 axes, respectively.

The components T_1, T_2, T_3 of $\mathbf{T}$ can be expressed in the general case as

$$\begin{aligned} T_1 &= \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3 \\ T_2 &= \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{32}n_3 \\ T_3 &= \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3 \end{aligned} \tag{1.19}$$

In these equations n_1, n_2 and n_3 are the direction cosines of the positive normal $\mathbf{n}$ to S at P. The quantities σ_{ij} are the nine components of the stress tensor at P. These components are symmetrical, i.e. $\sigma_{ij} = \sigma_{ji}$, like the components e_{ij}. An illustration is given in Figure 1.3 for a cube with faces of unit area parallel to the coordinate planes.

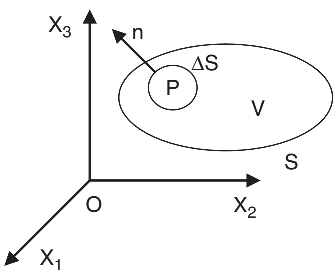

Figure 1.2 A volume V in a deformable medium, with an element ΔS belonging to the surface S limiting V.

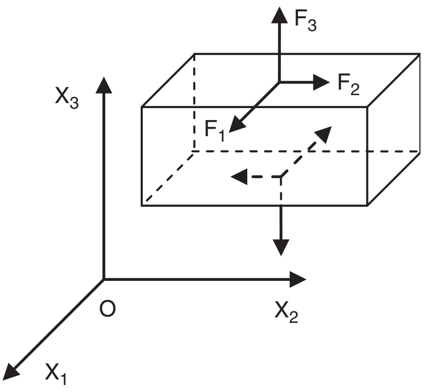

Figure 1.3 A cube with faces of unit area parallel to the coordinate planes. The three components of the forces acting on the upper and the lower faces are represented.

The variations of the components σ_{ij} are assumed to be negligible at the surface of the cube. With the components of the positive unit normal on the upper face being (0, 0, 1), Equations (1.19) reduce to

$$T_1 = \sigma_{31}, \qquad T_2 = \sigma_{32}, \qquad T_3 = \sigma_{33} \tag{1.20}$$

The force $\mathbf{F}(F_1, F_2, F_3)$ acting on the upper face is equal to $\mathbf{T}^3$. The components of the unit normal on the lower face are $(0, 0, -1)$. The forces on the lower and the upper face are equal in magnitude and lie in opposite directions. The same property holds for the two other pairs of opposite faces. The elements σ_{ij} where $i = j$ correspond to normal forces while those with $i \neq j$ correspond to tangential forces.

1.5 Stress–strain relations for an isotropic elastic medium

The stress–strain relations for an isotropic elastic medium are as follows:

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} \tag{1.21}$$

The quantities λ and μ are the Lamé coefficients and δ_{ij} is the Kronecker delta:

$$\begin{aligned} \delta_{ij} &= 1 \quad \textit{if} \quad i = j \\ \delta_{ij} &= 0 \quad \textit{if} \quad i \neq j \end{aligned} \tag{1.22}$$

In matrix form Equation (1.21) can be rewritten

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{13} \\ e_{23} \\ e_{12} \end{pmatrix} \tag{1.23}$$

$$
\begin{aligned}
C_{11} &= \lambda + 2\mu \\
C_{12} &= \lambda \\
C_{44} &= 2\mu = C_{11} - C_{12}
\end{aligned}
\tag{1.24}
$$

The strain elements are related to the stress elements by

$$
e_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{1}{2\mu}\sigma_{ij} \tag{1.25}
$$

$$
\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{13} \\ e_{23} \\ e_{12} \end{pmatrix} = \begin{pmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2\mu \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{12} \end{pmatrix} \tag{1.26}
$$

where E is the Young's modulus and ν is the Poisson ratio. They are related to the Lamé coefficients by

$$
\begin{aligned}
E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \\
\nu &= \frac{\lambda}{2(\lambda + \mu)}
\end{aligned}
\tag{1.27}
$$

The shear modulus G is related to E and ν via

$$
G = \mu = \frac{E}{2(1 + \nu)} \tag{1.28}
$$

Examples

Antiplane shear

The displacement field is represented in Figure 1.4. For this case, the two components e_{ij} which differ from zero are

$$
e_{32} = e_{23} = \frac{1}{2}\frac{\partial u_2}{\partial x_3} \tag{1.29}
$$

The angle α is equal to

$$
\alpha = \frac{\partial u_2}{\partial x_3} \tag{1.30}
$$

Using Equation (1.21), one obtains two components σ_{ij} which differ from zero:

$$
\sigma_{32} = \sigma_{23} = \mu\alpha \tag{1.31}
$$

The coefficient μ is the shear modulus of the medium, which relates the angle of deformation and the tangential force per unit area. The three components of the rotation

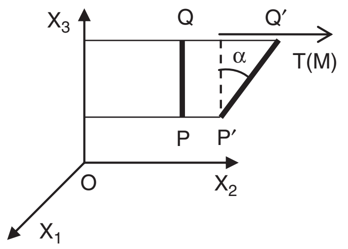

Figure 1.4 Antiplane shear in an elastic medium. A vector **PQ** initially parallel to x_3 becomes oblique with an angle α to the initial direction.

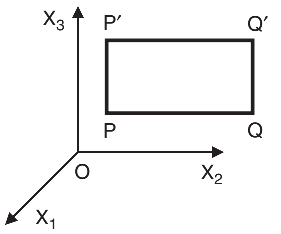

Figure 1.5 Longitudinal strain in the x_3 direction.

vector $\mathbf{\Omega}$ are

$$\Omega_1 = -\frac{1}{2}\frac{\partial u_2}{\partial x_3}, \qquad \Omega_2 = \Omega_3 = 0 \tag{1.32}$$

The deformation is equivoluminal, the dilatation θ being equal to zero, and there is a rotation around x_1.

Longitudinal strain

For this case only the component e_{33} of the strain tensor is different from zero. The vectors **PQ** and **P′Q′** are represented in Figure 1.5.

The stress tensor components that do not vanish are

$$\begin{aligned} \sigma_{33} &= (\lambda + 2\mu)e_{33} \\ \sigma_{11} &= \sigma_{22} = \lambda e_{33} \end{aligned} \tag{1.33}$$

Unidirectional stress

From Equation (1.26) the stress component σ_{33} transforms a vector **PQ** parallel to the axis x_3 into a vector **P′Q′** parallel to x_3. The ratio P′Q′/PQ is given by

$$P'Q'/PQ = \sigma_{33}/E \tag{1.34}$$

A vector **PQ** perpendicular to x_3 is transformed in a vector **P′Q′** parallel to **PQ** and the ratio P′Q′/PQ is now given by

$$P'Q'/PQ = -\nu\sigma_{33}/E \tag{1.35}$$

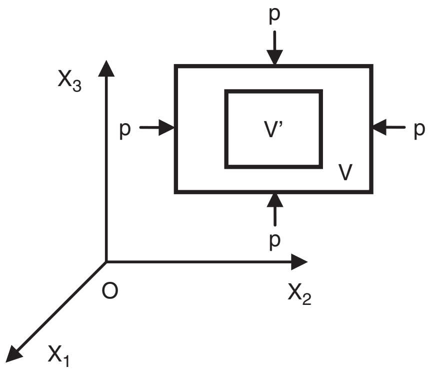

Figure 1.6 Compression of a volume V by a hydrostatic pressure.

Compression by a hydrostatic pressure

For this case, represented in Figure 1.6, the components of the stress tensor that do not vanish are

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p \tag{1.36}$$

From Equation (1.21) it follows that the dilatation θ is related to p by

$$\theta = -p \Big/ \left(\lambda + \frac{2\mu}{3}\right) \tag{1.37}$$

The ratio $-p/\theta$ is the bulk modulus K of the material, which is equal to

$$K = \lambda + \frac{2\mu}{3} \tag{1.38}$$

Contrary to the case of simple shear, $\mathbf{\Omega} = 0$ and θ is nonzero. The deformation is irrotational, as in the case with a longitudinal strain. Note that since a hydrostatic pressure leads to a negative volume change, the bulk modulus K is positive for all materials and in consequence Poisson's ratio is less than or equal to 0.5 for all materials.

1.6 Equations of motion

The total surface force $\mathbf{F_v}$ acting on the volume V represented in Figure 1.2 is

$$\mathbf{F_v} = \iint \mathbf{T}\, \mathrm{d}S \tag{1.39}$$

The projection of the force $\mathbf{F_v}$ on to the x_i axis is

$$F_{v_i} = \iint_S (\sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3)\, \mathrm{d}S \tag{1.40}$$

By using the divergence theorem, Equation (1.40) becomes

$$F_{v_i} = \iiint_V \left(\frac{\partial \sigma_{1i}}{\partial x_1} + \frac{\partial \sigma_{2i}}{\partial x_2} + \frac{\partial \sigma_{3i}}{\partial x_3} \right) \mathrm{d}V \tag{1.41}$$

Adding the component X_i of the body force per unit volume, the linearized Newton equation for V may be written as

$$\iiint_V \left(\frac{\partial \sigma_{1i}}{\partial x_1} + \frac{\partial \sigma_{2i}}{\partial x_2} + \frac{\partial \sigma_{3i}}{\partial x_3} + X_i - \rho \frac{\partial^2 u_i}{\partial t^2} \right) \mathrm{d}V = 0 \tag{1.42}$$

where ρ is the mass density of the material. This equation leads to the stress equations of motion

$$\frac{\partial \sigma_{1i}}{\partial x_1} + \frac{\partial \sigma_{2i}}{\partial x_2} + \frac{\partial \sigma_{3i}}{\partial x_3} + X_i - \rho \frac{\partial^2 u_i}{\partial t^2} = 0 \qquad i = 1, 2, 3 \tag{1.43}$$

With the aid of Equation (1.21) the equations of motion become

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \lambda \frac{\partial \theta}{\partial x_i} + 2\mu \frac{\partial e_{ii}}{\partial x_i} + \sum_{j \neq i} 2\mu \frac{\partial e_{ji}}{\partial x_j} + X_i \quad i = 1, 2, 3 \tag{1.44}$$

Replacing e_{ji} by $1/2(\partial u_j/\partial x_i + \partial u_i/\partial x_j)$, Equations (1.44) can be written in terms of displacement as

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial \boldsymbol{\nabla}.\mathbf{u}}{\partial x_i} + \mu \boldsymbol{\nabla}^2 u_i + X_i \quad i = 1, 2, 3 \tag{1.45}$$

where ∇^2 is the Laplacian operator $\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$.

Using vector notation, Equations (1.45) can be written

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \boldsymbol{\nabla\nabla} \cdot \mathbf{u} + \mu \boldsymbol{\nabla}^2 \mathbf{u} + \mathbf{X} \qquad i = 1, 2, 3 \tag{1.46}$$

In this equation, $\boldsymbol{\nabla\nabla} \cdot \mathbf{u}$ is the gradient of the divergence $\boldsymbol{\nabla} \cdot \mathbf{u}$ of the vector field $\mathbf{u}$, and its components are

$$\frac{\partial}{\partial x_i} \left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right] \qquad i = 1, 2, 3 \tag{1.47}$$

and the quantity $\boldsymbol{\nabla}^2 \mathbf{u}$ is the Laplacian of the vector field $\mathbf{u}$, having components

$$\sum_{j=1,2,3} \frac{\partial^2 u_i}{\partial x_j^2} \qquad i = 1, 2, 3 \tag{1.48}$$

as indicated in Section 1.2.

1.7 Wave equation in a fluid

In the case of an inviscid fluid, μ vanishes. The stress coefficients reduce to

$$\begin{aligned} \sigma_{11} = \sigma_{22} = \sigma_{33} = \lambda\theta \\ \sigma_{12} = \sigma_{13} = \sigma_{23} = 0 \end{aligned} \tag{1.49}$$

The three nonzero stress elements are equal to $-p$, where p is the pressure. The bulk modulus K, given by Equation (1.38), becomes simply λ:

$$K = \lambda \tag{1.50}$$

The stress field (Equation 1.49) generates only irrotational deformations such as $\mathbf{\Omega} = 0$.

A representation of the displacement vector **u** in the following form can be used:

$$u_1 = \partial\varphi/\partial x_1, \qquad u_2 = \partial\varphi/\partial x_2, \qquad u_3 = \partial\varphi/\partial x_3 \tag{1.51}$$

where φ is a displacement potential.

In vector form, Equations (1.51) can be written as

$$\boldsymbol{u} = \nabla\varphi \tag{1.52}$$

Using this representation, the rotation vector $\mathbf{\Omega}$ can be rewritten

$$\mathbf{\Omega} = \frac{1}{2}\mathbf{curl}\ \nabla\varphi = 0 \tag{1.53}$$

and the displacement field is irrotational.

Substitution of this displacement representation into Equation (1.46) with $\mu = 0$ and $\mathbf{X} = 0$ yields

$$\lambda\ \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi = \rho\frac{\partial^2}{\partial t^2}\boldsymbol{\nabla}\varphi \tag{1.54}$$

Since $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi = \nabla^2\varphi$, Equation (1.54) reduces, with Equation (1.50), to

$$\boldsymbol{\nabla}\left[K\nabla^2\varphi - \rho\frac{\partial^2}{\partial t^2}\varphi\right] = 0 \tag{1.55}$$

The displacement potential φ satisfies the equation of motion if

$$\nabla^2\varphi = \rho\frac{\partial^2\varphi}{K\partial t^2} \tag{1.56}$$

If the fluid is a perfectly elastic fluid, with no damping, K is a real number.

This displacement potential is related to pressure in a simple way. From Equations (1.49), (1.50) and (1.52), p can be written as

$$p = -K\theta = -K\nabla^2\varphi \tag{1.57}$$

By the use of Equations (1.56) and (1.57) one obtains

$$p = -\rho \frac{\partial^2 \varphi}{\partial t^2} \tag{1.58}$$

At an angular frequency ω ($\omega = 2\pi f$, where f is frequency), p can be rewritten as

$$p = \rho \omega^2 \varphi \tag{1.59}$$

As an example, a simple solution of Equation (1.56) is

$$\varphi = \frac{A}{\rho \omega^2} \exp[j(-kx_3 + \omega t) + \alpha] \tag{1.60}$$

In this equation, A and α are arbitrary constants, and k is the wave number

$$k = \omega(\rho/K)^{1/2} \tag{1.61}$$

The phase velocity is given by

$$c = \omega/\mathrm{Re}\ k \tag{1.62}$$

and $\mathrm{Im}(k)$ appears in the amplitude dependence on x_3, $\exp(\mathrm{Im}(k)x_3)$. In this example, u_3 is the only nonzero component of $\mathbf{u}$:

$$u_3 = \frac{\partial \varphi}{\partial x_3} = -\frac{jkA}{\rho \omega^2} \exp[j(-kx_3 + \omega t + \alpha)] \tag{1.63}$$

The pressure p is

$$p = -\rho \frac{\partial^2 \varphi}{\partial t^2} = A \exp[j(-kx_3 + \omega t + \alpha)] \tag{1.64}$$

This field of deformation corresponds to the propagation parallel to the x_3 axis of a longitudinal strain, with a phase velocity c.

1.8 Wave equations in an elastic solid

A scalar potential φ and a vector potential $\boldsymbol{\psi}(\psi_1, \psi_2, \psi_3)$ can be used to represent displacements in a solid

$$\begin{aligned} u_1 &= \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi_3}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3} \\ u_2 &= \frac{\partial \varphi}{\partial x_2} + \frac{\partial \psi_1}{\partial x_3} - \frac{\partial \psi_3}{\partial x_1} \\ u_3 &= \frac{\partial \varphi}{\partial x_3} + \frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2} \end{aligned} \tag{1.65}$$

In vector form, Equations (1.65) reduce to

$$\mathbf{u} = \mathbf{grad}\ \varphi + \mathbf{curl}\ \boldsymbol{\psi} \tag{1.66}$$

or, using the notation ∇ for the gradient operator

$$\mathbf{u} = \nabla\varphi + \nabla \wedge \boldsymbol{\psi} \tag{1.67}$$

The rotation vector $\boldsymbol{\Omega}$ in Equation (1.17) is then equal to

$$\boldsymbol{\Omega} = \frac{1}{2}\nabla \wedge \nabla \wedge \boldsymbol{\psi} \tag{1.68}$$

Therefore, the scalar potential involves dilatation while the vector potential describes infinitesimal rotations.

In the absence of body forces, the displacement equation of motion (1.46) is

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} \tag{1.69}$$

Substitution of the displacement representation given by Equation (1.67) into Equation (1.69) yields

$$\mu\nabla^2[\nabla\varphi + \nabla \wedge \boldsymbol{\psi}] + (\lambda + \mu)\nabla\nabla \cdot [\nabla\varphi + \nabla \wedge \boldsymbol{\psi}] = \rho\frac{\partial^2}{\partial t^2}[\nabla\varphi + \nabla \wedge \boldsymbol{\psi}] \tag{1.70}$$

In Equation (1.70), $\nabla \cdot \nabla\varphi$ can be replaced by $\nabla^2\varphi$, $\nabla \cdot \nabla \wedge \boldsymbol{\psi} = 0$, allowing this equation to reduce to

$$\mu\nabla^2\nabla\varphi + \lambda\nabla\nabla^{\mathbf{2}}\varphi + \mu\nabla\nabla^{\mathbf{2}}\varphi - \rho\frac{\partial^2}{\partial t^2}\nabla\varphi + \left(\mu\nabla^2 - \rho\frac{\partial^2}{\partial t^2}\right)\nabla \wedge \psi = \mathbf{0} \tag{1.71}$$

By using the relations $\nabla^2\nabla\varphi = \nabla\nabla^{\mathbf{2}}\varphi$ and $\nabla^2\nabla \wedge \boldsymbol{\psi} = \nabla \wedge \nabla^{\mathbf{2}}\boldsymbol{\psi}$, Equation (1.71) can be rewritten

$$\nabla\left[(\lambda + \mathbf{2}\mu)\nabla^2\varphi - \rho\frac{\partial^2\varphi}{\partial t^2}\right] + \nabla \wedge \left[\mu\nabla^{\mathbf{2}}\boldsymbol{\psi} - \rho\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}\right] = 0 \tag{1.72}$$

From this, we obtain two equations containing, respectively, the scalar and the vector potential

$$\nabla^2\varphi = \frac{\rho}{\lambda + 2\mu}\frac{\partial^2\varphi}{\partial t^2} \tag{1.73}$$

$$\nabla^2\boldsymbol{\psi} = \frac{\rho}{\mu}\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} \tag{1.74}$$

Equation (1.73) describes the propagation of irrotational waves travelling with a wave number vector k equal to

$$k = \omega(\rho/(\lambda + 2\mu))^{1/2} \tag{1.75}$$

The phase velocity c is always related to the wave number k by Equation (1.62). The quantity K_c defined as

$$K_c = \lambda + 2\mu \tag{1.76}$$

can be substituted in Equation (1.75), resulting in

$$k = \omega(\rho_c/K_c)^{1/2} \tag{1.77}$$

while the stress–strain relations (Equations (1.21) can be rewritten as

$$\sigma_{ij} = (K_c - 2\mu)\theta\delta_{ij} + 2\mu e_{ij} \tag{1.78}$$

Equation (1.74) describes the propagation of equivoluminal (shear) waves propagating with a wave number equal to

$$k' = \omega(\rho/\mu)^{1/2} \tag{1.79}$$

As an example, a simple vector potential $\boldsymbol{\psi}$ can be used:

$$\psi_2 = \psi_3 = 0 \qquad \psi_1 = B\exp[j(-k'x_3 + \omega t)] \tag{1.80}$$

In this case, u_2 is the only component of the displacement vector which is different from zero

$$u_2 = -jBk'\exp[j(-k'x_3 + \omega t)] \tag{1.81}$$

This field of deformation corresponds to propagation, parallel to the x_3 axis, of the antiplane shear.

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