ISOGEOMETRIC ANALYSIS
TOWARD INTEGRATION OF CAD AND FEA

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School of Athens, from the Stanza della Segnatura, 1510-11 (fresco), Raphael (Raffaello Sanzio of Urbino) (1483-1520)/Vatican Museums and Galleries, Vatican City, Italy/Giraudon/The Bridgeman Art Library. Legend has it that over the door to Plato’s Academy in Athens there was an inscription “Let no man ignorant of geometry enter here.” Words to live by, in antiquity and today.
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Preface

The initial work in isogeometric analysis was motivated by the existing gap between the worlds of finite element analysis (FEA) and computer-aided design (CAD); see Hughes et al., 2005. As the number of people involved with isogeometric analysis from both the FEA and the CAD communities has grown, this gap has become increasingly apparent to all involved. It is not only a shortcoming of the current technology but of the entire engineering process. Indeed, technological barriers are often easier to overcome than the inertia of the status quo. At this early stage, one of the most important contributions of the research in isogeometric analysis has been to initiate conversation between the design and analysis camps, and to begin to make each side aware of the hurdles that the other faces, as well as what each has to offer. This book is meant to be part of that dialogue.

What are we providing and for whom?

Isogeometric analysis seeks to unify the fields of CAD and FEA. In pursuing this end we have found, with very few exceptions, that FEA people know very little about computational geometry, and computational geometry people know very little about FEA. Our background is in FEA. We have attempted to cross the divide and learn from and work with computational geometers in order to orchestrate changes in CAD and FEA that will result in an agreed upon isogeometric technology satisfactory to both constituencies. That being said, we are neophytes in computational geometry so nothing fundamentally new on that topic will be found herein. Our most immediate goals are to encourage computational analysts to learn about isogeometric analysis and to begin to take advantage of it in their work. Specifically, we have attempted to build upon a knowledge of finite element analysis and to indicate what is new and different about isogeometric analysis. A background in finite element analysis at the level of Hughes, 2000 is ideal preparation for understanding this book. Most of the book, however, is sufficiently self-contained as to not require that much finite element background. We wrote this book so that the reader could learn how to do isogeometric analysis.

We wanted this book to be accessible, in fact, easy to read and learn from, but we did not want to superficially gloss over important details to achieve simplicity. Although computational mechanics has become a sophisticated and complex discipline, the essence of the finite element method is quite simple and straightforward. The same may be said of isogeometric analysis, and we endeavored to express this viewpoint in this book. Nevertheless, certain basics of computational geometry need to be learned, and these are not part of the traditional training and repertoire of finite element analysts. We have tried to present them in a clear and direct
manner. We at least hope the book is simple enough that most motivated readers will be able to learn the essential ideas.

We assumed that many readers would want to add isogeometric capabilities to existing finite element computer programs, so we developed this theme right from the start. The early chapters deal with the basic concepts, how to implement them, and how to handcraft isogeometric models. The latter chapters attempt to demonstrate convincingly why one might want to do so. By explaining the details of Non-Uniform Rational B-Spline (NURBS) basis functions and showing how their unique properties come to bear on a wide variety of applications, we hope to motivate others to consider how their own research might benefit from these powerful functions.

There are many computational geometry technologies that could serve as a basis for isogeometric analysis. The reason for selecting NURBS as the initial basis is compelling: It is the most widely used computational geometry technology in engineering design. Unfortunately, at this stage of the game, an isogeometric modeling toolset is not available. We hope that this void will be filled in the not-too-distant future and be made available to the community. Research projects are already underway with this as one of the goals.

Although we present some applications of isogeometric analysis that have appeared previously in research papers, a conscious effort has been made to present material not in research papers, in particular, detailed examples and data sets are presented that one needs to thoroughly understand to gain a working knowledge of the material. Another theme has been to only show examples and applications that exhibit some unique feature of isogeometric analysis not available in traditional finite element analysis. One might consider isogeometric analysis as simply an expansion and powerful generalization of traditional finite element analysis.

Channeling developments in order to make them more relevant to downstream engineering

We would like to help people on each side of the CAD/FEA divide to further the state of their respective arts. By being aware of the their own place in the idea-to-product process, both the geometer and analyst might strive to design technologies that are integrative and avoid creating bottlenecks at any stage of the engineering process. We have no doubt that the futures of CAD and FEA lie much closer together than do their pasts. The reader is invited to participate in the effort to unify these fields.

Organization of the text

This book begins in Chapter 1 with an historical perspective on the fields of finite element analysis and computer aided design. This provides a context from which the ideas throughout the book have emerged. Additionally, we briefly point out some of the issues of isogeometric analysis that seem to cause some confusion for researchers coming from a classical FEA background. Each of these issues is discussed in detail within the body of the text, but it may prove useful for the reader to be aware of them before embarking. We then introduce Non-Uniform Rational B-Splines (NURBS) in Chapter 2, with an initial focus on geometric design and the particular features that make this technology unique. A brief tutorial on the construction of a NURBS geometry is included. Chapter 3 describes how computer aided design technology can be used within an analysis framework. Particular attention is given
to a detailed explanation of the Galerkin finite element method as this is the framework within which the bulk of isogeometric analysis has been performed. Chapter 3 also includes a discussion of how classical finite element software might be modified to create isogeometric analysis software.

The bulk of the remainder of this book contains examples of the many different applications to which isogeometric analysis has been applied. The specific choice of material is meant to emphasize the interesting properties of NURBS basis functions and to display the unique capabilities of an analysis framework built upon them. The examples increase in complexity as the book progresses, loosely chronicling the evolution of the technology. For the most part, linear problems are discussed before nonlinear problems, and static problems precede time-dependent problems. Chapters 6 and 7 provide general discussions of time-dependent problems and nonlinear problems, respectively. The reader unfamiliar with these topics may want to review these chapters before proceeding to chapters on such applications. We attempt to be quite thorough on the simpler examples, providing everything needed for an individual just getting started to be able to perform a calculation. Contrastingly, there is a bias towards brevity for the more complex problems. The treatment of examples from the forefront of research is meant to highlight the specific features of isogeometric analysis upon which these applications rely. Whenever details necessary to replicate the work are omitted, references to the literature where those details may be found are included. Still, every effort is made to tie the implementation used and the results obtained to the features of isogeometric analysis that differ from classical finite elements.

Chapter 4 discusses linear elasticity, with a particular emphasis on the analysis of thin-walled structures. Chapter 5 covers vibrations and wave propagation. Whereas the examples considered in Chapter 4 particularly benefit from the geometrical accuracy of isogeometric analysis, the examples in Chapter 5 demonstrate the accuracy advantages NURBS have over classical finite elements due to their increased smoothness. In Chapter 6 we move from static to dynamic problems and discuss various time-integration techniques that are in common usage. Chapter 7 discusses the solution of nonlinear equations, and it expands on the discussion of Chapter 6 to address solving nonlinear, time-dependent problems by means of the generalized-$\alpha$ method. Chapter 8 discusses one approach to addressing the locking phenomenon common in the analysis of both linear and nonlinear nearly incompressible solids. Chapter 9 features many examples from the field of computational fluid dynamics, ranging from the linear advection-diffusion equation to turbulence. In all cases, smooth NURBS basis functions are shown to achieve superior accuracy per degree-of-freedom than the classical FEA basis functions of the same order. Chapter 9 also presents the variational multiscale method. Fluid-structure interaction and fluids problems posed on moving domains are discussed in Chapter 10. Each of these problems requires care in tracking the motion of the mesh and correctly formulating the equations on the moving domain. Chapter 11 discusses partial differential equations in which the highest order derivative is greater than two. A traditional variational treatment of such problems requires the use of basis functions that are smoother than $C^0$. This is frequently difficult or impossible in a classical FEA setting, but is quite easy within isogeometric analysis. Chapter 12 discusses polar forms, which offer an alternative mathematical description of splines. The use of polar forms has been instrumental in the development of efficient algorithms for the manipulation of spline objects. Lastly, Chapter 13 discusses the current state-of-the-art in isogeometric analysis, as well as many promising directions for future work in the subject.
Preface

Additional resources

There are many places for the interested reader to seek more information about the topics discussed in this book. Though an effort has been made to make this book as self-contained as possible, it is not possible to address every topic in the full generality that it deserves. For a more thorough discussion of NURBS we suggest starting with Rogers, 2001 and then going on to Piegl and Tiller, 1997. The former is quite readable and features many historical perspectives on NURBS and those whose work has led to their development; the latter is quite comprehensive and served as an indispensable guide when we were developing our initial software. Here is a list of geometry books we have found helpful, including the two already mentioned. It is by no means complete, and we are still learning from them.

- Geometric Modeling with Splines: An Introduction, E. Cohen et al., 2001
- The NURBS Book, L. Piegl and W. Tiller, 1997
- Bézier and B-Spline Techniques, H. Prautzsch, W. Boehm and M. Paluszny, 2002
- An Introduction to NURBS: With Historical Perspective, D.F. Rogers, 2001

For an introductory but thorough treatise on the finite element method, see Hughes, 2000. We attempt as far as possible to be consistent with the notation of Hughes, 2000, which we will make reference to many times throughout this book. The best source for information on the many applications contained herein is in the research papers upon which much of the content is based. Each chapter provides references to original journal articles, which frequently discuss the topics in a great deal more depth, and with many more examples, than is possible here.

Notation

A word of caution is in order. Notational conventions that are very illustrative in simple settings, particularly when introducing a concept for the first time, frequently become unwieldy as things become more complex. For this reason, we attempt to use the notation that provides the most clarity in a given situation, though this choice is sometimes at odds with other usage. Whenever there is the potential for confusion, the issue is addressed directly herein.

How work on isogeometric analysis began

Isogeometric analysis began when Tom Hughes was privy to a conversation concerning the creation of finite element models from CAD representations. The gist of the conversation expressed the theme that despite years of research into mesh generation, the model creation problem was a significant bottleneck to the effective use of FEA and, for complex engineering designs, the problem seemed to be getting worse. It appeared to Tom that if the situation was that bad, the problem must either be very difficult or the research community was pursuing a solution from the wrong perspective. After some study he concluded that the problem as
posed was indeed very difficult, but not only was the research community pursuing it from the wrong perspective, it was pursuing the wrong problem.

CAD and FEA grew up independently. Despite dealing with the same objects, engineering designs, they represent them with entirely different geometrical constructs. This seemed to be the fundamental problem. Tom hoped to replace this situation with a single, agreed upon, geometrical description. He thought that he might be able to reconstitute analysis within the geometric framework of CAD technologies. This seemed doable, but it also became apparent that CAD representations would have to be enhanced. He was surprised to find that newer technologies emanating from the computational geometry research literature were actually moving in that direction and that some of these technologies were finding their way into commercial products. The final piece of the puzzle, developing analysis suitable trivariate parameterizations from surface representations, is an open problem but one that is beginning to be addressed by the computational geometry community with new and promising mathematical approaches. The confluence of all these activities is Isogeometric Analysis. Through the combined efforts of the computational geometry and computational analysis communities, we believe the potential of isogeometric analysis can be realized.

Work on isogeometric analysis began in earnest in 2003 almost a year after Tom Hughes joined the University of Texas at Austin. He had received a research grant to pursue the topic, but did not have a PhD student assigned to it. A then first-year graduate student, Austin Cottrell, in the Computational and Applied Mathematics program in the Institute for Computational Engineering and Sciences came to talk to Tom about research topics and possibly doing his PhD under Tom’s supervision. Among other topics, Tom described to Austin his vision of this as yet nameless new approach to analysis and geometry. After thinking things over, Austin said he would like to pursue it and could get started in the summer of 2003. Tom gave Austin a copy of Rogers’ book on NURBS.

As Austin was making progress with NURBS technology, another of Tom’s graduate students, Yuri Bazilevs, started to interact with him on it, and the two of them implemented the first NURBS based finite element codes. By the fall of 2003, linear problems were being solved and good results were being obtained. It was around that time that the name “isogeometric analysis” was coined. Rapid progress was being made developing the technology. Before long, isogeometric analysis became an integral part of all work in Tom’s group. After completing his PhD, Victor Calo also joined the effort, as did a number of other students, post-docs, and visitors to the Institute, including Ido Akkerman, Laurenco Beirão da Veiga, David Benson, Thomas Elguedj, John Evans, Héctor Gómez, Scott Lipton, Alessandro Reali, Giancarlo Sangalli, Mike Scott, and Jessica Zhang. The effectiveness of the procedures and the richness of the subject exceeded everyone’s expectations.

How this book was written

Discussions about writing a book occurred frequently during the course of the work. It was decided that a good time to start would be after Austin and Yuri completed their PhDs. The project began in earnest in September of 2007. The plan was to release Austin from all other obligations and have him rough draft as much as possible, as quickly as possible, and then he and Tom would begin to iterate on the drafts. Austin and Tom put together an outline and set as a goal to be finished, or at least declare they were finished, by the end of July, 2008, when Austin was scheduled to leave for New York City. Realizing this schedule might be a bit
optimistic, it was intended that Yuri, who provided numerous results and helped in a variety of ways throughout the project, would step in after Austin left and that he and Tom would complete the project. Things more or less unfolded as planned.

Acknowledgments

We would like to thank our collaborators on the work contained in this volume. In particular, the efforts of Ido Akkerman, Laurenco Beirão da Veiga, Victor Calo, Thomas Elgueudj, John Evans, Héctor Gómez, Scott Lipton, Alessandro Reali, Mike Scott, Guglielmo Scovazzi, and Jessica Zhang have all led directly to the examples in this book. Your efforts are all greatly appreciated, and we look forward to many fruitful collaborations again in the future. We would also like to thank Tom Sederberg for making geometry “analysis-suitable,” Omar Ghattas for providing insights into current trends in supercomputing and describing how isogeometric analysis may be a key enabling technology for taking advantage of modern computer hardware, and Ted Belytschko, Elaine Cohen, Tom Lyche, Rich Riesenfeld, and Tom Sederberg for helpful comments and suggestions concerning an initial draft of this book.

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J. Austin Cottrell
Thomas J. R. Hughes
Yuri Bazilevs
1

From CAD and FEA to Isogeometric Analysis: An Historical Perspective

1.1 Introduction

1.1.1 The need for isogeometric analysis

It may seem inconceivable to young engineers, but it was not long ago that computers were nowhere to be seen in design offices. Designers worked at drawing boards and designs were drawn with pencils on vellum or Mylar\(^1\). The design drawings were passed to stress analysts and the interaction between designer and analyst was simple and direct. Times have changed. Designers now generate CAD (Computer Aided Design) files and these must be translated into analysis-suitable geometries, meshed and input to large-scale finite element analysis (FEA) codes. This task is far from trivial and for complex engineering designs is now estimated to take over 80% of the overall analysis time, and engineering designs are becoming increasingly more complex; see Figure 1.1. For example, presently, a typical automobile consists of about 3,000 parts, a fighter jet over 30,000, the Boeing 777 over 100,000, and a modern nuclear submarine over 1,000,000. Engineering design and analysis are not separate endeavors. Design of sophisticated engineering systems is based on a wide range of computational analysis and simulation methods, such as structural mechanics, fluid dynamics, acoustics, electromagnetics, heat transfer, etc. Design speaks to analysis, and analysis speaks to design. However, analysis-suitable models are not automatically created or readily meshed from CAD geometry. Although not always appreciated in the academic analysis community, model generation is much more involved than simply generating a mesh. There are many time consuming, preparatory steps involved. And one mesh is no longer enough. According to Steve Gordon, Principal Engineer, General Dynamics / Electric Boat Corporation, “We find that today’s bottleneck in CAD-CAE integration is not only automated mesh generation, it lies with efficient creation of appropriate ‘simulation-specific’ geometry.” (In the commercial sector analysis is usually referred to as CAE, which stands for Computer Aided Engineering.) The anatomy of the process has been studied by Ted Blacker, Manager of Simulation Sciences, Sandia National Laboratories. At
Sandia, mesh generation accounts for about 20% of overall analysis time, whereas creation of the analysis-suitable geometry requires about 60%, and only 20% of overall time is actually devoted to analysis per se; see Figure 1.2. The 80/20 modeling/analysis ratio seems to be a very common industrial experience, and there is a strong desire to reverse it, but so far little progress has been made, despite enormous effort to do so. The integration of CAD and FEA has proven a formidable problem. It seems that fundamental changes must take place to fully integrate engineering design and analysis processes.

Recent trends taking place in engineering analysis and high-performance computing are also demanding greater precision and tighter integration of the overall modeling-analysis process. We note that a finite element mesh is only an approximation of the CAD geometry, which we view as “exact.” This approximation can in many situations create errors in analytical results. The following examples may be mentioned: Shell buckling analysis is very sensitive to geometric imperfections (see Figure 1.3), boundary layer phenomena are sensitive to the precise geometry of aerodynamic and hydrodynamic configurations (see Figures 1.4 and 1.5), and sliding contact between bodies cannot be accurately represented without precise geometric descriptions (see Figure 1.6). The Babuška paradox (see Birkhoff and Lynch, 1987) is another example of the pitfalls of polygonal approximations to curved boundaries. Automatic adaptive mesh refinement has not been as widely adopted in industry as one might assume from the extensive academic literature, because mesh refinement requires access to the exact geometry and thus seamless and automatic communication with CAD, which simply does not exist. Without accurate geometry and mesh adaptivity, convergence and high-precision results are impossible.
Deficiencies in current engineering analysis procedures also preclude successful application of important pace-setting technologies, such as design optimization, verification and validation (V&V), uncertainty quantification (UQ), and petascale computing.

The benefits of design optimization have been largely unavailable to industry. The bottleneck is that to do shape optimization the CAD geometry-to-mesh mapping needs to be automatic, differentiable, and tightly integrated with the solver and optimizer. This is simply not the case as meshes are disconnected from the CAD geometries from which they were generated.

V&V requires error estimation and adaptivity, which in turn requires tight integration of CAD, geometry, meshing, and analysis. UQ requires simulations with numerous samples of models needed to characterize probability distributions. Sampling puts a premium on the ability to rapidly generate geometry models, meshes, and analyses, which again leads to the need for tightly integrated geometry, meshing, and analysis.

The era of petaflop computing is on the horizon. Parallelism keeps increasing, but the largest unstructured mesh simulations have stalled, because no one truly knows how to generate and adapt massive meshes that keep up with increasing concurrency. To be able to capitalize on the era of $O(100,000)$ core parallel systems, CAD, geometry, meshing, analysis, adaptivity, and visualization all have to run in a tightly integrated way, in parallel, and in a scalable fashion.
Figure 1.3 Thin shell structures exhibit significant imperfection sensitivity. (a) Faceted geometry of typical finite element meshes introduces geometric imperfections (adapted from Gee et al., 2005). (b) Buckling of cylindrical shell with random geometric imperfections. The buckling load depends significantly upon the magnitude of the imperfections (from Stanley, 1985).
It is apparent that the way to break down the barriers between engineering design and analysis is to reconstitute the entire process, but at the same time maintain compatibility with existing practices. A fundamental step is to focus on one, and only one, geometric model, which can be utilized directly as an analysis model, or from which geometrically precise analysis models can be automatically built. This will require a change from classical finite

**Figure 1.4** Isodensity contours of Galerkin/least-squares (GLS) discretization of Ringleb flow. (a) Isoparametric linear triangular element approximation: both solution and geometry are represented by piecewise linear functions. (b) Super-isoparametric element approximation: solution is piecewise linear, while geometry is piecewise quadratic. Smooth geometry avoids spurious entropy layers associated with piecewise-linear geometric approximations (from Barth, 1998).

**Figure 1.5** The two-dimensional Boussinesq equations. The $x$-component of velocity obtained using 552 triangles with fifth order polynomials on each triangle. On the left, the elements are straight-sided. The spurious oscillations in the solution on the left are due to the use of straight-sided elements for the geometric approximation. On the right, the cylinder is approximated by elements with curved edges, and the oscillations are eliminated (from Eskilsson and Sherwin, 2006).
Isogeometric Analysis: Toward Integration of CAD and FEA

Figure 1.6 Sliding contact. (a) Faceted polynomial finite elements create problems in sliding contact (see Laursen, 2002 and Wriggers, 2002). (b) NURBS geometries can attain the smoothness of real bodies.

Element analysis to an analysis procedure based on CAD representations. This concept is referred to as Isogeometric Analysis, and it was introduced in Hughes et al., 2005. Since then a number of additional papers have appeared (Bazilevs et al., 2006a, 2006b; Cottrell et al., 2006, 2007; Zhang et al., 2007; Gomez et al., 2008).

Here are the reasons why the time may be right to transform design and analysis technologies: Initiatory investigations of the isogeometric concept have proven very successful. Backward compatibility with existing design and analysis technologies is attainable. There is interest in both the computational geometry and analysis communities to embark on isogeometric research. Several mini-symposia and workshops at international meetings have been held and several very large multi-institutional research projects have begun in Europe. In particular, EXCITING – exact geometry simulation for optimized design of vehicles and vessels – is a three year, six million dollar project focused on developing computational tools for the optimized design of functional free-form surfaces, and the Integrated Computer Aided Design and Analysis (ICADA) project is a five year, five million dollar initiative focused on bridging the gap between design and analysis in industry through isogeometric analysis.

There is an inexorable march toward higher precision and greater reality. New technologies are being introduced and adopted rapidly in design software to gain competitive advantage. New and better analysis technologies can be built upon and influence these new CAD technologies. Engineering analysis can leverage these developments as a basis for the isogeometric concept.
Anyone who has lived the last 60 years is acutely aware of the profound changes that have occurred due to the emergence of new technologies. History has demonstrated repeatedly that statements to the effect that “people will not change” are false. An interesting example of a paradigm shift concerns the slide rule, a mechanical device that dominated computing for approximately 350 years. In the 20th century alone nearly 40 million slide rules were produced throughout the world. The first transistorized electronic calculators emerged in the early 1960s, with portable four-function models available by the end of the decade. The first hand-held scientific calculator, Hewlett-Packard’s HP35, became commercially available in 1972. Keuffel and Esser Co., the world’s largest producer of slide rules, manufactured its last slide rule in 1975, just 3 years later (see Stoll, 2006).

1.1.2 Computational geometry

There are a number of candidate computational geometry technologies that may be used in isogeometric analysis. The most widely used in engineering design are NURBS (non-uniform rational B-splines), the industry standard (see, Piegl and Tiller, 1997; Farin, 1999a, 1999b; Cohen et al., 2001; Rogers, 2001). The major strengths of NURBS are that they are convenient for free-form surface modeling, can exactly represent all conic sections, and therefore circles, cylinders, spheres, ellipsoids, etc., and that there exist many efficient and numerically stable algorithms to generate NURBS objects. They also possess useful mathematical properties, such as the ability to be refined through knot insertion, $C^{p-1}$-continuity for $p$th-order NURBS, and the variation diminishing and convex hull properties. NURBS are ubiquitous in CAD systems, representing billions of dollars in development investment. One may argue the merits of NURBS versus other computational geometry technologies, but their preeminence in engineering design is indisputable. As such, they were the natural starting point for isogeometric analysis and their use in an analysis setting is the focus of this book.

T-splines (Sederberg et al., 2003; Sederberg et al., 2004) are a recently developed forward and backward generalization of NURBS technology. T-splines extend NURBS to permit local refinement and coarsening, and are very robust in their ability to efficiently sew together adjacent patches. Commercial T-spline plug-ins have been introduced in Maya and Rhino, two NURBS-based design systems (see references T-Splines, Inc., 2008a and T-Splines, Inc., 2008b). Initiatory investigations of T-splines in an isogeometric analysis context have been undertaken by Bazilevs et al., 2009 and Dorfel et al., 2008. These works point to a promising future for T-splines as an isogeometric technology.

There are other computational geometry technologies that also warrant investigation as a basis of isogeometric analysis. One is subdivision surfaces which use a limiting process to define a smooth surface from a mesh of triangles or quadrilaterals (see, e.g., Warren and Weimer, 2002; Peters and Reif, 2008). They have already been used in analysis of shell structures by Cirak et al., 2000; Cirak and Ortiz, 2001, 2002. The appeal of subdivision surfaces is there is no restriction on the topology of the control grid. Like T-splines, they also create gap-free models. Most of the characters in Pixar animations are modeled using subdivision models. The CAD industry has not adopted subdivision surfaces very widely because they are not compatible with NURBS. With billions of dollars of infrastructure invested in NURBS, the financial cost would be prohibitive. Nevertheless, subdivision surfaces should play an
important role in isogeometric technology. Subdivision solids have been studied by Bajaj et al., 2002.

Other geometric technologies that may play a role in the future of isogeometric analysis include Gordon patches (Gordon, 1969), Gregory patches (Gregory, 1983), S-patches (Loop and DeRose, 1989), and A-patches (Bajaj et al., 1995). Provatidis has recently solved a number of problems using Coons patches (see Provatidis, 2009, and references therein). Others may be invented specifically with the intent of fostering the isogeometric concept, namely, to use the surface design model directly in analysis. This would only suffice if analysis only requires the surface geometry, such as in the stress or buckling analysis of a shell. In many cases, the surface will enclose a volume and an analysis model will need to be created for the volume. The basic problem is to develop a three-dimensional (trivariate) representation of the solid in such a way that the surface representation is preserved. This is far from a trivial problem. Surface differential and computational geometry and topology are now fairly well understood, but the three-dimensional problem is still open (the Thurston conjecture characterizing its solution remains to be proven, see Thurston, 1982, 1997). The hope is that through the use of new technologies, such as, for example, Ricci flows and polycube splines (see Gu and Yau, 2008), progress will be forthcoming.

1.2 The evolution of FEA basis functions

Solution of partial differential equations by the finite element method consists, roughly speaking, of a variational formulation and trial and weighting function spaces defined by their respective basis functions. These basis functions are defined in turn by finite elements, local representations of the spaces. The elements are a non-overlapping decomposition of the problem domain into simple shapes (e.g., triangles, quadrilaterals, tetrahedra, hexahedra, etc.). In the most widely used variational methods, the trial and weighting functions are essentially the same. Specifically, the same elements are used in their construction. There are three ways to improve a finite element method:

1. Improve the variational method. Sometimes this can be done in such a way as to correct a shortcoming in the finite elements for the problem under consideration, such as, for example, through the use of selective integration (see Hughes, 2000). Another way is to use an alternative variational formulation with improved properties, an example being “stabilized methods.” See Brooks and Hughes, 1982; Hughes et al., 2004.
2. Improve the finite element spaces, that is, the elements themselves.
3. Improve both, that is, the variational method and the elements.

Our focus here is on finite element spaces and ultimately how they perform in comparison to spaces of functions built from NURBS, T-splines, etc. Consequently, we will give a brief review of the historical milestones in finite elements.

Typically, finite elements are defined in terms of interpolatory polynomials. The classical families of polynomials, especially the Lagrange and Hermite polynomials, are widely utilized (see Hughes, 2000). These may be considered the historical antecedents of finite elements.
Figure 1.7  Finite element picture gallery.
Early publications in the engineering literature describing what is now known as the finite element method were Argyris and Kelsey, 1960, which is a collection of articles by those authors dating from 1954 and 1955, and Turner et al., 1956. The term “finite elements” was coined by Clough, 1960. However, the first finite element, the linear triangle, can be traced all the way back to Courant, 1943. It is perhaps the simplest element and is still widely used today. It is interesting to note that the engineering finite element literature was unaware of this reference until sometime in the late 1960s by which time the essential features of the finite element method were well established. The linear tetrahedron appeared in Gallagher et al., 1962. Through the use of triangular and tetrahedral coordinates (i.e., barycentric coordinates) and the Pascal triangle and tetrahedron, it became a simple matter to generate \( C^0 \)-continuous finite elements for straight-edged triangles and flat-surfaced tetrahedra. The bilinear quadrilateral was developed by Taig, 1961, and it presaged the development of \textit{isoparametric elements} (Irons, 1966; Zienkiewicz and Cheung, 1968), perhaps the most important concept in the history of element technology.

The idea of isoparametric elements immediately generalized elements which could be developed on a regular parent domain, such as a square, or a cube, to an element which could take on a smoothly curved shape in physical space. Furthermore, it was applicable to any element topology, including triangles, tetrahedra, etc. An essential feature was that the spaces so constructed satisfied basic mathematical convergence criteria, as well as physical attributes in problems of mechanics, namely, the ability to represent all affine motions (i.e., rigid translations and rotations, uniform stretchings and shearings) exactly. Curved quadrilateral and