STOCHASTIC DYNAMICS OF STRUCTURES

Jie Li and Jianbing Chen

Tongji University, China
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OF STRUCTURES
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Tongji University, China
To Min Xie, My wife
Jie Li

To My Parents
Jianbing Chen
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Foreword

It is a great pleasure to introduce *Stochastic Dynamics of Structures* by Jie Li and Jianbing Chen. The book begins with a brief history of the early discovery and developments of the field, starting with Einstein's introduction of the Brownian motion, followed by the classical developments, including the mathematical formulations of Fokker, Planck, and Kolmogorov. It is a timely and much needed exposition of the existing state of knowledge of stochastic dynamics and its potential applications in structural dynamics and the reliability of dynamical systems.

The topical coverage of stochastic dynamics starts properly with an introduction of the fundamentals of random variables, random vectors, and stochastic processes including random fields, which are the essentials necessary for the study of random vibration and stochastic structural analysis, and culminates with the presentation of the probability density evolution theory and its corollary the equivalent extreme value distribution; the latter is especially significant for evaluating the dynamic reliability of structures and other engineering systems.

This book is a valuable contribution to the continuing development of the field of stochastic structural dynamics, including the recent discoveries and developments by the authors of the probability density evolution method (PDEM) and its applications in the assessment of the dynamic reliability and control of complex structures through the equivalent extreme-value distribution. The traditional analytical approach to such a dynamic reliability problem is to formulate it as a “barrier-crossing problem” that leads to the solution of the Fokker-Planck equation; the limitations of this approach are well known, even for single-degree-of-freedom systems. The authors thoroughly discuss this classical approach and show its limitations, following with the PDEM, including the numerical solution of complex multi-degree-of-freedom systems. These are preceded with new insights, derivations, and interpretations of the classical formulations and solutions—such as the Liouville equation, the Kolmogorov equation, and the Itô stochastic equations—are provided through the concept of the preservation of probability.

Besides elucidating the principles of stochastic dynamics from an engineer’s viewpoint, the most significant contribution of this book is its lucid presentation of the PDEM and its applications for the assessment of the dynamic reliability and control of structures under earthquake excitations and wind and wave forces. In this regard, the PDEM should serve to spur further developments of stochastic structural dynamics; with the PDEM, solutions to the dynamic reliability of multi-degree-of-freedom systems can be evaluated numerically, including non-linear systems. Innovative numerical schemes are proposed; besides finite difference schemes, spherical packing schemes are also suggested for solutions of highly complex problems.
In other words, this book includes a novel approach to the field of stochastic dynamics with special emphasis on the applications to the dynamic response and reliability of structures. It should serve well to advance the research in the field of stochastic structural dynamics in general and dynamic reliability in particular.

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Preface

As a scientific discipline, stochastic dynamics of structures has evolved from its infancy in the early 1940s to a relatively mature branch of dynamics today. In the process, basic random vibration theory is believed to have been established in the late 1950s and mainly deals with the response analysis of structures to stochastic excitations, such as the response of buildings and bridges to wind loading and earthquakes, the vibration of vehicles traveling over rough ground and the dynamic behavior of aircraft induced by atmospheric turbulence and jet noise. In the late 1960s, the importance of the effect of randomness in structural parameters on the structural response was recognized gradually and this led to stochastic structural analysis, or stochastic finite-element analysis as termed by many researchers. There has been a large amount of literature published in the past 40 years; however, careful people may find that random vibration theory and stochastic finite element analysis seem to have developed in two parallel ways. It is very hard for most engineers, even those specialists who are familiar with stochastic analysis, to organize their knowledge of the two branches of dynamics in a systematic framework. Therefore, the first aim of this book is to present a coherent and reasonably self-contained theoretical framework of the stochastic dynamics of structures which may bridge the gap between traditional random vibration theory and the stochastic finite-element analysis method. We hope such a treatment will provide a comprehensive account for stochastic dynamic response analysis, reliability evaluation and system control.

The second aim, which may be more important and seems a little bit ambitious, is to deal with the basic content of stochastic dynamics of structures in a unified new theoretical framework. We refer to this as the frame of the physical stochastic system. Most people know that, in many practical applications, the system of concern usually exhibits nonlinearities. However, it is just for nonlinear dynamical systems that the foundational stochastic dynamics theory involves huge complexities. After considerable research efforts in the field of random vibration and stochastic finite-element analysis, although some important progress has been made for simple structural models, people still cannot solve the problem of nonlinear stochastic dynamical systems rationally, especially for practical complex structures. Motivated by the need to provide a rational description of a nonlinear stochastic system and of developing appropriate analytical tools, we undertook a systematic investigation on the difficult area in the past 15 years. Tracing back to the source of the discipline, we find that there are two historical traditions in the study of stochastic dynamics: the phenomenological tradition and the physical tradition. Because of the introduction of the Wiener process, the two traditions gain an intrinsic relation. However, if we return to the physical processes themselves (that is, investigating random phenomena from a physical viewpoint), then we will be led to another possible way: approaching the stochastic system based on physics. Using this approach, we give a rational
description of the relationship between the physical sample trajectories of a dynamic system and its probabilistic description and, therefore, establish a family of generalized probability density evolution equations for dynamical systems which could deal with both linear and nonlinear systems in a unified form. Furthermore, bearing in mind the physical stochastic system, we find that traditional random vibration theory and the stochastic finite-element methods can be appropriately brought into a new theoretical frame. Obviously, this provides a foundation to rearrange the content of stochastic dynamics of structures in a comprehensive framework. This book tries to present such a development, as well as pragmatic methods and algorithms where possible.

We assume that the book will be used by graduate students and professionals in civil engineering, mechanical engineering, aircraft and marine engineering, as well as in mechanics. The level of the preparation assumed of the reader corresponds to that of the bachelor’s degree in science or engineering, especially those who have a basic understanding of the concept of probability theory and structural dynamics. In addition, to make the book self-sufficient, the essential concepts of random variables, stochastic processes and random fields are presented in the book.

Our sincere appreciations go first to Professor P.D. Spanos at Rice University, for his friendly encouragement, and to Professor R.G. Ghanem at the University of Southern California, for his constructive comments and fruitful discussions. For their valuable help and advice, special thanks are also due to Professor W.D. Iwan at California Institute of Technology, Professor Jinping Ou at Dalian University of Technology and Professor Yangang Zhao at Nagaya Institute of Technology. The first author would also like to take this opportunity to express his deep appreciation to Professor J.B. Roberts of Sussex University, who is greatly missed, for the generous support given for the chance to complete the investigation on stochastic analysis and modeling during 1993 and 1994 when the author spent one year as a senior visiting scholar at Sussex University, and to his colleagues Professor Xilin Lu, Professor Guoqiang Li, Professor Ming Gu, Professor Yiyi Chen and Professor Menglin Lou at Tongji University for their continuous cooperation and support.

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1

Introduction

1.1 Motivations and Historical Clues

Structural dynamics deals with the problems of response analysis, reliability evaluation and system control of any given type of structure subjected to dynamic actions. Structures (such as buildings, bridges, aircraft, ships and so on) refer to those bodies or systems composed of various materials in a certain way that are capable of bearing loads and actions. On the other hand, when we say an action applied on structures is dynamic, this not only indicates that the action is time varying, but also that the induced inertial effects cannot be ignored. For example, earthquakes, wind, sea waves, jet noise and turbulence in the boundary layer and the like are typical dynamic actions. The task of dynamic response analysis of structures is to capture the internal forces, deformations or other state quantities of structures when they are subjected to dynamic actions. At the same time, we may need to study whether the structural response meets some specified limit in a sense, which is generally referred to as reliability evaluation. Furthermore, to make a structure subjected to dynamic actions response in a desired way to an extent is what to be done in system control.

Most dynamic actions exhibit appreciable randomness. Actually, investigators frequently find that the results observed under almost identical conditions have obvious deviation, but simultaneously exhibit some statistical rules. In essence, the randomness results from the uncontrollability of causation of the realized phenomenon. For example, consider wind turbulence in the atmospheric boundary layer. It is well known that the observed wind speeds recorded at the same position but during different time intervals are quite different (Figure 1.1). However, if the statistics of a large number of samples are examined, then we find that the probabilistic characteristics of the wind speed are relatively stable (Figure 1.2). In fact, the randomness involved stems from a complicated physical mechanism in the wind flows, say the mechanism of turbulence. The underlying reason is the uncontrollable nature of the motion of air molecules.

In addition, the randomness involved in the physical parameters of structures is also one of the sources that induce randomness in the dynamic responses of structures. For instance, in the

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1 The dynamic properties of structures, such as the frequencies and mode shapes, are also research topics of structural dynamics. But, in a general sense, the dynamic properties of structures can be regarded as part of the dynamic analysis of structures.

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dynamic response analysis of building structures, the soil–structure interaction is one of the basic problems where the properties of soil must be considered in the establishment of a reasonable structural analysis model. Evidently, it is impossible to measure the physical properties of soil completely at all points in the groundwork. Thus, a reasonable modeling approach is to regard the physical properties of soil, such as the shear wave speed and the damping ratio, as random variables or random fields. This will lead to the structural analysis involving random parameters, usually known as stochastic structural analysis.

Stochastic dynamic response analysis, reliability evaluation and system control compose the basic research scope of the stochastic dynamics of structures.

Although the studies on stochastic dynamical systems can be dated back to the investigations on statistical mechanics by Gibbs and Boltzmann (Gibbs, 1902; Cercignani, 1998), it is generally considered more reasonable to regard the studies on Brownian motion by Einstein (1905) as the origin of the discipline.
In 1905, Einstein studied the problem of the irregular motion of particles suspended in fluids, which was first observed by the Scottish botanist Robert Brown in 1827 (Figure 1.3). Einstein believed that Brownian motion of the particles was induced by the highly frequent random impacts of the fluid molecules. Based on this physical interpretation, Einstein made the following assumptions:

(a) the motion of different Brownian particles is mutually independent;
(b) the motion of Brownian particles is isotropic and no external actions except the collision of fluid molecules are applied;
(c) the collision of fluid molecules is instantaneous, such that the time of collision can be ignored (rigid collision).

Based on the above assumptions, the probability density of the particle group at two different instants of time can be derived by examining the phenomenological evolution process of the particle group; that is:

\[ f(x, t + \tau) = \int_{-\infty}^{+\infty} f(x + r, t) \Phi(r) \, dr \quad (1.1) \]

where \( f(x, t + \tau) \) is the probability density of the position of the particles at time \( t + \tau \), \( f(x + r, t) \) is the probability density by transition of the particles with distance \( r \) during the time interval \( \tau \), and \( \Phi(r) \) is the probability density of displacement of the particles.

Using the rigid collision assumption, expanding the functions by using the Taylor series and retaining the first-order term with respect to \( f(x, t + \tau) \) and the second-order term with respect to \( f(x + r, t) \) will yield

\[ \frac{\partial f(x, t)}{\partial t} = D \frac{\partial^2 f(x, t)}{\partial x^2} \quad (1.2) \]

where

\[ D = \frac{1}{\tau} \int_{-\infty}^{+\infty} \frac{1}{2} r^2 \Phi(r) \, dr \quad (1.3) \]

Clearly, Equation (1.2) is a diffusion equation, where \( D \) is the diffusion coefficient.
In 1914 and 1917, Fokker and Planck respectively introduced the drift term for a similar physical problem, leading to the so-called Fokker–Planck equation (Fokker, 1914; Planck, 1917; Gardiner, 1983), of which the rigorous mathematical basis was later established by Kolmogorov (1931).²

We note that, initially, the studies on Brownian motion were based on physical concepts; however, a statistical phenomenological interpretation was subsequently introduced in the deductions. In this book, we call this historical clue the Einstein–Fokker–Planck tradition or phenomenological tradition. In this tradition, a large number of studies on the probability density evolution of stochastic dynamical systems have been done (Kozin, 1961; Lin, 1967; Roberts and Spanos, 1990; Zhu, 1992, 2003; Lin and Cai, 1995). However, for the multi-degree-of-freedom (MDOF) systems or multidimensional problems, advancement is still quite limited (Schüeller, 1997, 2001).

Soon after Einstein’s work, Langevin (1908) came up with a completely different research approach. In his investigation, the physical interpretation of Brownian motion is the same as that of Einstein, but Langevin contributed to two basic aspects. He:

(a) introduced the assumption of random forces;
(b) employed Newton’s equation of motion to govern the motion of the Brownian particles.

Based on this, he established the stochastic dynamics equation, which was later called the Langevin equation:

\[ m\ddot{x} = -\gamma \dot{x} + \xi(t) \quad (1.4) \]

where \( m \) is the mass of the Brownian particles, \( \ddot{x} \) and \( \dot{x} \) are the acceleration and velocity of motion respectively, \( \gamma \) is the viscous damping coefficient and \( \xi(t) \) is the force induced by the collision of the fluid molecules, which is randomly fluctuating.

Using the ensemble average, Langevin obtained a diffusion coefficient identical to that given by Einstein.

In contrast to the diffusion equation derived by Einstein, the Langevin equation is more direct and more physically intuitive. However, the physical features of the random forces are not completely clear in Langevin’s work.

In 1923, Wiener proposed a stochastic process model for Brownian motion (Wiener, 1923). Around 20 years later, Itô introduced the Itô integral and gave the more generic Langevin equation based on the Wiener process (Itô, 1942, 1944; Itô and McKean, 1965):

\[ dx(t) = a[x(t), t] \, dt + b[x(t), t] \, dW(t) \quad (1.5) \]

where \( a(\cdot) \) and \( b(\cdot) \) are known deterministic functions and \( W(t) \) is a Wiener process.

The form of Equation 1.5 is nowadays called the Itô stochastic differential equation. Clearly, this equation is in essence a physical equation. It is generally believed that the Itô equation²

²Interestingly, Kolmogorov did not at first know about the work of Fokker and Planck and developed his equation independently.
provides a sample trajectory description for stochastic dynamical systems. In this book, we refer to this historical clue as the Langevin–Itô tradition or physical tradition. In this approach, the mean-square calculus theory was established, based on which correlation analysis and spectral analysis in classical random vibration analysis were well developed (Crandall, 1958, 2006; Lin, 1967; Zhu, 1992; Øksendal, 2005).

There were intrinsic and countless ties between the phenomenological tradition and the physical tradition in stochastic dynamics. As a matter of fact, upon the assumption that the system inputs are white-noise processes, it is easy to obtain the Fokker–Planck–Kolmogorov (FPK) equation via the Itô equation. This demonstrates that in the physics approach we can discover the intrinsic arcaneum of the evolution of stochastic systems. Unfortunately, white noise is physically unrealizable. In other words, although mathematically it plays a fundamental role in a sense, the various singular or even ridiculous features of white noise (say, continuous but indifferentiable everywhere) are rare in the real world.

The white-noise process is, of course, an idealized model for various real physical processes. Noticing this, we naturally hope to return to the real physical processes themselves. For a specific physical dynamical process, the problem is usually easily resolvable. Thus, once further introducing the intrinsic ties between the sample trajectories and the probabilistic description, we will be led to an approach of studying stochastic systems based on physics. In this approach, we not only can establish the generalized probability density evolution equation (Li and Chen, 2003, 2006c, 2008), but also find that the nowadays available major research results, such as traditional random vibration theory and stochastic finite element methods, can be appropriately brought into the new theoretical frame (Li, 2006). In fact, correlation analysis and the spectral analysis in classical random vibration theory can be regarded as the results of combining the formal solution of physical equations and the evolution of moment characteristics of the response processes. Perturbation theory and orthogonal expansion theory in the analysis of structures with random parameters can also be reasonably interpreted in this sense. The classical FPK equation, as mentioned before, can be viewed as the result of the idealization of physical processes. In addition, the thoughts of physical stochastic system can also be used in modeling of general stochastic process, such as seismic ground motion, wind turbulence and the like (Li and Ai, 2006; Li and Zhang, 2007).

On the basis of the above thoughts on physical stochastic systems, we prefer to entitle this book Stochastic Dynamics of Structures: a Physical Approach.

1.2 Contents of the Book

This book deals with the basic problems of the stochastic dynamics of structures in the theoretical frame of physical stochastic systems.

In Chapter 2 the prerequisite fundamentals of probability theory are outlined, including the basic concepts of random variables, stochastic processes, random fields and the orthogonal expansion of random functions.

Chapter 3 deals with stochastic process models for typical dynamic excitations of structures, including the phenomenological and physical modeling of seismic ground motions, fluctuating wind speed and sea waves. Simultaneously, we introduce the standard orthogonal expansion of stochastic processes, which can be applied to random vibration analysis of structures.
The approaches for analysis of structures with random parameters mainly include the random simulation method, the perturbation method and the orthogonal expansion method. These approaches are discussed in detail in Chapter 4.

Chapter 5 deals with the response analysis of deterministic structures subjected to stochastic dynamic excitations, including correlation analysis, spectral analysis, the statistical linearization method and the FPK equation approach. In particular, in this chapter we introduce the pseudo-excitation method for linear systems. We believe these contents are valuable to in-depth understanding of classical random vibration theory.

Probability density evolution analysis of stochastic responses of dynamical systems is an important topic of the book. We will deal with this topic in Chapters 6 and 7. In Chapter 6 we trace in some detail the historical origin of probability density evolution analysis of stochastic dynamical systems. Using the principle of preservation of probability as a unified basis, we derive the Liouville equation, the FPK equation, the Dostupov–Pugachev equation and the generalized probability density evolution equation proposed by the authors. In Chapter 7 we study the numerical methods for probability density evolution analysis in detail, including the finite difference method, the strategy of selecting representative points via tangent spheres, lattices and the number theoretical method. For all these methods, we discuss the problems of numerical convergence and stability where possible.

The aim of structural dynamical analysis is to realize reliability-based design and performance control of structures. We discuss the problem of dynamic reliability and global reliability of structures in Chapter 8. Based on the random event description of the evolution of probability density, the absorbing boundary condition for the first-passage problem is introduced. The theory on evaluation of the extreme value distribution is elaborated through introducing a virtual stochastic process related to the extreme value of the response process. Furthermore, the principle of equivalent extreme value and its application to the global reliability evaluation of structures is discussed. It is worth pointing out that the principle of equivalent extreme value is of significance and applicable to static reliability evaluation of generic systems.

We come to the problem of the dynamic control of structures in Chapter 9. On the basis of classical dynamic control, the concept of stochastic optimal control is introduced and the approach for design of the control systems based on probability density evolution analysis is proposed. For realization of ‘real’ stochastic optimal control of dynamical systems, the proposed approach is undoubtedly promising.
2

Stochastic Processes and Random Fields

2.1 Random Variables

2.1.1 Introduction

By an experiment we mean taking a kind of action or operation devised to seek for a certain truth or fact. For some experiments, the results possess basic properties of deterministic phenomena once all underlying conditions are well controlled and all experimental phenomena are exactly observed. In other words, the results of these experiments are predictable. Owing to uncontrollable or immeasurable facts, however, those experiments may obtain varied results, though fundamental conditions remain invariant in some respects. This is the so-called random phenomenon. The results occurring in a set of random experiments are generally called random events, or events for simplicity. The basic property of a random event lies in that the predicted event may be observed or not when the observational conditions differ by a small amount. However, we can always identify the set whose elements consist of all the possible results for a given experiment. In other words, the union of all experimental results can be determined beforehand. This set is called the sample space and denoted by \( \Omega \). Each possible result in \( \Omega \) is called a sample point, denoted by \( \omega \). Each event \( A \) can be understood as a subset of \( \Omega \). An event is called an elementary event if it contains only a single sample point. On the other hand, we can say that a compound event is a certain set of sample points. A family of events, denoted by \( \mathcal{F} \) or termed \( \sigma \)-algebra, refers to the subset of \( A \) which satisfies the following statements:

(a) \( \Omega \in \mathcal{F} \);
(b) \( A \in \mathcal{F} \) implies \( \bar{A} \in \mathcal{F} \), where \( \bar{A} \) is the complement of \( A \);
(c) \( A_n \in \mathcal{F} \ (n = 1, 2, \ldots) \) implies \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \).

To measure a sample space, we need to assign a numerical value to measure the possibility of an event occurring. This gives rise to the concept of probability measure, by which every event in \( \mathcal{F} \) is mapped into the unit interval \([0, 1]\). That is, the possibility of each occurrence can be represented by a nonnegative number smaller than unity. In general, we call this number the
probability measure of $\Omega$, or the probability of the given event $A$, denoted by $P(A)$ or $\Pr\{A\}$. In addition, in Kolmogorov’s *Foundations of the Theory of Probability*, the triple $(\Omega, \mathcal{F}, P)$ is defined as the probability space (Kolmogorov, 1933; Loeve, 1977; Kallenberg, 2002).

We have already noted that the probability measure gives a gauge of an event’s occurrence, but does not give a similar one for the sample points. In mathematics, this problem is resolved by defining on the probability space a measurable function $X(\omega)$, which is generally called a random variable, and denoted by $X$ for short.$^1$ It has two basic properties:

(a) A random variable is a single-valued real function of sample points. That is, each random variable produces a mapping from a probability space to a field of real numbers.

(b) For any real number, $\{\omega : X(\omega) < x\}$ is a random event.

With the concept of a random variable, we can adopt numerical values to describe the results of any random experiment. For instance, an elementary event is expressed in the form that a random variable $X$ is equal to one deterministic number (i.e. $X = x$), while any arbitrary event can be expressed in a way that $X$ takes values over an interval $x_1 \leq X \leq x_2$ and its probability of occurring is denoted by $\Pr\{x_1 \leq X \leq x_2\}$. There are two basic types of random variable: discrete random variable and continuous random variable. The former take values in a finite or countable infinite set, while the latter can be assigned any value in one or several intervals. When the Dirac delta function is introduced later, the discrete random variable and continuous random variable will be seen to operate in a unified way (see Appendix A), but in this book it is mainly the continuous random variables that are discussed.

In general, $F_X(x) = \Pr\{X(\omega) < x\}$ ($-\infty < x < \infty$) is called the cumulative distribution function (CDF) of the random variable $X$. It satisfies the following basic properties:

(a) $\lim_{x \to -\infty} F_X(x) = 0, \lim_{x \to \infty} F_X(x) = 1$;

(b) if $x_1 < x_2$, then $F_X(x_1) \leq F_X(x_2)$;

(c) $F_X(x - 0) = F_X(x)$;

(d) $\Pr\{x_1 \leq X < x_2\} = F_X(x_2) - F_X(x_1)$.

By introducing random variables, we can further deal with probability measure problems of complicated systems. This is done by the use of operations performed on random variables.

### 2.1.2 Operations with Random Variables

Two of the most important operations are the distributions of random variables’ functions and the moments of random variables. They are both based on calculations of the probability density functions (PDFs). Thus, the concept of the PDF is introduced first.

For a continuous random variable $X$, the PDF is defined as the derivative of its CDF:

$$p_X(x) = \frac{d}{dx} F_X(x)$$

(2.1)

where $p_X(x)$ is a nonnegative function; that is, there always exists $p_X(x) \geq 0$.

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$^1$ A random variable is usually denoted by a capital letter or Greek character, say $X$ or $\xi$, while the sample value of a random variable is usually denoted by the corresponding lower case character, say $x$. The convention is used in the book except for special statements.
The inversion of Equation 2.1 gives

\[ F_X(x) = \int_{-\infty}^{x} p_X(x) \, dx \]  \hspace{1cm} (2.2)

where the condition \( F_X(-\infty) = 0 \) has been used.

As the upper limit of the integral goes to infinity, we have

\[ \int_{-\infty}^{\infty} p_X(x) \, dx = 1 \]  \hspace{1cm} (2.3)

Figure 2.1 depicts a typical PDF and the CDF.

If a random variable \( Y \) is the function of another one \( X \), namely \( Y = f(X) \), and \( f(\cdot) \) only has a finite number of discontinuity points, then the CDF of \( Y \) is given by

\[ F_Y(y) = \text{Pr}\{f(X) < y\} = \int_{f(x) < y} p_X(x) \, dx \]  \hspace{1cm} (2.4)

This integral is calculated over all the segments in the \( x \) axis which satisfy the inequality below the integral symbol.

Theoretically, according to Equation 2.1, it is easy to obtain the PDF of \( Y \) from Equation 2.4. However, we may encounter difficulties when making specific operations, because sometimes \( f(\cdot) \) may be a very complicated function. Thus, in general we only consider two cases as follows:
(a) Suppose \( f(x) \) is a monotonic function. Then there exists \( g(y) \) as the unique inverse function of \( f(x) \). Using Equations 2.1 and 2.4, the PDF of \( Y \) is given by

\[
p_Y(y) = p_X[g(y)] \left| \frac{dg(y)}{dy} \right| \quad (2.5)
\]

(b) Suppose \( f(x) \) is not monotonic but a single-valued function (see Figure 2.2). In this case, we can try to divide the domain of \( x \)-values into several intervals such that, over each interval, \( f(x) \) is a monotonic function. Then, similar to Equation 2.4, there is

\[
p_Y(y) = \sum_k p_X[g_k(y)] \left| \frac{dg_k(y)}{dy} \right| \quad (2.6)
\]

where \( g_k(y) \) is the inverse function of \( f(x) \) in the \( k \)th interval.

As already noted, the CDF or PDF describes the distribution properties of random variables in a precise way. On the other hand, somewhat rough descriptions of random variables are the moments, among which two of the most useful ones are the expectation and the variance.

The expectation of a continuous random variable is defined as the first origin moment of its density function; that is,

\[
\mathcal{E}[X] = \int_{-\infty}^{\infty} xp_X(x) \, dx \quad (2.7)
\]

Its variance is the second central moment of its density function:

\[
\mathcal{D}[X] = \mathcal{E}\{(X - \mathcal{E}[X])^2\} = \int_{-\infty}^{\infty} (x - \mathcal{E}[X])^2 p_X(x) \, dx \quad (2.8)
\]

The basic property of the expectation is its linear superposition; that is:

\[
\mathcal{E}[aX + b] = a\mathcal{E}[X] + b \quad (2.9)
\]

where \( a \) and \( b \) are any two constants.

Correspondingly, the variance obeys

\[
\mathcal{D}[aX + b] = a^2\mathcal{D}[X] \quad (2.10)
\]
In general, we call

$$m_n = \mathcal{E}[X^n] = \int_{-\infty}^{\infty} x^n p_X(x) \, dx$$  \hspace{1cm} (2.11)$$

the $n$th origin moment of $X$ and denote the expectation $m_1$ by $\mu$.

$$K_n = \mathcal{E}[(X - \mu)^n] = \int_{-\infty}^{\infty} (x - \mu)^n p_X(x) \, dx$$  \hspace{1cm} (2.12)$$
is called the $n$th central moment of $X$, and $\sigma^2$ is used to denote $K_2$ or $\mathcal{D}[X]$. $\sigma = \sqrt{\mathcal{D}[X]}$ is usually called the standard deviation of $X$.

The central moments can be expressed by the linear combination of origin moments:

$$\mathcal{E}[(X - \mu)^n] = \sum_{i=0}^{n} \binom{n}{i} (-\mathcal{E}[X])^{n-i} \mathcal{E}[X^i]$$  \hspace{1cm} (2.13)$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-1)!}$$

Similarly, the origin moments can also be computed by the central moments.

For a continuous random variable $X$, the characteristic function, denoted by $f_X(\vartheta)$, is the Fourier transform of its PDF; that is:

$$f_X(\vartheta) = \int_{-\infty}^{\infty} e^{i\vartheta x} p_X(x) \, dx$$  \hspace{1cm} (2.14)$$

As noted, the characteristic function can serve as a mode of describing random variables like the PDF. More significantly, moment functions of a random variable can be given by derivatives of its characteristic function. In fact:

$$\frac{d^n f_X(\vartheta)}{d\vartheta^n} = i^n \int_{-\infty}^{\infty} e^{i\vartheta x} x^n p_X(x) \, dx$$  \hspace{1cm} (2.15)$$

Let $\vartheta = 0$, then

$$\left. \frac{d^n f_X(\vartheta)}{d\vartheta^n} \right|_{\vartheta=0} = i^n \int_{-\infty}^{\infty} x^n p_X(x) \, dx = i^n \mathcal{E}[X^n]$$  \hspace{1cm} (2.16)$$

where $i$ is the imaginary number unit. Meanwhile, we obtain the Maclaurin series expansion of $f_X(\vartheta)$:

$$f_X(\vartheta) = f_X(0) + \sum_{n=1}^{\infty} \frac{d^n f_X}{d\vartheta^n} \bigg|_{\vartheta=0} \frac{\vartheta^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(i\vartheta)^n}{n!} \mathcal{E}[X^n]$$  \hspace{1cm} (2.17)$$

Equation 2.17 implies that the lower-order moments contain the major parts of information about a distribution. For many practical problems, second-order statistics are enough to describe them.
2.1.3 Random Vectors

In many cases there is more than one random variable of interest. If the random variables \( X_1(\omega), X_2(\omega), \ldots, X_n(\omega) \) belong to the same probability space \( (\Omega, \mathcal{F}, P) \), then

\[
X = (X_1(\omega), X_2(\omega), \ldots, X_n(\omega))
\]

is an \( n \)-dimensional random vector.

The joint CDF of a random vector is defined by

\[
F_X(x_1, x_2, \ldots, x_n) = \Pr\{X_1 < x_1, X_2 < x_2, \ldots, X_n < x_n\} = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \ldots \int_{-\infty}^{x_n} p_X(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \ldots dx_n
\]

where \( p_X(x_1, x_2, \ldots, x_n) \) is the joint PDF of \( X \). The joint density function satisfies the following properties:

\[
p_X(x_1, x_2, \ldots, x_n) \geq 0
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} p_X(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \ldots dx_n = 1
\]

and there exists

\[
p_X(x_1, x_2, \ldots, x_n) = \frac{\partial^n F_X(x_1, x_2, \ldots, x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}
\]

For a certain component \( X_i \), the marginal distribution and the marginal density function are respectively defined by

\[
F_{X_i}(x_i) = \Pr\{X_i < x_i\} = F_X(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty)
\]

and

\[
p_{X_i}(x_i) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} p_X(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \ldots dx_{i-1} \, dx_{i+1} \ldots dx_n
\]

Generally speaking, the marginal distribution can be uniquely determined by the joint probability distribution function, but the converse is not true. In other words, the joint PDF contains more information than each marginal density function separately, since the latter can be obtained from the former. This implies that the correlation between random variables is an important profile of a random vector.

For an \( n \)-dimensional random vector, the conditional cumulative distribution and the conditional PDF with respect to a certain component \( X_i \) are respectively defined by

\[
F_{X|X_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n|x_i) = \Pr\{X_1 < x_1, \ldots, X_{i-1} < x_{i-1}, X_{i+1} < x_{i+1}, \ldots, X_n < x_n|x_i = x_i\}
\]