Copula Methods in Finance

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Preface

Copula functions represent a methodology which has recently become the most significant new tool to handle in a flexible way the comovement between markets, risk factors and other relevant variables studied in finance. While the tool is borrowed from the theory of statistics, it has been gathering more and more popularity both among academics and practitioners in the field of finance principally because of the huge increase of volatility and erratic behavior of financial markets. These new developments have caused standard tools of financial mathematics, such as the Black and Scholes formula, to become suddenly obsolete. The reason has to be traced back to the overwhelming evidence of non-normality of the probability distribution of financial assets returns, which has become popular well beyond the academia and in the dealing rooms. Maybe for this reason, and these new environments, non-normality has been described using curious terms such as the "smile effect", which traders now commonly use to define strategies, and the "fat-tails" problem, which is the major topic of debate among risk managers and regulators. The result is that nowadays no one would dare to address any financial or statistical problem connected to financial markets without taking care of the issue of departures from normality.

For one-dimensional problems many effective answers have been given, both in the field of pricing and risk measurement, even though no model has emerged as the heir of the traditional standard models of the Gaussian world.

On top of that, people in the field have now begun to realize that abandoning the normality assumption for multidimensional problems was a much more involved issue. The multidimensional extension of the techniques devised at the univariate level has also grown all the more as a necessity in the market practice. On the one hand, the massive use of derivatives in asset management, in particular from hedge funds, has made the non-normality of returns an investment tool, rather than a mere statistical problem: using non-linear derivatives any hedge fund can design an appropriate probability distribution for any market. As a counterpart, it has the problem of determining the joint probability distribution of those exposures to such markets and risk factors. On the other hand, the need to reach effective diversification has led to new investment products, bound to exploit the credit risk features of the assets. It is particularly for the evaluation of these new products, such as securitized assets (asset-backed securities, such as CDO and the like) and basket credit derivatives (*n*th to default options) that the need to account for comovement among non-normally distributed variables has become an unavoidable task.

Copula functions have been first applied to the solution of these problems, and have been later applied to the multidimensional non-normality problem throughout all the fields in mathematical finance. In fact, the use of copula functions enables the task of specifying the marginal distributions to be decoupled from the dependence structure of variables. This allows us to exploit univariate techniques at the first step, and is directly linked to non-parametric dependence measures at the second step. This avoids the flaws of linear correlation that have, by now, become well known.

This book is an introduction to the use of copula functions from the viewpoint of mathematical finance applications. Our method intends to explain copulas by means of applications to major topics such as asset pricing, risk management and credit risk analysis. Our target is to enable the readers to devise their own applications, following the strategies illustrated throughout the book. In the text we concentrate all the information concerning mathematics, statistics and finance that one needs to build an application to a financial problem. Examples of applications include the pricing of multivariate derivatives and exotic contracts (basket, rainbow, barrier options and so on), as well as risk-management applications. Beyond that, references to financial topics and market data are pervasively present throughout the book, to make the mathematical and statistical concepts, and particularly the estimation issues, easier for the reader to grasp.

The audience target of our work consists of academics and practitioners who are eager to master and construct copula applications to financial problems. For this applied focus, this book is, to the best of our knowledge, the first initiative in the market. Of course, the novelty of the topic and the growing number of research papers on the subject presented at finance conferences all over the world allows us to predict that our book will not remain the only one for too long, and that, on the contrary, this topic will be one of the major issues to be studied in the mathematical finance field in the near future.

Outline of the book

Chapter 1 reviews the state of the art in asset pricing and risk management, going over the major frontier issues and providing justifications for introducing copula functions.

Chapter 2 introduces the reader to the bivariate copula case. It presents the mathematical and probabilistic background on which the applications are built and gives some first examples in finance.

Chapter 3 discusses the flaws of linear correlation and highlights how copula functions, along with non-parametric association measures, may provide a much more flexible way to represent market comovements.

Chapter 4 extends the technical tools to a multivariate setting. Readers who are not already familiar with copulas are advised to skip this chapter at first reading (or to read it at their own risk!).

Chapter 5 explains the statistical inference for copulas. It covers both methodological aspects and applications from market data, such as calibration of actual risk factors comovements and VaR measurement. Here the readers can find details on the classical estimation methods as well as on most recent approaches, such as the conditional copula.

Chapter 6 is devoted to an exhaustive account of simulation algorithms for a large class of multivariate copulas. It is enhanced by financial examples.

Chapter 7 presents credit risk applications, besides giving a brief introduction to credit derivative markets and instruments. It applies copulas to the pricing of complex credit structures such as basket default swaps and CDOs. It is shown how to calibrate the pricing model to market data. Its sensitivity with respect to the copula choice is accounted for in concrete examples.

Chapter 8 covers option pricing applications. Starting from the bivariate pricing kernel, copulas are used to evaluate counterparty risk in derivative transactions and bivariate rainbow options, such as options to exchange. We also show how the barrier option pricing problem can be cast in a bivariate setting and can be represented in terms of copulas. Finally, the estimation and simulation techniques presented in Chapters 5 and 6 are put at work to solve the evaluation problem of a multivariate basket option.

List of Common Symbols and Notations

$\square$ = end of proof
N = the set of natural numbers
I = $[0, 1]$ the unit interval of the real line
$\Re$ = $(-\infty, +\infty)$ the real line
$\Re^*$ = $[-\infty, +\infty]$ the extended real line
$\Re^{*+}$ = $[0, +\infty]$ the non-negative extended real line
$\Re^{*+} \setminus \{0\}$ = $(0, +\infty]$ the positive extended real line
$[a, b] \times [c, d]$ = Cartesian product of the intervals $[a, b]$, $[c, d]$

$$\Re^n = \underbrace{(-\infty, +\infty) \times (-\infty, +\infty) \times \cdots \times (-\infty, +\infty)}_{n \text{ times}}$$

the n-dimensional Euclidean vector space

$$\Re^{*n} = \underbrace{[-\infty, +\infty] \times [-\infty, +\infty] \times \cdots \times [-\infty, +\infty]}_{n \text{ times}}$$

the n-dimensional extended Euclidean vector space

$$I^n = \underbrace{[0, 1] \times [0, 1] \times \cdots \times [0, 1]}_{n \text{ times}} \text{ unit cube in } R^n, n \geqslant 2$$

$\mathbf{x}$ = $[x_1 x_2 \ldots x_n]^T$ n-dimensional (column) vector
$\mathbf{x}'$ = the transpose of the vector $\mathbf{x}$
C = copula function
$\mathcal{C}$ = subcopula function
ϕ = generator of an Archimedean copula
$\phi^{[-1]}$ = pseudo-inverse of ϕ
$F(x, y)$ = bivariate distribution function (cumulative probability function) of the random vector $[X, Y]$, computed at (x, y)
F_i = univariate distribution function (cumulative probability function) of the ith random variable
F_i^{-1} = generalized inverse of F_i
Dom F = the domain of the F function
Ran F = the range of the F function

f_i = density of F_i (if it exists)
C^∞ = the space of functions $f : R \to R$ with derivatives of all orders
L^2 = the space of random variables with finite first two moments
C^+ = upper Fréchet bound
C^- = lower Fréchet bound (copula for $n = 2$)
$C^\perp$ = product copula
$\overline{C}$ = survival copula
C = joint survivor or survival function
$\tilde{C}$ = co-copula
$\mathbf{1}_{\{E\}}$ = indicator function of the event E

$$\mathrm{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \quad \text{signum function}$$

a.e. = almost everywhere (other than in a set of Lebesgue measure zero)
a.s. = almost surely
c.d.f. = cumulative distribution function
d.o.f. = degrees of freedom
i.i.d. = independent identically distributed
l.h.s. = left-hand side
p.d.f. = probability distribution function
r.h.s. = right-hand side
r.v. = random variable
w.r.t. = with respect to
iff = if and only if
monotone increasing = monotone strictly increasing
monotone non-decreasing = monotone weakly increasing
$\equiv$ = equal by definition
$\sim$ = equal in distribution
$\mathrm{E}(X)$ = the expectation of the r.v. X
$\mathrm{Var}(X)$ = its variance

1 Derivatives Pricing, Hedging and Risk Management: The State of the Art

1.1 INTRODUCTION

The purpose of this chapter is to give a brief review of the basic concepts used in finance for the purpose of pricing contingent claims. As our book is focusing on the use of copula functions in financial applications, most of the content of this chapter should be considered as a prerequisite to the book. Readers who are not familiar with the concepts exposed here are referred for a detailed treatment to standard textbooks on the subject. Here our purpose is mainly to describe the basic tools that represent the state of the art of finance, as well as general problems, and to provide a brief, mainly non-technical, introduction to copula functions and the reason why they may be so useful in financial applications. It is particularly important that we address three hot issues in finance. The first is the non-normality of returns, which makes the standard Black and Scholes option pricing approach obsolete. The second is the incomplete market issue, which introduces a new dimension to the asset pricing problem – that of the choice of the right pricing kernel both in asset pricing and risk management. The third is credit risk, which has seen a huge development of products and techniques in asset pricing.

This discussion would naturally lead to a first understanding of how copula functions can be used to tackle some of these issues. Asset pricing and risk evaluation techniques rely heavily on tools borrowed from probability theory. The prices of derivative products may be written, at least in the standard complete market setting, as the discounted expected values of their future pay-offs under a specific probability measure derived from non-arbitrage arguments. The risk of a position is instead evaluated by studying the negative tail of the probability distribution of profit and loss. Since copula functions provide a useful way to represent multivariate probability distributions, it is no surprise that they may be of great assistance in financial applications. More than this, one can even wonder why it is only recently that they have been discovered and massively applied in finance. The answer has to do with the main developments of market dynamics and financial products over the last decade of the past century.

The main change that has been responsible for the discovery of copula methods in finance has to do with the standard hypothesis assumed for the stochastic dynamics of the rates of returns on financial products. Until the 1987 crash, a normal distribution for these returns was held as a reasonable guess. This concept represented a basic pillar on which most of modern finance theory has been built. In the field of pricing, this assumption corresponds to the standard Black and Scholes approach to contingent claim evaluation. In risk management, assuming normality leads to the standard parametric approach to risk measurement that has been diffused by J.P. Morgan under the trading mark of RiskMetrics since 1994, and is still in use in many financial institutions: due to the assumption of normality, the approach only relies on volatilities and correlations among the returns on the assets in the portfolio. Unfortunately, the assumption of normally distributed returns has been severely challenged by the data and the reality of the markets. On one hand, even evidence on the returns of standard financial products such as stocks and bonds can be easily proved to be at odds with this assumption. On the other hand, financial innovation has spurred the development of products that are specifically targeted to provide non-normal returns. Plain vanilla options are only the most trivial example of this trend, and the development of the structured finance business has made the presence of non-linear products, both plain vanilla and exotic, a pervasive phenomenon in bank balance sheets. This trend has even more been fueled by the pervasive growth in the market for credit derivatives and credit-linked products, whose returns are inherently non-Gaussian. Moreover, the task to exploit the benefits of diversification has caused both equity-linked and credit-linked products to be typically referred to baskets of stocks or credit exposures. As we will see throughout this book, tackling these issues of non-normality and non-linearity in products and portfolios composed by many assets would be a hopeless task without the use of copula functions.

1.2 DERIVATIVE PRICING BASICS: THE BINOMIAL MODEL

Here we give a brief description of the basic pillar behind pricing techniques, that is the use of risk-neutral probability measures to evaluate contingent claims, versus the objective measure observed from the time series of market data. We will see that the existence of such risk measures is directly linked to the basic pricing principle used in modern finance to evaluate financial products. This requirement imposes that prices must ensure that arbitrage gains, also called "free lunches", cannot be obtained by trading the securities in the market. An arbitrage deal is a trading strategy yielding positive returns at no risk. Intuitively, the idea is that if we can set up two positions or trading strategies giving identical pay-offs at some future date, they must also have the same value prior to that date, otherwise one could exploit arbitrage profits by buying the cheaper and selling the more expensive before that date, and unwinding the deal as soon as they are worth the same. Ruling out arbitrage gains then imposes a relationship among the prices of the financial assets involved in the trading strategies. These are called "fair" or "arbitrage-free" prices. It is also worth noting that these prices are not based on any assumption concerning utility maximizing behavior of the agents or equilibrium of the capital markets. The only requirement concerning utility is that traders "prefer more to less", so that they would be ready to exploit whatever arbitrage opportunity was available in the market. In this section we show what the no-arbitrage principle implies for the risk-neutral measure and the objective measure in a discrete setting, before extending it to a continuous time model.

The main results of modern asset pricing theory, as well as some of its major problems, can be presented in a very simple form in a binomial model. For the sake of simplicity, assume that the market is open on two dates, t and T, and that the information structure of the economy is such that, at the future time T, only two states of the world $\{H, L\}$ are possible. A risky asset is traded on the market at the current time t for a price equal to $S(t)$, while at time T the price is represented by a random variable taking values $\{S(H), S(L)\}$ in the two states of the world. A risk-free asset gives instead a value equal to 1 unit of currency at time T no matter which state of the world occurs: we assume that the price at time t of the risk-free asset is equal to B. Our problem is to price another risky asset taking values $\{G(H), G(L)\}$ at time T. As we said before, the price $g(t)$ must be consistent with the prices $S(t)$ and B observed on the market.

1.2.1 Replicating portfolios

In order to check for arbitrage opportunities, assume that we construct a position in Δ_g units of the risky security $S(t)$ and Π_g units of the risk-free asset in such a way that at time T

$$\Delta_g S(H) + \Pi_g = G(H)$$
$$\Delta_g S(L) + \Pi_g = G(L)$$

So, the portfolio has the same value of asset G at time T. We say that it is the "replicating portfolio" of asset G. Obviously we have

$$\Delta_g = \frac{G(H) - G(L)}{S(H) - S(L)}$$
$$\Pi_g = \frac{G(L)S(H) - G(H)S(L)}{S(H) - S(L)}$$

1.2.2 No-arbitrage and the risk-neutral probability measure

If we substitute Δ_g and Π_g in the no-arbitrage equation

$$g(t) = \Delta_g S(t) + B\Pi_g$$

we may rewrite the price, after naive algebraic manipulation, as

$$g(t) = B[QG(H) + (1 - Q)G(L)]$$

with

$$Q \equiv \frac{S(t)/B - S(L)}{S(H) - S(L)}$$

Notice that we have

$$0 < Q < 1 \Leftrightarrow S(L) < \frac{S(t)}{B} < S(H)$$

It is straightforward to check that if the inequality does not hold there are arbitrage opportunities: in fact, if, for example, $S(t)/B \leqslant S(L)$ one could exploit a free-lunch by borrowing and buying the asset. So, in the absence of arbitrage opportunities it follows that $0 < Q < 1$, and Q is a probability measure. We may then write the no-arbitrage price as

$$g(t) = BE_Q[G(T)]$$

In order to rule out arbitrage, then, the above relationship must hold for all the contingent claims and the financial products in the economy. In fact, even for the risky asset S we must have

$$S(t) = BE_Q[S(T)]$$

Notice that the probability measure Q was recovered from the no-arbitrage requirement only. To understand the nature of this measure, it is sufficient to compute the expected rate of return of the different assets under this probability. We have that

$$E_Q\left[\frac{G(T)}{g(t)} - 1\right] = E_Q\left[\frac{S(T)}{S(t)} - 1\right] = \frac{1}{B} - 1 \equiv i$$

where i is the interest rate earned on the risk-free asset for an investment horizon from t to T. So, under the measure Q all of the risky assets in the economy are expected to yield the same return as the risk-free asset. For this reason such a measure is called *risk-neutral* probability.

Alternatively, the measure can be characterized in a more technical sense in the following way. Let us assume that we measure each risky asset in the economy using the risk-free asset as numeraire. Recalling that the value of the riskless asset is B at time t and 1 at time T, we have

$$\frac{g(t)}{B(t)} = E_Q\left[\frac{G(T)}{B(T)}\right] = E_Q[G(T)]$$

A process endowed with this property (i.e. $z(t) = E_Q(z(T))$) is called a *martingale*. For this reason, the measure Q is also called an *equivalent martingale measure* (EMM).[1]

1.2.3 No-arbitrage and the objective probability measure

For comparison with the results above, it may be useful to address the question of which constraints are imposed by the no-arbitrage requirements on expected returns under the objective probability measure. The answer to this question may be found in the well-known *arbitrage pricing theory* (APT). Define the rates of return of an investment on assets S and g over the horizon from t to T as

$$i_g \equiv \frac{G(T)}{g(t)} - 1 \qquad i_S \equiv \frac{S(T)}{S(t)} - 1$$

and the rate of return on the risk-free asset as $i \equiv 1/B - 1$.

The rate of returns on the risky assets are assumed to be driven by a linear data-generating process

$$i_g = a_g + b_g f \qquad i_S = a_S + b_S f$$

where the risk factor f is taken with zero mean and unit variance with no loss of generality.

[1] The term *equivalent* is a technical requirement referring to the fact that the risk-neutral measure and the objective measure must agree on the same subset of zero measure events.

Of course this implies $a_g = E\left(i_g\right)$ and $a_S = E\left(i_S\right)$. Notice that the expectation is now taken under the original probability measure associated with the data-generating process of the returns. We define this measure P. Under the same measure, of course, b_g and b_S represent the standard deviations of the returns. Following a standard no-arbitrage argument we may build a zero volatility portfolio from the two risky assets and equate its return to that of the risk-free asset. This yields

$$\frac{a_S - i}{b_S} = \frac{a_g - i}{b_g} = \lambda$$

where λ is a parameter, which may be constant, time-varying or even stochastic, but has to be the same for all the assets. This relationship, that avoids arbitrage gains, could be rewritten as

$$E\left(i_S\right) = i + \lambda b_S \qquad E\left(i_g\right) = i + \lambda b_g$$

In words, the expected rate of return of each and every risky asset under the objective measure must be equal to the risk-free rate of return plus a risk premium. The risk premium is the product of the volatility of the risky asset times the market price of risk parameter λ. Notice that in order to prevent arbitrage gains the key requirement is that the market price of risk must be the same for all of the risky assets in the economy.

1.2.4 Discounting under different probability measures

The no-arbitrage requirement implies different restrictions under the objective probability measures. The relationship between the two measures can get involved in more complex pricing models, depending on the structure imposed on the dynamics of the market price of risk. To understand what is going on, however, it may be instructive to recover this relationship in a binomial setting. Assuming that P is the objective measure, one can easily prove that

$$Q = P - \lambda\sqrt{P\left(1 - P\right)}$$

and the risk-neutral measure Q is obtained by shifting probability from state H to state L.

To get an intuitive assessment of the relationship between the two measures, one could say that under risk-neutral valuation the probability is adjusted for risk in such a way as to guarantee that all of the assets are expected to yield the risk-free rate; on the contrary, under the objective risk-neutral measure the expected rate of return is adjusted to account for risk. In both cases, the amount of adjustment is determined by the market price of risk parameter λ.

To avoid mistakes in the evaluation of uncertain cash flows, it is essential to take into consideration the kind of probability measure under which one is working. In fact, the discount factor applied to expected cash flows must be adjusted for risk if the expectation is computed under the objective measure, while it must be the risk-free discount factor if the expectation is taken under the risk-neutral probability. Indeed, one can also check that

$$g\left(t\right) = \frac{E\left[G\left(T\right)\right]}{1 + i + \lambda b_g} = \frac{E_Q\left[G\left(T\right)\right]}{1 + i}$$

and using the wrong interest rate to discount the expected cash flow would get the wrong evaluation.

1.2.5 Multiple states of the world

Consider the case in which three scenarios are possible at time T, say $\{S(HH), S(HL), S(LL)\}$. The crucial, albeit obvious, thing to notice is that it is not possible to replicate an asset by a portfolio of only two other assets. To continue with the example above, whatever amount Δ_g of the asset S we choose, and whatever the position of Π_g in the risk-free asset, we are not able to perfectly replicate the pay-off of the contract g in all the three states of the world: whatever replicating portfolio was used would lead to some *hedging error*. Technically, we say that contract g is not *attainable* and we have an *incomplete market* problem. The discussion of this problem has been at the center of the analysis of modern finance theory for some years, and will be tackled in more detail below. Here we want to stress in which way the model above can be extended to this multiple scenario setting. There are basically two ways to do so. The first is to assume that there is a third asset, whose pay-off is independent of the first two, so that a replicating portfolio can be constructed using three assets instead of two. For an infinitely large number of scenarios, an infinitely large set of independent assets is needed to ensure perfect hedging. The second way to go is to assume that the market for the underlying opens at some intermediate time τ prior to T and the underlying on that date may take values $\{S(H), S(L)\}$. If this is the case, one could use the following strategy:

- Evaluate $g(\tau)$ under both scenarios $\{S(H), S(L)\}$, yielding $\{g(H), g(L)\}$: this will result in the computation of the risk-neutral probabilities $\{Q(H), Q(L)\}$ and the replicating portfolios consisting of $\{\Delta_g(H), \Delta_g(L)\}$ units of the underlying and $\{\Pi_g(H), \Pi_g(L)\}$ units of the risk-free asset.
- Evaluate $g(t)$ as a derivative product giving a pay-off $\{g(H), g(L)\}$ at time τ, depending on the state of the world: this will result in a risk-neutral probability Q, and a replicating portfolio with Δ_g units of the underlying and Π_g units of the risk-free asset.

The result is that the value of the product will be again set equal to its replicating portfolio

$$g(t) = \Delta_g S(t) + B\Pi_g$$

but at time τ it will be *rebalanced*, depending on the price observed for the underlying asset. We will then have

$$g(H) = \Delta_g(H) S(H) + B\Pi_g(H)$$
$$g(L) = \Delta_g(L) S(L) + B\Pi_g(L)$$

and both the position on the underlying asset and the risk-free asset will be changed following the change of the underlying price. We see that even though we have three possible scenarios, we can replicate the product g by a replicating portfolio of only two assets, thanks to the possibility of changing it at an intermediate date. We say that we follow a *dynamic* replication trading strategy, opposed to the *static* replication portfolio of the simple example above. The replication trading strategy has a peculiar feature: the value of the replicating portfolio set up at t and re-evaluated using the prices of time τ is, in any circumstances, equal to that of the new replicating portfolio which will be set up at time τ. We have in fact that

$$\Delta_g S(H) + \Pi_g = g(H) = \Delta_g(H) S(H) + B\Pi_g(H)$$
$$\Delta_g S(L) + \Pi_g = g(L) = \Delta_g(L) S(L) + B\Pi_g(L)$$

This means that once the replicating portfolio is set up at time t, no further expense or withdrawal will be required to rebalance it, and the sums to be paid to buy more of an asset will be exactly those made available by the selling of the other. For this reason the replicating portfolio is called *self-financing*.

1.3 THE BLACK–SCHOLES MODEL

Let us think of a multiperiod binomial model, with a time difference between one date and the following equal to h. The gain or loss on an investment on asset S over every period will be given by

$$S(t+h) - S(t) = i_S(t) S(t)$$

Now assume that the rates of return are serially uncorrelated and normally distributed as

$$i_S(t) = \mu^* + \sigma^* \varepsilon(t)$$

with μ^* and σ^* constant parameters and $\varepsilon(t) \sim N(0, 1)$, i.e. a series of uncorrelated standard normal variables. Substituting in the dynamics of S we get

$$S(t+h) - S(t) = \mu^* S(t) + \sigma^* S(t) \varepsilon(t)$$

Taking the limit for h that tends to zero, we may write the stochastic dynamics of S in continuous time as

$$\mathrm{d}S(t) = \mu S(t)\ \mathrm{d}t + \sigma S(t)\ \mathrm{d}z(t)$$

The stochastic process is called *geometric brownian motion*, and it is a specific case of a *diffusive* process. $z(t)$ is a Wiener process, defined by $\mathrm{d}z(t) \sim N(0, \mathrm{d}t)$ and the terms $\mu S(t)$ and $\sigma S(t)$ are known as the *drift* and *diffusion* of the process. Intuitively, they represent the expected value and the volatility (standard deviation) of instantaneous changes of $S(t)$.

Technically, a stochastic process in continuous time $S(t), t \leqslant T$, is defined with respect to a filtered probability space $\{\Omega, \Im_t, P\}$, where $\Im_t = \sigma(S(u), u \leqslant t)$ is the smallest σ-field containing sets of the form $\{a \leqslant S(u) \leqslant b\}$, $0 \leqslant u \leqslant t$: more intuitively, $\Im_t$ represents the amount of information available at time t.

The increasing σ-fields $\{\Im_t\}$ form a so-called filtration F:

$$\Im_0 \subset \Im_1 \subset \cdots \subset \Im_T$$

Not only is the filtration increasing, but $\Im_0$ also contains all the events with zero measure; and these are typically referred to as "the usual assumptions". The increasing property corresponds to the fact that, at least in financial applications, the amount of information is continuously increasing as time elapses.

A variable observed at time t is said to be measurable with respect to $\Im_t$ if the set of events, such that the random variable belongs to a Borel set on the line, is contained in $\Im_t$, for every Borel set: in other words, $\Im_t$ contains all the amount of information needed to recover the value of the variable at time t. If a process $S(t)$ is measurable with respect to $\Im_t$ for all $t \geqslant 0$, it is said to be adapted with respect to $\Im_t$. At time t, the values of a variable at any time $\tau > t$ can instead be characterized only in terms of the last object, i.e. the probability measure P, conditional on the information set $\Im_t$.

In this setting, a diffusive process is defined, assuming that the limit of the first and second moments of $S(t+h) - S(t)$ exist and are finite, and that finite jumps have zero probability in the limit. Technically,

$$\lim_{h \to 0} \frac{1}{h} E\left[S(t+h) - S(t) \mid S(t) = S\right] = \mu(S, t)$$

$$\lim_{h \to 0} \frac{1}{h} E\left[\left[S(t+h) - S(t)\right]^2 \mid S(t) = S\right] = \sigma^2(S, t)$$

and

$$\lim_{h \to 0} \frac{1}{h} \Pr\left(\left|S(t+h) - S(t)\right| > \varepsilon \mid S(t) = S\right) = 0$$

Of course the moments in the equations above are tacitly assumed to exist. For further and detailed discussion of the matter, the reader is referred to standard textbooks on stochastic processes (see, for example, Karlin & Taylor, 1981).

1.3.1 Ito's lemma

A paramount result that is used again and again in financial applications is Ito's lemma. Say $y(t)$ is a diffusive stochastic process

$$\mathrm{d}y(t) = \mu_y \, \mathrm{d}t + \sigma_y \, \mathrm{d}z(t)$$

and $f(y, t)$ is a function differentiable twice in the first argument and once in the second. Then f also follows a diffusive process

$$\mathrm{d}f(y, t) = \mu_f \, \mathrm{d}t + \sigma_f \, \mathrm{d}z(t)$$

with drift and diffusion terms given by

$$\mu_f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\mu_y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\sigma_y^2$$

$$\sigma_f = \frac{\partial f}{\partial y}\sigma_y$$

Example 1.1 Notice that, given

$$dS(t) = \mu S(t)\ dt + \sigma S(t)\ dz(t)$$

we can set $f(S,t) = \ln S(t)$ to obtain

$$d\ln S(t) = (\mu - \tfrac{1}{2}\sigma^2)\, dt + \sigma dz(t)$$

If μ and σ are constant parameters, it is easy to obtain

$$\ln S(\tau) \mid \Im_t \sim N(\ln S(t) + (\mu - \tfrac{1}{2}\sigma^2)(\tau - t), \sigma^2(\tau - t))$$

where $N(m,s)$ is the normal distribution with mean m and variance s. Then, $\Pr(S(\tau) \mid \Im_t)$ is described by the lognormal distribution.

It is worth stressing that the *geometric brownian motion* assumption used in the Black–Scholes model implies that the log-returns on the asset S are normally distributed, and this is the same as saying that their volatility is assumed to be constant.

1.3.2 Girsanov theorem

A second technique that is mandatory to know for the application of diffusive processes to financial problems is the result known as the Girsanov theorem (or Cameron–Martin–Girsanov theorem). The main idea is that given a Wiener process $z(t)$ defined under the filtration $\{\Omega, \Im_t, P\}$ we may construct another process $\widetilde{z}(t)$ which is a Wiener process under another probability space $\{\Omega, \Im_t, Q\}$. Of course, the latter process will have a drift under the original measure P. Under such measure it will be in fact

$$d\widetilde{z}(t) = dz(t) + \gamma\ dt$$

for γ deterministic or stochastic and satisfying regularity conditions. In plain words, changing the probability measure is the same as changing the drift of the process.

The application of this principle to our problem is straightforward. Assume there is an opportunity to invest in a *money market mutual fund* yielding a constant instantaneous risk-free yield equal to r. In other words, let us assume that the dynamics of the investment in the risk-free asset is

$$dB(t) = rB(t)$$

where the constant r is also called the interest rate intensity ($r \equiv \ln(1 + i)$). We saw before that under the objective measure P the no-arbitrage requirement implies

$$E\left[\frac{dS(t)}{S(t)}\right] = \mu\, dt = (r + \lambda\sigma)\ dt$$

where λ is the market price of risk. Substituting in the process followed by $S(t)$ we have

$$\begin{aligned} \mathrm{d}S(t) &= (r + \lambda\sigma) S(t)\ \mathrm{d}t + \sigma S(t)\ \mathrm{d}z(t) \\ &= S(t)(r\,\mathrm{d}t + \sigma(\mathrm{d}z(t) + \lambda\,\mathrm{d}t)) \\ &= S(t)(r\,\mathrm{d}t + \sigma\,\mathrm{d}\widetilde{z}(t)) \end{aligned}$$

where $\mathrm{d}\widetilde{z}(t) = \mathrm{d}z(t) + \lambda\,\mathrm{d}t$ is a Wiener process under some new measure Q. Under such a measure, the dynamics of the underlying is then

$$\mathrm{d}S(t) = rS(t)\ \mathrm{d}t + \sigma S(t)\ \mathrm{d}\widetilde{z}(t)$$

meaning that the instantaneous expected rate of the return on asset $S(t)$ is equal to the instantaneous yield on the risk-free asset

$$E_Q\left[\frac{\mathrm{d}S(t)}{S(t)}\right] = r\,\mathrm{d}t$$

i.e. that Q is the so-called risk-neutral measure. It is easy to check that the same holds for any derivative written on $S(t)$. Define $g(S,t)$ the price of a derivative contract giving pay-off $G(S(T),T)$. Indeed, using Ito's lemma we have

$$\mathrm{d}g(t) = \mu_g g(t)\ \mathrm{d}t + \sigma_g g(t)\ \mathrm{d}z(t)$$

with

$$\begin{aligned} \mu_g g &= \frac{\partial g}{\partial t} + \frac{\partial g}{\partial S}(r + \lambda\sigma) S(t) + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}\sigma^2(t) S^2 \\ \sigma_g g &= \frac{\partial g}{\partial S}\sigma \end{aligned}$$

Notice that under the original measure we then have

$$\mathrm{d}g(t) = \left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial S}\mu S(t) + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}\sigma^2(t) S^2\right]\mathrm{d}t + \frac{\partial g}{\partial S}\sigma\,\mathrm{d}z(t)$$

However, the no-arbitrage requirement implies

$$\mu_g g = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial S}(r + \lambda\sigma) S(t) + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}\sigma^2(t) S^2 = rg + \lambda\frac{\partial g}{\partial S}\sigma$$

so it follows that

$$\frac{\partial g}{\partial t} + \frac{\partial g}{\partial S} rS(t) + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}\sigma^2(t) S^2 = rg$$

This is the fundamental partial differential equation (PDE) of the Black–Scholes model. Notice that by substituting this result into the risk-neutral dynamics of g under measure Q we get

$$\mathrm{d}g(t) = rg(t)\ \mathrm{d}t + \frac{\partial g}{\partial S}\sigma\,\mathrm{d}\widetilde{z}(t)$$

and the product g is expected to yield the instantaneous risk-free rate. We reach the conclusion that under the risk-neutral measure Q

$$E_Q\left[\frac{\mathrm{d}S(t)}{S(t)}\right] = E_Q\left[\frac{\mathrm{d}g(t)}{g(t)}\right] = r\,\mathrm{d}t$$

that is, all the risky assets are assumed to yield the instantaneous risk-free rate.

1.3.3 The martingale property

The price of any contingent claim g can be recovered solving the fundamental PDE. An alternative way is to exploit the martingale property embedded in the measure Q. Define Z as the value of a product expressed using the riskless money market account as the numeraire, i.e. $Z(t) \equiv g(t)/B(t)$. Given the dynamics of the risky asset under the risk-neutral measure Q we have that

$$\begin{aligned}\mathrm{d}S(t) &= rS(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}\widetilde{z}(t)\\ \mathrm{d}B(t) &= rB(t)\,\mathrm{d}t\end{aligned}$$

and it is easy to check that

$$\mathrm{d}Z(t) = \sigma Z(t)\,\mathrm{d}\widetilde{z}(t)$$

The process $Z(t)$ then follows a martingale, so that $E_Q(Z(T)) = Z(t)$. This directly provides us with a pricing formula. In fact we have

$$Z(t) = \frac{g(S,t)}{B(t)} = E_Q(Z(T)) = E_Q\left(\frac{G(S,T)}{B(T)}\right)$$

Considering that $B(T)$ is a deterministic function, we have

$$g(S,t) = \frac{B(t)}{B(T)}E_Q(G(S,T)) = \exp(-r(T-t))\,E_Q(G(S,T))$$

The price of a contingent claim is obtained by taking the relevant expectation under the risk-neutral measure and discounting it back to the current time t. Under the assumption of log-normal distribution of the future price of the underlying asset S, we may recover for instance the basic Black–Scholes formula for a plain vanilla call option

$$\begin{aligned}\mathrm{CALL}(S,t;K,T) &= S(t)\,\Phi(d_1) - \exp[-r(T-t)]\,K\Phi(d_2)\\ d_1 &= \frac{\ln(S(t)/K) + \left(r+\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\\ d_2 &= d_1 - \sigma\sqrt{T-t}\end{aligned}$$

where $\Phi(x)$ is the standard normal distribution function evaluated at x

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{x}\exp\left[-\frac{u^2}{2}\right]\mathrm{d}u$$

The formula for the put option is, instead,

$$\mathrm{PUT}(S, t; K, T) = -S(t)\,\Phi(-d_1) + \exp[-r(T-t)]\,K\Phi(-d_2)$$

Notice that a long position in a call option corresponds to a long position in the underlying and a debt position, while a long position in a put option corresponds to a short position in the underlying and an investment in the risk-free asset. As $S(t)$ tends to infinity, the value of a call tends to that of a long position in a forward and the value of the put tends to zero; as $S(t)$ tends to zero, the value of the put tends to the value of a short position in a forward and the price of the call option tends to zero.

The sensitivity of the option price with respect to the underlying is called *delta* (Δ) and is equal to $\Phi(d_1)$ for the call option and $\Phi(d_1) - 1$ for the put. The sensitivity of the delta with respect to the underlying is called *gamma* (Γ), and that of the option price with respect to time is called *theta* (Θ). These derivatives, called the *greek letters*, can be used to approximate, in general, the value of any derivative contract by a Taylor expansion as

$$\begin{aligned} g(S(t+h), t+h) \simeq\ & g(S(t), t) + \Delta_g (S(t+h) - S(t)) \\ & + \tfrac{1}{2}\Gamma_g (S(t+h) - S(t))^2 + \Theta_g h \end{aligned}$$

Notice that the *greek letters* are linked one to the others by the *fundamental PDE* ruling out arbitrage. Indeed, this condition can be rewritten as

$$\Theta_g + \Delta_g r S(t) + \tfrac{1}{2}\Gamma_g \sigma^2(t)\,S^2 - rg = 0$$

1.3.4 Digital options

A way to understand the probabilistic meaning of the Black–Scholes formula is to compute the price of digital options. Digital options pay a fixed sum or a unit of the underlying if the underlying asset is above some strike level at the exercise date. Digital options, which pay a fixed sum, are called *cash-or-nothing* (CoN) options, while those paying the asset are called *asset-or-nothing* (AoN) options. Under the log-normal assumption of the conditional distribution of the underlying held under the Black–Scholes model, we easily obtain

$$\mathrm{CoN}(S, t; K, T) = \exp[-r(T-t)]\,\Phi(d_2)$$

The asset-or-nothing price can be recovered by arbitrage observing that at time T

$$\mathrm{CALL}(S, T; K, T) + K\ \mathrm{CoN}(S, T; K, T) = \mathbf{1}_{\{S(T)>K\}} S(T) = \mathrm{AoN}(S, T; K, T)$$

where $\mathbf{1}_{\{S(T)>K\}}$ is the indicator function assigning 1 to the case $S(T) > K$. So, to avoid arbitrage we must have

$$\mathrm{AoN}(S, t; K, T) = S(t)\,\Phi(d_1)$$

Beyond the formulas deriving from the Black–Scholes model, it is important to stress that this result – that a call option is the sum of a long position in a digital *asset-or-nothing* option and a short position in K *cash-or-nothing* options – remains true for all the option