GREEN'S FUNCTIONS AND BOUNDARY VALUE PROBLEMS

Third Edition

Ivar Stakgold
Department of Mathematical Sciences
University of Delaware
Newark, Delaware
and
Department of Mathematics
University of California, San Diego
La Jolla, California

Michael Holst
Departments of Mathematics and Physics
University of California, San Diego
La Jolla, California

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To Lainie and Alissa
I.S.

For Mai,
Mason, and Makenna
M.H.

CONTENTS

PREFACE TO THE THIRD EDITION

Why a third edition? The principal reason is to include more material from analysis, approximation theory, partial differential equations, and numerical analysis as needed for understanding modern computational methods that play such a vital role in the solution of boundary value problems. As I am not an expert in computational mathematics, it was essential to find a highly qualified coauthor. When I moved to San Diego in early 2008, I was offered an office at the University of California, San Diego (UCSD), which, luckily, was next to the office of Michael Holst. Here was the perfect coauthor, and it was my good fortune that he agreed to collaborate on the new edition! The most substantial change for the new third edition is a fairly extensive new chapter (Chapter 10), which covers the new material listed above. The sections of the new chapter are:

10.1 Nonlinear Analysis Tools for Banach Spaces
10.2 Best and Near-Best Approximation in Banach Spaces
10.3 Overview of Sobolev and Besov Spaces
10.4 Applications to Nonlinear Elliptic Equations
10.5 Finite Element and Related Discretization Methods
10.6 Iterative Methods for Discretized Linear Equations
10.7 Methods for Nonlinear Equations

To support the inclusion of this new chapter, and to help connect the presentation of the analysis material to standard references, we have added an additional final section to four of the chapters that appeared in the second edition of the book. These completely new sections for the third edition are:

2.6 Weak Derivatives and Sobolev Spaces
4.8 The Hahn-Banach Theorem and Reflexive Banach Spaces
5.9 The Banach-Schauder and Banach-Steinhaus Theorems
8.5 The Lax-Milgram Theorem

We have also added a final subsection on Lebesgue integration at the end of Chapter 0, listing a few of the main concepts and results on Lebesgue integration in $\mathbb{R}^n$. In addition, the titles of a few sections from the second edition have been changed slightly to more clearly bring out the material already contained in the sections, again to help connect the material in the sections to presentations of these topics appearing in standard references. The new section titles are:

4.4 Contractions and the Banach Fixed-Point Theorem
4.5 Hilbert Spaces and the Projection Theorem
4.7 Linear Functionals and the Riesz Representation Theorem
9.1 Introduction and Basic Fixed-Point Techniques

The bibliographies at the end of the chapters in the second edition have also been updated for the third edition, but we have likely left out many outstanding new books and papers that should have been included, and we apologize in advance for all such omissions.

IVAR STAKGOLD

La Jolla, California
November 2010

When Ivar asked me to consider joining him on a third edition of his well-known and popular book, *Green's Functions and Boundary-Value Problems*, I was a bit intimidated; not only had it been a standard reference for me for many years, but it is also used as the main text for the first-year graduate applied analysis sequence in a number of applied mathematics doctoral programs around the country. However, I soon realized it was an opportunity for me to add the material that I feel is often missing from first-year graduate courses in modern applied mathematics, namely, additional foundational material from analysis and approximation theory to support the design, development, and analysis of effective and reliable computational methods for partial differential equations. Although there are some wonderful books covering applied mathematics (such as Ivar's) and some equally strong books on numerical analysis, the bridge between them (built with linear functional analysis, approximation theory, and nonlinear analysis) is often mostly missing in these same books. There are a number of books devoted entirely to building this bridge; however, our goal for the third edition was to add just the right subset of this material so that a course based on this single book, combined with a course based on a strong graduate numerical analysis book, would provide a solid foundation for applied mathematics students in our mathematics doctoral program and in our interdisciplinary Computational Science, Mathematics, and Engineering Graduate Program at UCSD.

After spending substantial time with the second edition of the book over the last year, my appreciation for Ivar's original book has only grown. The second edition is a unique combination of modeling, real analysis, linear functional analysis and operator theory, partial differential equations, integral equations, nonlinear functional analysis, and applications. The book manages to present the topics in a friendly, informal way, and at the same time gives the real theorems, with real proofs, when they are called for. The changes that I recommended we make to the second edition (as Ivar outlined above) were mostly to draw out the existing structure of the book, and also to add in a few results from linear functional analysis to complete the material where it was needed to support the new final chapter of the book. Since those of us who have worked closely with the second edition are very familiar with exactly where to find particular topics, one of my goals for the third edition was to preserve as much of the second edition as possible, right down to theorem, equation, and exercise numbers within the sections of each chapter. This is why I have tried to fit all of the new material into new sections appearing at the end of existing chapters, and into the new final chapter appearing at the end of the book. The index to the second edition also provided finer-grained access to the book than did the table of contents; I always found this a very valuable part of the second edition, so I attempted to preserve the entire second edition index as a subset of the third edition index. My hope is that as a consequence of our efforts, the third edition of the book will be viewed as a useful superset of the second edition, with new material on approximation theory and methods, together with some additional supporting analysis material.

The third edition contains approximately 30% new material not found in the second edition. The longest chapter is now the new final chapter (Chapter 10) on approximation theory and methods. We considered splitting it into two chapters, but it seems to hold together well as a single chapter. In addition to the new material in Chapter 10, we have added material to Chapters 2, 4, 5, and 8 as Ivar outlined above. Chapter 2 now contains an early introduction to Sobolev spaces based on weak differentiation, and Chapter 8 now includes the Lax-Milgram Theorem and some related tools. Chapters 4 and 5 now provide a gentle introduction to many of the central concepts and theorems in linear functional analysis and operator theory, as needed by most first-year graduate students working in applied analysis and applied partial differential equations. Some of the new material in Chapter 10 is a bit more advanced than some of the other sections of the book; however, this material builds only on (old and new) material found in Chapters 2, 4, 5, and 8, with the support of a few new paragraphs added to the end of Chapter 0 (on Lebesgue integration). The only exception is perhaps the last example in Section 10.4, chosen from mathematical physics to illustrate the combined use of several tools from nonlinear analysis and approximation theory; it requires a bit of familiarity with the notation used in differential geometry.

A brief word about the numbering system used in third edition is in order, since we are departing substantially from the convention used in the previous two editions (as outlined in the preface to the first edition). The book is now divided into eleven chapters (beginning with Chapter 0), with the inclusion of a new final chapter (Chapter 10). Each chapter is divided into numbered sections, and equations are numbered by chapter, section, and equation within each section. For example, a reference to equation (8.5.2) is to the second numbered equation appearing in Section 5 of Chapter 8. Similarly, all definitions, theorems, corollaries, lemmas, and the like, as well as exercises, are numbered using the same convention. This convention makes the third edition easier to navigate than the first two editions, with a simple glance at a typical page revealing precisely the section and chapter in which the page appears. However, it also preserves the numbering of items from the second edition; for example, equation (5.2) of Chapter 8 in the second edition is numbered as (8.5.2) in the third edition. Note that some objects remain unnumbered if they were unnumbered in the first two editions (for example, a theorem that is not referred to later in the book). To simplify the presentation without losing the advantages of this numbering convention, we make three consistent exceptions: Figures are numbered only by chapter and figure within the chapter; examples and remarks are numbered only within the section; and the Bibliography continues to consist of a chapter-specific list of references immediately following the chapter, ordered alphabetically. Citations to references are now also numbered within the referring text; for example, a citation to reference [3] occurring within a chapter refers to the third reference appearing in the list of references at the end of the chapter.

I would like to thank my family (Mai, Mason, and Makenna) for their patience during the last few months as I focused on the book. I would also like to thank the faculty in the Center for Computational Mathematics at UCSD, and in particular Randy Bank, Philip Gill, and Jim Bunch, for the support and encouragement they have given me over the last ten years. I am also indebted to the Center for Theoretical Biological Physics, the National Biomedical Computation Resource, the National Science Foundation, the National Institutes of Health, the Department of Energy, and the Department of Defense for their ongoing support of my research. I must express my appreciation for the interactions I have had with Randy Bank, Long Chen, Don Estep, Gabriel Nagy, Gantumur Tsogtgerel, and Jinchao Xu, as each played a role in the development of my understanding of much of the material I wrote for the book. I would also like to thank Ari Stern, Ryan Szypowski, Yunrong Zhu, and Jonny Serencsa for reading the new material carefully and catching mistakes. Finally, I am grateful to my friend and mentor Herb Keller, who greatly influenced my work over the last fifteen years, and this is reflected in the topics that I chose to include in the book. Herb was my postdoctoral advisor at Caltech from 1993 to 1997, and after retiring from Caltech around 2000, he moved to San Diego to join our research group at UCSD. We thoroughly enjoyed the years Herb was with us at the Center (attending the weekly seminars in his biking outfit, after biking down the coast from Leucadia). Unfortunately, Herb passed away just before Ivar joined our research group in 2008; otherwise, we might have had three authors on this new edition of the book.

MICHAEL HOLST

La Jolla, California
November 2010

PREFACE TO THE SECOND EDITION

The field of applied mathematics has evolved considerably in the nearly twenty years since this book's first edition. To incorporate some of these changes, the publishers and I decided to undertake a second edition. Although many fine books on related subjects have appeared in recent years, we believe that the favorable reception accorded the first edition— as measured by adoptions and reviews— justifies the effort involved in a new edition.

My basic purpose is still to prepare the reader to use differential and integral equations to attack significant problems in the physical sciences, engineering, and applied mathematics. Throughout, I try to maintain a balance between sound mathematics and meaningful applications. The principal changes in the second edition are in the areas of modeling, Fourier analysis, fixed-point theorems, inverse problems, asymptotics, and nonlinear methods. The exercises, quite a few of which are new, are rarely routine and occasionally can even be considered extensions of the text. Let me now turn to a chapter-by-chapter list of the major changes.

Chapter 0 [Preliminaries] has assumed a more important role. It is now the starting point for a discussion of the relation among the four alternative formulations of physical problems: integral balance law, boundary value problem, weak form (also known as the principle of virtual work), and variational principle. I have also added new modeling examples in climatology, population dynamics, and fluid flow.

Chapter 1 [Green's functions: intuitive ideas] contains some revisions in exposition, particularly in regard to continuous dependence on the data.

In Chapter 2 [The theory of distributions], the treatment of Fourier analysis has been extended to include Discrete and Fast transforms, band-limited functions, and the sampling theorem using the sinc function.

Chapter 3 [One-dimensional boundary value problems] now includes a more thorough treatment of least-squares solutions and pseudo-inverses. The ideas are introduced through a discussion of unbalanced systems (underdetermined or overdetermined).

Chapter 4 has been retitled "Hilbert and Banach spaces," reflecting an increased emphasis on normed spaces at the expense of general metric spaces. The material on contractions is rewritten from this point of view with some new examples.

Chapter 5 [Operator theory] is virtually unchanged.

Chapter 6 [Integral equations] now includes a treatment of Tychonov regularization for integral equations of the first kind, an important aspect of the study of ill-posed inverse problems. Some new examples of integral equations are presented and there is a short discussion of singular-value decomposition. Part of the material on integrodifferential equations has been deleted.

Chapter 7 [Spectral theory of second-order differential operators] is basically unchanged.

In Chapter 8 [Partial differential equations], I have added a more comprehensive treatment of the spectral properties of the Laplacian, including a discussion of recent results on isospectral problems. The asymptotic behavior of the heat equation is examined. A brief introduction to the finite element method is incorporated in a slightly revised section on variational principles.

Chapter 9 [Nonlinear problems] contains a new subsection comparing the three major fixed-point theorems: the Schauder theorem, the contraction theorem of Chapter 4, and the theorem for order-preserving maps, which is used extensively in the remainder of Chapter 9. I have also included a study of the phenomena of finite-time extinction and blow-up for nonlinear reaction-diffusion problems.

There now remains the pleasant task of acknowledging my debt to the students and teachers who commented on the first edition and diplomatically muted their criticism! I am particularly grateful to my friends Stuart Antman of the University of Maryland, W. Edward Olmstead of Northwestern University, and David Colton and M. Zuhair Nashed of the University of Delaware, who generously provided me with ideas and encouragement. The new material in Chapter 9 owes much to my overseas collaborators, Catherine Bandle (University of Basel) and J. Ildefonso Diaz (Universidad Complutense, Madrid). The TEX preparation of the manuscript was in the highly skilled hands of Linda Kelly and Pamela Haverland.

IVAR STAKGOLD

Newark, Delaware
September 1997

PREFACE TO THE FIRST EDITION

As a result of graduate-level adoptions of my earlier two-volume book, *Boundary Value Problems of Mathematical Physics*, I received many constructive suggestions from users. One frequent recommendation was to consolidate and reorganize the topics into a single volume that could be covered in a one-year course. Another was to place additional emphasis on modeling and to choose examples from a wider variety of physical applications, particularly some emerging ones. In the meantime my own research interests had turned to nonlinear problems, so that, inescapably, some of these would also have to be included in any revision. The only way to incorporate these changes, as well as others, was to write a new book, whose main thrust, however, remains the systematic analysis of boundary value problems. Of course some topics had to be dropped and others curtailed, but I can only hope that your favorite ones are not among them.

My book is aimed at graduate students in the physical sciences, engineering, and applied mathematics who have taken the typical "methods" course that includes vector analysis, elementary complex variables, and an introduction to Fourier series and boundary value problems. Why go beyond this? A glance at modern publications in science and engineering provides the answer. To the lament of some and the delight of others, much of this literature is deeply mathematical. I am referring not only to areas such as mechanics and electromagnetic theory that are traditionally mathematical but also to relative newcomers to mathematization, such as chemical engineering, materials science, soil mechanics, environmental engineering, biomedical engineering, and nuclear engineering. These fields give rise to challenging mathematical problems whose flavor can be sensed from the following short list of examples; integrodifferential equations of neutron transport theory, combined diffusion and reaction in chemical and environmental engineering, phase transitions in metallurgy, free boundary problems for dams in soil mechanics, propagation of impulses along nerves in biology. It would be irresponsible and foolish to claim that readers of my book will become instantaneous experts in these fields, but they will be prepared to tackle many of the mathematical aspects of the relevant literature.

Next, let me say a few words about the numbering system. The book is divided into ten chapters, and each chapter is divided into sections. Equations do *not* carry a chapter designation. A reference to, say, equation 4.32 is to the thirty-second numbered equation in Section 4 of the chapter you happen to be reading. The same system is used for figures and exercises, the latter being found at the end of sections. The exercises, by the way, are rarely routine and, on occasion, contain substantial extensions of the main text. Examples do not carry any section designation and are numbered consecutively within a section, even though there may be separate clusters of examples within the same section. Some theorems have numbers and others do not; those that do are numbered in a sequence within a section— Theorem 1, Theorem 2, and so on.

A brief description of the book's contents follows. No attempt is made to mention all topics covered; only the general thread of the development is indicated.

Chapter 0 presents background material that consists principally of careful derivations of several of the equations of mathematical physics. Among them are the equations of heat conduction, of neutron transport, and of vibrations of rods. In the last-named derivation an effort is made to show how the usual linear equations for beams and strings can be regarded as first approximations to nonlinear problems. There are also two short sections on modes of convergence and on Lebesgue integration.

Many of the principal ideas related to boundary value problems are introduced on an intuitive level in Chapter 1. A boundary value problem (BVP, for short) consists of a differential equation $Lu = f$ with boundary conditions of the form $Bu = h$. The pair (f, h) is known collectively as the data for the problem, and u is the response to be determined. Green's function is the response when f represents a concentrated unit source and $h = 0$. In terms of Green's function, the BVP with arbitrary data can be solved in a form that shows clearly the dependence of the solution on the data. Various examples are given, including some multidimensional ones, some involving interface conditions, and some initial value problems. The useful notion of a well-posed problem is discussed, and a first look is taken at maximum principles for differential equations.

Chapter 2 deals with the theory of distributions, which provides a rigorous mathematical framework for singular sources such as the point charges, dipoles, line charges, and surface layers of electrostatics. The notion of response to such sources is made precise by defining the distributional solution of a differential equation. The related concepts of weak solution, adjoint, and fundamental solution are also introduced. Fourier series and Fourier transforms are presented in both classical and distributional settings.

Chapter 3 returns to a more detailed study of one-dimensional linear boundary value problems. To an equation of order p there are usually associated p independent boundary conditions involving derivatives of order less than p at the endpoints a and b of a bounded interval. If the corresponding BVP with 0 data has only the trivial solution, then the BVP with arbitrary data has one and only one solution which can be expressed in terms of Green's function. If, however, the BVP with 0 data has a nontrivial solution, certain solvability conditions must be satisfied for the BVP with arbitrary data to have a solution. These statements are formulated precisely in an alternative theorem, which recurs throughout the book in various forms. When the BVP with 0 data has a nontrivial solution, Green's function cannot be constructed in the ordinary way, but some of its properties can be salvaged by using a modified Green's function, defined in Section 5.

Chapter 4 begins the study of Hilbert spaces. A Hilbert space is the proper setting for many of the linear problems of applied analysis. Though its elements may be functions or abstract "vectors," a Hilbert space enjoys all the algebraic and geometric properties of ordinary Euclidean space. A Hilbert space is a linear space equipped with an inner product that induces a natural notion of distance between elements, thereby converting it into a metric space which is required to be complete. Some of the important geometric properties of Hilbert spaces are developed, including the projection theorem and the existence of orthonormal bases for separable spaces. Metric spaces can be useful quite apart from any linear structure. A contraction is a transformation on a metric space that uniformly reduces distances between pairs of points. A contraction on a complete metric space has a unique fixed point that can be calculated by iteration from any initial approximation. Examples demonstrate how to use these ideas to prove uniqueness and constructive existence for certain classes of nonlinear differential equations and integral equations.

Chapter 5 examines the theory of linear operators on a separable Hilbert space, particularly integral and differential operators, the latter being unbounded operators. The principal problem of operator theory is the solution of the equation $Au = f$, where A is a linear operator and f an element of the space. A thorough discussion of this problem leads again to adjoint operators, solvability conditions, and alternative theorems. Additional insight is obtained by considering the inversion of the equation $Au - \lambda u = f$, which leads to the idea of the spectrum, a generalization of the more familiar concept of eigenvalue. For compact operators (which include most integral operators) the inversion problem is essentially solved by the Riesz-Schauder theory of Section 7. Section 8 relates the spectrum of symmetric operators to extremal principles for the Rayleigh quotient. Throughout, the theory is illustrated by specific examples.

In Chapter 6 the general ideas of operator theory are specialized to integral equations. Integral equations are particularly important as alternative formulations of boundary value problems. Special emphasis is given to Fredholm equations with symmetric Hilbert-Schmidt kernels. For the corresponding class of operators, the nonzero eigenvalues and associated eigenfunctions can be characterized through successive extremal principles, and it is then possible to give a complete treatment of the inhomogeneous equation. The last section discusses the Ritz procedure for estimating eigenvalues, as well as other approximation methods for eigenvalues and eigenfunctions. There is also a brief introduction to integrodifferential operators in Exercises 5.3 to 5.8.

Chapter 7 extends the Sturm-Liouville theory of second-order ordinary differential equations to the case of singular endpoints. It is shown, beginning with the regular case, how the necessarily discrete spectrum can be constructed from Green's function. A formal extension of this relationship to the singular case makes it possible to calculate the spectrum, which may now be partly continuous. The transition from regular to singular is analyzed rigorously for equations of the first order, but the Weyl classification for second-order equations is given without proof. The eigenfunction expansion in the singular case can lead to integral transforms such as Fourier, Hankel, Mellin, and Weber. It is shown how to use these transforms and their inversion formulas to solve partial differential equations in particular geometries by separation of variables.

Although partial differential equations have appeared frequently as examples in earlier chapters, they are treated more systematically in Chapter 8. Examination of the Cauchy problem— the appropriate generalization of the initial value problem to higher dimensions— gives rise to a natural classification of partial differential equations into hyperbolic, parabolic, and elliptic types. The theory of characteristics for hyperbolic equations is introduced and applied to simple linear and nonlinear examples. In the second and third sections various methods (Green's functions, Laplace transforms, images, etc.) are used to solve BVPs for the wave equation, the heat equation, and Laplace's equation. The simple and double layers of potential theory make it possible to reduce the Dirichlet problem to an integral equation on the boundary of the domain, thereby providing a rather weak existence proof. In Section 4 a stronger existence proof is given, using variational principles. Two-sided bounds for some functionals of physical interest, such as capacity and torsional rigidity, are obtained by introducing complementary principles. Another application involving level-line analysis is also given, and there is a very brief treatment of unilateral constraints and variational inequalities.

Finally, in Chapter 9, a number of methods applicable to nonlinear problems are developed. Section 1 points out some of the features that distinguish nonlinear problems from linear ones and illustrates these differences through some simple examples. In Section 2 the principal qualitative results of branching theory (also known as bifurcation theory) are presented. The phenomenon of bifurcation is understood most easily in terms of the buckling of a rod under compressive thrust. As the thrust is increased beyond a certain critical value, the state of simple compression gives way to the buckled state with its appreciable transverse deflection. Section 3 shows how a variety of linear problems can be handled by perturbation theory (inhomogeneous problems, eigenvalue problems, change in boundary conditions, domain perturbations). These techniques, as well as monotone methods, are then adapted to the solution of nonlinear BVPs. The concluding section discusses the possible loss of stability of the basic steady state when an underlying parameter is allowed to vary.

I have already acknowledged my debt to the students and teachers who were kind enough to comment on my earlier book. There are, however, two colleagues to whom I am particularly grateful: Stuart Antman, who generously contributed the ideas underlying the derivation of the equations for rods in Chapter 0, and W. Edward Olmstead, who suggested some of the examples on contractions in Chapter 4 and on branching in Chapter 9.

IVAR STAKGOLD

Newark, Delaware
September 1979

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CHAPTER 0

PRELIMINARIES

As its name and number indicate, this chapter contains background material having no precise place in the subsequent, systematic, mathematical development. Readers already familiar with some of the topics in the present chapter may nevertheless profit from a new presentation; they are particularly urged to read Sections 0.1, 0.5, and 0.6 before proceeding to the later chapters.

The principal purpose here is to give fairly careful derivations of some of the equations of mathematical physics which will be studied more extensively in the rest of the book. The attention paid to modeling in the present chapter could, regrettably, not be sustained in the subsequent ones. Readers who want to further explore aspects of modeling are encouraged to consult the books by Aris [4], Lin and Segel [19], Segel [28], Tayler [32], Keener [16], and Logan [20]. Extensive surveys in mathematical physics, including modern geometric tools, can be found in the recent books of Hassani [13] and of Szekeres [31].

Even when agreement exists on the proper modeling of the physical problem, there are still a number of different possible mathematical descriptions. Although the four formulations we use can be shown to be more or less equivalent (see Section 0.5), each has its distinct advantages. The first, and closest in spirit to the underlying physical law, is the so-called *integral balance* written for a field quantity

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of interest, such as mass, energy, charge, or momentum. The integral balance is formulated over an arbitrary subregion (control region) of the region in space-time where the physical process takes place. In the absence of external inputs, the integral balance becomes a *conservation law*. The second formulation requires additional regularity assumptions for the inputs and the field quantity; the integral balance can then be transformed into a partial differential equation (PDE) governing the local behavior of the field quantity. Constitutive relations as well as boundary and initial conditions supplement the PDE to yield an *initial boundary value problem* (BVP), which, under normal circumstances, will have one and only one solution. When there is doubt as to the range of validity of the PDE, it is often helpful to return to the integral balance for inspiration and verification.

The third formulation is called the *weak form* of the BVP (also known, in special contexts, as the *variational equation* or the *principle of virtual work*). In many ways this is the most powerful mathematical formulation, as it lends itself to the use of modern techniques of functional analysis and also forms the basis for many numerical methods. As the term *variational equation* suggests, the weak form is often related to a *variational principle* (the fourth formulation), such as a principle of minimum energy. The vanishing of the first variation of the functional being minimized is then just the variational equation or weak form of the BVP.

In Section 0.5 we show, for a simple example, how these various formulations are interconnected. In Sections 0.1 through 0.4 we develop integral balances of energy, mass, and momentum in various physical contexts and show how they lead to the respective BVPs.

The chapter ends with two sections (0.6 and 0.7) of a mathematical nature. Section 0.6 reviews fundamental ideas of convergence and norm which are widely used in the rest of the book. Section 0.7 presents a short treatment of Lebesgue integration. Although only a few essential properties of the Lebesgue integral will be needed, it seemed worthwhile to spend a few pages explaining its construction. These limited goals made it convenient to use Tonnelli's approach (as described, for instance, in Shilov [29]). Another recent approach due to Lax [18] involves defining L^1 as the completion of $C(K)$ in the L^1 norm, where K is a ball in $\mathbb{R}^n$. In this approach, measure is a derived concept.

A few words about terminology are in order. $\mathbb{R}^n$ stands for n-dimensional Euclidean space. The definitions below are given for $\mathbb{R}^3$ but are easily modified for $\mathbb{R}^n$. A point in $\mathbb{R}^3$ is identified by its position vector $\mathbf{x} = (x_1, x_2, x_3)$, where x_1, x_2, x_3 are *Cartesian* coordinates; $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, where the nonnegative square root is understood; $d\mathbf{x}$ stands for a volume element $dx_1\,dx_2\,dx_3$. In later chapters the distinguishing notation for vectors is dropped.

An *open ball* of radius a, centered at the origin, is the set of points $\mathbf{x}$ such that $|\mathbf{x}| < a$. The set $|\mathbf{x}| \leqslant a$ is a *closed ball*, and the set $|\mathbf{x}| = a$ is a *sphere*. In $\mathbb{R}^2$ the words *disk* and *circle* are often substituted for ball and sphere, respectively. An *open set* Ω has the property that whenever $\mathbf{x} \in \Omega$, so does some sufficiently small ball with center at $\mathbf{x}$. A point $\mathbf{x}$ belongs to the *boundary* Γ of an open set Ω if $\mathbf{x}$ is not in Ω but if every open ball centered at $\mathbf{x}$ contains a point of Ω. The *closure* $\bar{\Omega}$ of Ω is the union of Ω and Γ. These ideas are best illustrated by an egg with a very thin shell. The interior of the egg is an open set Ω, the shell is Γ, and the egg with shell is $\bar{\Omega}$. An open set Ω is *connected* if each pair of points in Ω can be connected by a curve lying entirely in Ω. A *domain* is an open connected set. Thus an open ball is a domain, but the union of two disjoint open balls is not.

In the definition of the function spaces below, there are some distinctions which are best understood through examples: (a) the function $1/x$ is continuous on $0 < x \leqslant 1$ but cannot be extended continuously to $0 \leqslant x \leqslant 1$; (b) the function $\sqrt{x(1-x)}$ is continuous on $0 \leqslant x \leqslant 1$ with a continuous derivative on $0 < x < 1$ which cannot be extended continuously to $0 \leqslant x \leqslant 1$.

Definition. *Let Ω be a domain in $\mathbb{R}^n$. Then $C^k(\Omega)$ is the set of functions $f(\mathbf{x})$ with continuous derivatives of order 0, 1, 2, ..., k on Ω. (The derivative of order 0 of f is understood to be f itself.) $C^k(\bar{\Omega})$ is the set of functions $f(x) \in C^k(\Omega)$ each of whose derivatives of order 0, 1, 2, ..., k can be extended continuously to $\bar{\Omega}$.*

The sets $C^0(\Omega)$ and $C^0(\bar{\Omega})$ are usually written $C(\Omega)$ and $C(\bar{\Omega})$, respectively.

If Ω is the open interval $a < x < b$ in $\mathbb{R}$, we usually prefer the notation $C^k(a, b)$ for $C^k(\Omega)$ and $C^k[a, b]$ for $C^k(\bar{\Omega})$. Thus the function

$$\sqrt{x(1-x)} \in C[0,1] \cap C^1(0,1) \quad \text{but} \quad \notin C^1[0,1].$$

We shall encounter other function spaces in the sequel (such as the space of piecewise continuous functions, L_p spaces, and Sobolev spaces) with definitions given at the appropriate time.

The symbol $\doteq$ means "set equal to." It is occasionally used to define a new expression. For instance, in writing $D \doteq dS/dx$ we are defining D as dS/dx, which, in turn, is presumably known from earlier discussion.

The terms

$$\inf_{\mathbf{x}\in\Omega} f(\mathbf{x}), \qquad \sup_{\mathbf{x}\in\Omega} f(\mathbf{x})$$

stand for the infimum (greatest lower bound) and supremum (least upper bound) of the real-valued function f on Ω. For instance, if Ω is the open ball in $\mathbb{R}^n$ with radius a and center at the origin, and $f(\mathbf{x}) = |\mathbf{x}|$, then

$$\inf_{\mathbf{x}\in\Omega} f(\mathbf{x}) = 0, \qquad \sup_{\mathbf{x}\in\Omega} f(\mathbf{x}) = a,$$

even though the supremum is not attained for any element $\mathbf{x}$ in Ω.

0.1 HEAT CONDUCTION

Consider heat conduction taking place during a time interval (0, T) in a medium, possibly inhomogeneous, occupying the three-dimensional domain Ω with boundary Γ. There are thus four independent variables x_1, x_2, x_3, t, which we write as $(\mathbf{x}, t)$ since it is convenient to distinguish the time variable from the space variables. The basic domain in space-time is the Cartesian product of Ω and (0, T), written as

$\Omega \times (0, T)$. We now take an energy balance, not over $\Omega \times (0, T)$, but over a subset of the form $R \times (t, t + h)$, where R is an arbitrary portion of Ω and $(t, t + h)$ is an arbitrary interval contained in (0, T). We need flexibility in the choice of the subset $R \times (t, t + h)$ so that we can obtain sufficient information for our purposes. We postulate that only heat energy plays a significant role in the energy budget. The terms which contribute to the heat balance are (a) the change in the heat content of R from time t to time $t + h$, caused by a change in temperature, (b) the heat generated by sources, called *body sources*, in $R \times (t, t + h)$, and (c) the heat flowing in or out through the boundary B of R over the time interval $(t, t + h)$. All of these terms are measured in appropriate units of heat, say calories. The body sources may stem, for instance, from a chemical reaction liberating heat (positive source) or absorbing heat (negative source or sink). Body sources can be of three types: distributed, impulsive, or concentrated. A distributed source is characterized by a density $p(\mathbf{x}, t)$ measured in calories/cm^3 sec, and can generate a finite amount of heat only by acting in a finite volume over a finite time interval. An impulsive source is instantaneous in time and generates a finite amount of heat over an infinitesimal time interval. Concentrated sources are localized in space at points, curves, or surfaces and generate a finite amount of heat over regions of infinitesimal volume.

Whatever the type of source, a heat balance for $R \times (t, t + h)$ gives, in calories,

$$\begin{aligned}&\text{rise in heat content}\\&\qquad = \text{heat generated by body sources} - \text{outflow of heat through } B.\end{aligned} \tag{0.1.1}$$

With $E_R(t)$ representing the heat content of R at time t, the left side of (0.1.1) becomes $E_R(t + h) - E_R(t)$. Assuming no impulsive sources, we can divide (0.1.1) by h and take the limit as $h \to 0$ to obtain, for any t in (0, T),

$$\frac{dE_R}{dt} = P_R(t) - Q_B(t) \quad \left(\frac{\text{cal}}{\text{sec}}\right), \tag{0.1.2}$$

where $P_R(t)$ is the *rate* of heat generation in R and $Q_B(t)$ is the *rate* of outflow through B. Next, we exclude concentrated sources by expressing P_R and E_R in terms of densities $p(\mathbf{x}, t)$ and $e(\mathbf{x}, t)$ defined on $\Omega \times (0, T)$:

$$P_R(t) = \int_R p(\mathbf{x}, t)\, d\mathbf{x}, \quad E_R(t) = \int_R e(\mathbf{x}, t)\, d\mathbf{x},$$

where e is measured in cal/cm^3 and p in cal/cm^3 sec.

The rate of outflow Q_B is expressed in terms of a *heat flux vector* $\mathbf{J}(\mathbf{x}, t)$ defined on $\Omega \times (0, T)$. The amount of heat flowing per unit time across a surface element of area dS with unit normal $\mathbf{n}$ is given by $\mathbf{J} \cdot \mathbf{n}\; dS$, so that

$$Q_B(t) = \int_B \mathbf{J} \cdot \mathbf{n}\; dS, \tag{0.1.3}$$

where $\mathbf{n}$ is the outward normal to B. Use of the divergence theorem on this term transforms (0.1.2) to

$$\int_R \left(\frac{\partial e}{\partial t} + \operatorname{div} \mathbf{J} - p\right) d\mathbf{x} = 0, \quad 0 < t < T. \tag{0.1.4}$$

If (0.1.4) held only for a particular region R, little information could be extracted, but instead we know that it is true for every subregion R of Ω. We claim that this implies that the integrand vanishes at every $\mathbf{x}$ and t (assuming that the integrand is a continuous function of $\mathbf{x}$ and t). Indeed, suppose that the integrand were positive at $\mathbf{x}$, t; we can then surround $\mathbf{x}$ by a sufficiently small region R in which the integrand is positive, thereby violating (0.1.4). We therefore conclude that

$$\frac{\partial e}{\partial t} + \operatorname{div} \mathbf{J} = p, \quad (\mathbf{x},\ t) \text{ in } \Omega \times (0,\ T). \tag{0.1.5}$$

There are too many unknowns in (0.1.5), but both e and $\mathbf{J}$ are related to the temperature $u(\mathbf{x}, t)$ through the following constitutive relations:

1. When a homogeneous material element of volume $d\mathbf{x}$ is raised from the temperature u to the temperature $u + du$, its heat content is raised by $C\, du\, d\mathbf{x}$, where C is the *specific heat* of the material measured in calories per degree per cm^3. Note that C depends on the material and may also depend on u.

2. Fourier's law of heat conduction for a homogeneous material:

$$\mathbf{J} = -k \text{ grad } u, \tag{0.1.6}$$

where k is the thermal conductivity (which may depend on u) and has units of cal per sec per cm^3 per degree. Thus, the heat flowing across an element of surface per unit time is

$$\mathbf{J} \cdot \mathbf{n}\ dS = -k \text{ grad } u \cdot \mathbf{n}\ dS = -k\frac{\partial u}{\partial n}\, dS,$$

where the minus sign is consistent with the fact that heat flows in the direction of decreasing temperature. Note that if we also wanted to include convection, we would have to modify (0.1.6)—see the remarks below.

Since our medium may be inhomogeneous, both C and k may depend on $\mathbf{x}$ as well as u. Then

$$\frac{\partial e}{\partial t} = C\frac{\partial u}{\partial t} \quad \text{and} \quad \operatorname{div} \mathbf{J} = -\operatorname{div}(k \text{ grad } u),$$

so that (0.1.5) becomes the usual equation of heat conduction:

$$C\frac{\partial u}{\partial t} - \operatorname{div}(k \text{ grad } u) = p, \quad (\mathbf{x},\ t) \in \Omega \times (0,\ T). \tag{0.1.7}$$

If C and k are constants, the equation reduces to

$$\frac{\partial u}{\partial t} - a\,\Delta u = \frac{P}{C}, \tag{0.1.8}$$

where $a = k/C$ is the thermal diffusivity in cm^2/sec, and $\Delta = \text{div grad}$ is the Laplacian operator whose form in Cartesian coordinates is

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

REMARKS

1. The term div **J** in (0.1.5) stems from the outflow–inflow of heat through the boundary of a test region—the third term in (0.1.1)—but is now expressed as a point function. We regard div **J** as representing *redistribution* of heat through whatever mechanisms are available for that purpose. If the only such mechanism is heat conduction, then Fourier's law applies: $\mathbf{J} = -k \text{ grad } u$. If the medium is a fluid moving with velocity $\mathbf{v}(\mathbf{x}, t)$, then $\mathbf{J} = -k \text{ grad } u + Cu\mathbf{v}$, which incorporates convection. Then $\text{div } \mathbf{J} = -\text{div } (k \text{ grad } u) + \text{div } (Cu\mathbf{v})$; if the fluid is *incompressible*, $\text{div } \mathbf{v} = 0$ [see (0.2.9) with constant ρ and $p = 0$] and
$$\text{div } \mathbf{J} = -\text{div}(k \text{ grad } u) + \mathbf{v} \cdot \text{grad } u,$$
so that (0.1.7) becomes
$$C\frac{\partial u}{\partial t} - \text{div}(k \text{ grad } u) + \mathbf{v} \cdot \text{grad } u = p. \tag{0.1.9}$$

2. Suppose that we consider (0.1.8) for a medium covering all of $\mathbb{R}^3$, with $p = 0$, and with initial temperature positive over a small part of $\mathbb{R}^3$ and vanishing elsewhere. Then, we shall see in Chapter 8 that $u(\mathbf{x}, t) > 0$ for all of $\mathbb{R}^3$ when $t > 0$. Of course, $u(\mathbf{x}, t)$ will be small for large $|\mathbf{x}|$, but nevertheless there is something disturbing about our model since a localized initial temperature propagates with infinite velocity to give a positive temperature everywhere for $t > 0$. (See, however, the article by Day [8], who gives a spirited defense of Fourier's law.) One possible remedy is to modify Fourier's law (0.1.6) to
$$\mathbf{J} + \tau\frac{\partial \mathbf{J}}{\partial t} = -k \text{ grad } u,$$
with τ a relaxation time. This leads to an equation of the form
$$\varepsilon\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - b\,\Delta u = 0,$$
where ε and b are positive constants. The new equation for u has finite propagation velocity, so that a localized initial temperature is only felt within some bounded region which depends on t. Since $\varepsilon \ll 1$, the effect is noticeable only for small times.

3. If the production term p is prescribed as a function of $\mathbf{x}$ and t, (0.1.8) is a linear equation [and so is (0.1.7) if C and k do not depend on u]. There are