THE FRANK J. FABOZZI SERIES

INTRODUCTION *to* FIXED INCOME ANALYTICS

SECOND EDITION

Relative Value Analysis, Risk Measures, and Valuation

FRANK J. FABOZZI • STEVEN V. MANN

Introduction to Fixed Income Analytics

Second Edition

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FJF
To my wife Donna
and my children Patricia, Karly, and Francesco

SVM
To my wife, Mary – TDA

Contents

Preface

Participants in the fixed income market are inundated with terms and concepts in both the popular press and, more typically, in research reports and professional journal articles. Making life more difficult for professionals in this market sector is the fact that for some important analytical concepts, the same concept is referred to in different ways by different dealer firms and asset management firms. The purpose of this book is to describe the key analytical concepts used in the fixed income market and illustrate how they are computed. The book is not only intended for professionals but also newcomers to the field. It is for this reason that we provide end of chapter questions.

Although market professionals often want a walk through demonstration of how a metric is computed, once they are comfortable with the concept and its computation, professionals then rely on vendors of analytical systems. Probably the most popular system relied upon by fixed income professionals is the Bloomberg System. For this reason, every chapter ties in the analytical concepts that are available on Bloomberg and walks the reader through the relevant Bloomberg screens. We want to thank Bloomberg Financial for granting us permission to reproduce the screens that we used in our exhibits.

We begin the book with an explanation of the most basic concept in finance: the time value of money. In Chapter 2, we describe yield curve analysis, discussing the importance of spot rates and forward rates. The fixed income market has adopted various conventions for determining the number of days when computing accrued interest when trades are settled. These market conventions are the subject of Chapter 3.

The basics of bond valuation are covered in Chapter 4. Our focus in this chapter is on option-free bonds (i.e., bonds that are not callable, putable or convertible) and that have a fixed coupon rate. Yield measures for bonds are covered in Chapter 5.

The analysis of floating rate securities and bonds whose coupon interest is linked to some inflation measure are the subjects of Chapters 6 and 14, respectively. Bonds with embedded options are the subjects of Chapters 7, 9, and 10. Chapter 7 explains how to analyze callable and putable agency and corporate bonds. All residential mortgage-backed securities and certain asset-backed securities grant borrowers a prepayment option and, therefore, these securities have an embedded call option. Chapter 9 explains how these bonds are valued. For those readers unfamiliar with mortgage-backed and asset-backed securities, Chapter 8 explains them and how their cash flows are estimated. Convertible bond valuation is the subject of Chapter 10.

While one often hears about yield measures, portfolio managers are assessed based on their performance, which is measured in terms of total return. Chapter 11 demonstrates the calculation of this measure for individual bonds and portfolios.

A key analytical concept for quantifying and controlling the interest risk of a portfolio or trading position is duration and convexity. These measures of interest rate risk are explained in Chapter 12. One of the limitations of these two measures for use in portfolio risk management is that they assume that if interest rates change, the interest rate for all maturities change by the same amount. This is known as the parallel yield curve shift assumption. An analytical framework for assessing how a portfolio's value might change if this assumption is relaxed is the calculation of a portfolio's key rate durations, which is also explained in Chapter 12.

There are other measures used frequently for quantifying a portfolio's risk exposure. The most popular one is the value-at-risk (VaR) metric. In Chapter 13 we explain not only the reason for the popularity of this metric and alternatives methodologies for calculating it, but the severe limitations of this measure. We explain a superior metric for quantifying risk, conditional VaR.

The approach to bond valuation described in the earlier chapters of the book are based on the discounted cash flow framework. Another approach to valuing bonds for inclusion in a portfolio or positioning for a trade is relative valuation. When properly interpreted, the tools of relative value analysis offer investors some clues about how similar bonds are currently priced in the market on a relative basis. Relative value analysis is the subject of Chapter 15.

An important derivative instrument in the fixed income market for controlling risk is an interest rate swap and is the subject of Chapter 16. After describing a swap's counterparties, risk-return profile, and economic interpretation, we illustrate how to value it.

As explained in several chapters, a key input into a valuation model is the expected interest rate volatility or expected yield volatility. How this measure is estimated is covered in Chapter 17.

We would like to thank Kimberly Bradshaw for her editorial assistance and Megan Orem for her patience in typesetting this book.

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CHAPTER 1

Time Value of Money

A security is a package of cash flows. The cash flows are delivered across time with varying degrees of uncertainty. To value a security, we must determine how much this package of cash flows is worth today. This process employs a fundamental finance principle—the ***time value of money***. Simply stated, one dollar today is worth more than one dollar to be received in the future. The reason is that the money has a time value. One dollar today can be invested, start earning interest immediately, and grow to a larger amount in the future. Conversely, one dollar to be received one year from today is worth less than one dollar delivered today. This is true because an individual can invest an amount of money less than one dollar today and at some interest rate it will grow to one dollar in a year's time.

The purpose of this chapter is to introduce the fundamental principles of future value (i.e., compounding cash flows) and present value (i.e., discounting cash flows). These principles will be employed in every chapter in the remainder of the book. To be sure, no matter how complicated the security's cash flows become (e.g., bonds with embedded options, interest rate swaps, etc.), determining how much they are worth today involves taking present values. In addition, we introduce the concept of yield, which is a measure of potential return and explain how to compute the yield on any investment.

FUTURE VALUE OF A SINGLE CASH FLOW

Suppose an individual invests $100 at 5% compounded annually for three years. We call the $100 invested the ***original principal*** and denote it as P. In this example, the annual interest rate is 5% and is the compensation the investor receives for giving up the use of his or her money for one year's time. Intuitively, the interest rate is a bribe offered to induce an individual to postpone their consumption of one dollar until some time in the future. If interest is compounded annually, this means that interest is paid for use of the money only once per year.

We denote the interest rate as i and put it in decimal form. In addition, N is the number of years the individual gives up use of his or her funds and FV_N is the future value or what the original principal will grow to after N years. In our example,

P = \$100
i = 0.05
N = 3 years

So the question at hand is how much \$100 will be worth at the end of three years if it earns interest at 5% compounded annually?

To answer this question, let's first determine what the \$100 will grow to after one year if it earns 5% interest annually. This amount is determined with the following expression

$$FV_1 = P(1 + i)$$

Using the numbers in our example

$$FV_1 = \$100(1.05) = \$105$$

In words, if an individual invests \$100 that earns 5% compounded annually, at the end of one year the amount invested will grow to \$105 (i.e., the original principal of \$100 plus \$5 interest).

To find out how much the \$100 will be worth at the end of two years, we repeat the process one more time

$$FV_2 = FV_1(1 + i)$$

From the expression above, we know that

$$FV_1 = P(1 + i)$$

Substituting this in the expression and then simplifying, we obtain

$$FV_2 = P(1 + i)(1 + i) = P(1 + i)^2$$

Using the numbers in our example, we find that

$$FV_2 = \$100(1.05)^2 = \$110.25$$

Note that during the second year, we earn \$5.25 in interest rather than \$5 because we are earning interest on our interest from the first year. This example illustrates an important point about how securities' returns work; returns reproduce multiplicatively rather than additively.

To find out how much the original principal will be worth at the end of three years, we repeat the process one last time

$$FV_3 = FV_2(1 + i)$$

Like before, we have already determined FV_2, so making this substitution and simplifying gives us

$$FV_3 = P(1 + i)^2(1 + i)$$

$$FV_3 = P(1 + i)^3$$

Using the numbers in our example, we find that

$$FV_3 = \$100(1.05)^3 = \$115.7625$$

The future value of \$100 invested for three years earning 5% interest compounded annually is \$115.7625.

The general formula for the future value of a single cash flow N years in the future given an interest rate i is

$$FV_N = P(1 + i)^N \tag{1.1}$$

From this expression, it is easy to see that for a given original principal P the future value will depend on the interest rate (i) and the number of years (N) that the cash flow is allowed to grow at that rate. For example, suppose we take the same \$100 and invest it at 5% interest for 10 years rather than five years, what is the future value? Using the expression presented above, we find that the future value is

$$FV_N = \$100(1.05)^{10} = \$162.8894$$

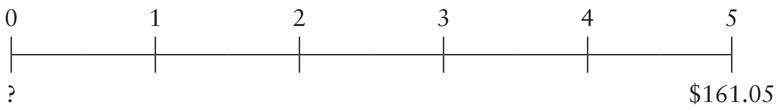
Now let us leave everything unchanged except the interest rate. What is the future value of \$100 invested for 10 years at 6%? The future value is now

$$FV_N = \$100(1.06)^{10} = \$179.0848$$

As we will see in due course, the longer the investment, the more dramatic the impact of even relatively small changes in interest rates on future values.

PRESENT VALUE OF A SINGLE CASH FLOW

The present value of a single cash flow asks the opposite question. Namely, how much is a single cash flow to be received in the future worth today given a particular interest rate? Suppose the interest rate is 10%, how much is \$161.05 to be received five years hence worth today? This question can be easily visualized on the time line presented below:



Alternatively, given the interest rate is 10%, how much would one have to invest today to have \$161.05 in five years? The process is called "discounting" because as long as interest rates are positive, the amount invested (the present value) will be less than \$161.05 (the future value) because of the time value of money.[1]

Since finding present values or discounting asks the opposite question from the future value, the mathematics should be opposite as well. We know the expression for the future value for a single cash flow is given by the expression:

$$FV_N = P(1 + i)^N$$

Let us plug in the information from the question above

$$\$161.05 = P(1.10)^5$$

In order to answer the question of how much we would have to invest today at 10% to have \$161.05 in five years, we must solve for P

$$P = \frac{\$161.05}{(1.10)^5} = \$100$$

So, the present value of \$161.05 delivered five years hence at 10% is \$100.

It is easy to see that the mathematics conform to our intuition. When we calculate a future value, we ask how much will the dollars invested today be worth in the future given a particular interest rate. So, the mathematics of future value involve multiplication by a value greater than one (i.e., making things bigger). Correspondingly, when we find present values, we ask how much a future amount of dollars is worth today given a particular interest

[1]The interest rates used to determine present values are often called "discount rates."

rate. Thus, the mathematics of present value involve division by a value greater than one (i.e., making things smaller).

The general formula for the present value (PV) of a single cash flow N years in the future given an interest rate i is

$$PV = \frac{FV_N}{(1+i)^N} \tag{1.2}$$

Note that we have replaced P with PV. In addition, PV does not have a subscript because we assume it is the value at time 0 (i.e., today).

It is instructive to write the expression for the present value of a single cash flow as follows

$$PV = FV_N\left[\frac{1}{(1+i)^N}\right]$$

The term in brackets is equal to the present value of one dollar to be received N years hence given interest rate i and is often called a ***discount factor***. The present value of a single cash flow is the product of the cash flow to be received (FV_N) and the discount factor. Essentially, the discount factor is today's value of one dollar that is expected to be delivered at some time in the future given a particular interest rate. An analogy will illustrate the point.

Suppose a U.S. investor receives cash payments of \$200,000, ¥500,000, and £600,000. How much does the investor receive? We cannot simply add up the cash flows since the three cash flows are denominated in different currencies. In order to determine how much the investor receives, we would convert the three cash flows into a common currency (say, U.S. dollars) using currency exchange rates. Similarly, we cannot value cash flows to be received at different dates in the future merely by taking their sum. The expected cash flows are delivered at different times and are denominated in different "currencies" (Year 1 dollars, Year 2 dollars, etc.). We use discount factors just like exchange rates to convert cash flows to be received across time into a "common currency" called the present value (i.e., Year 0 dollars).

To illustrate this, we return to the last example—what is the present value of \$161.05 to be received five years from today given that the interest rate is 10%? The present value can be written as

$$PV = \$161.05\left[\frac{1}{(1.10)^5}\right] = \$161.05(0.6209) = \$100$$

One dollar to be received in five years is worth \$0.6209 today given the interest rate is 10%. We expect to receive \$161.05 Year 5 dollars each worth 0.6209 dollars today. The present value is \$100, which is the quantity (\$161.05) multiplied by the price per unit (\$0.6209).

As can be easily seen from the present value expression, the discount factor depends on two things. First, holding the interest rate constant, the longer the time until the cash flow is to be received, the lower the discount factor. To illustrate this, suppose we have \$100 to be received 10 years from now and the interest rate is 10%. What is the present value?

$$PV = \$100\left[\frac{1}{(1.10)^{10}}\right] = \$100(0.3855) = \$38.55$$

Now suppose the cash flow is to be received 20 years hence instead, all else the same. What is the present value?

$$PV = \$100\left[\frac{1}{(1.10)^{20}}\right] = \$100(0.1486) = \$14.86$$

The discount factor falls 0.3855 to 0.1486. This is simply the time value of money at work. The present value is lower the farther into the future the cash flow will be received.

Why this occurs is apparent from looking at the present value equation. The numerator remains the same and is being divided by a larger number in the denominator as one plus the discount rate is being raised to ever higher powers. This is an important property of the present value: for a given interest rate, the farther into the future a cash flow is received, the lower its present value. Simply put, as cash flows move away from the present, they are worth less to us today. Intuitively, we can invest an even smaller amount now (\$14.86) today and it will have more time to grow (20 years versus 10 years) to be equal in size to the payment to be received, \$100.

The second factor driving the discount factor is the level of the interest rate. Specifically, holding the time to receipt constant, the discount factor is inversely related to the interest rate. Suppose, once again, we have \$100 to be received 10 years from now at 10%. From our previous calculations, we know that the present value is \$38.55. Now suppose everything is the same except that the interest rate is 12%. What is the present value when the interest rate increases?

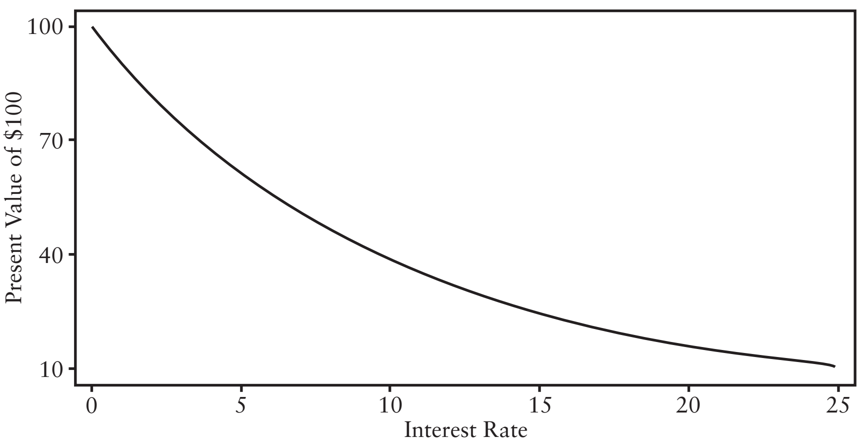
$$PV = \$100\left[\frac{1}{(1.12)^{10}}\right] = \$100(0.3220) = \$32.20$$

As the interest rate rises from 10% to 12%, the present value of $100 to be received 10 years from today falls from $38.55 to $32.20. The reasoning is equally straightforward. If the amount invested compounds at a faster rate (12% versus 10%), we can invest a smaller amount now ($32.20 versus $38.55) and still have $100 after 10 years.

The relationship between the present value of a single cash flow ($100 to be received 10 years hence) and the level of the interest rate is presented in Exhibit 1.1. For now, there are two things to note about present value/interest rate relationship depicted in the exhibit. First, the relationship is downward sloping. This is simply the inverse relationship between present values and interest rates at work. Second, the relationship is a curve rather than a straight line. In fact, the shape of the curve in Exhibit 1.1 is referred to as convex. By convex, it simply means the curve is "bowed in" relative to the origin.

This second observation raises two questions about the convex or curved shape of the present value/interest rate relationship. First, why is it curved? Second, what is the significance of the curvature? The answer to the first question is mathematical. The answer lies in the denominator of the present value formula. Since we are raising one plus the discount rate to powers greater than one, it should not be surprising that the relationship between the present value and the interest rate is not linear. The answer to the second question requires an entire chapter. Specifically, as we see in Chapter 12, this convexity or bowed shape has implications for the price volatility of a bond when interest rates change. What is important to understand at this point is that the relationship is not linear.

EXHIBIT 1.1 PV/Interest Rate Relationship



Note: Present value of $100 to be received in 10 years compounded semiannually.

COMPOUNDING/DISCOUNTING WHEN INTEREST IS PAID MORE THAN ANNUALLY

An investment may pay interest more frequently than once per year (e.g., semiannually, quarterly, monthly, weekly). If an investment pays interest compounded semiannually, then interest is added to the principal twice a year. To account for this, the future value and present value computations presented above require two simple modifications. First, the annual interest rate is adjusted by dividing by the number of times that interest is paid per year. The adjusted interest rate is called a ***periodic interest rate***. Second, the number of years, N, is replaced with the number of periods, n, which is found by multiplying the number of years by the number of times that interest is paid per year.

Future Value of a Single Cash Flow with More Frequent Compounding

The future value of a single cash flow when interest is paid m times per year is as follows:

$$FV_n = P(1 + i)^n \tag{1.3}$$

where

i = annual interest rate divided by m
n = number of interest payments (= $N \times m$)

To illustrate, suppose that a portfolio manager invests \$500,000 in an investment that promises to pay an annual interest rate of 6.8% for five years. Interest is paid on this investment semiannually. What is the future value of this single cash flow given semiannual compounding? The answer is \$698,514.45 as shown below:

PV = \$500,000
m = 2
i = 0.034 (= 0.068/2)
N = 5
n = 10 (5 × 2)

Plugging this information into the future value expression gives us:

$$FV_{10} = \$500{,}000(1.034)^{10} = \$500{,}000(1.397029) = \$698{,}514.50$$

This future value is larger than if interest were compounded annually. With annual compounding, the future value would be \$694,746.34. The higher future value when interest is paid semiannually reflects the fact that the interest is being added to principal more frequently, which in turn earns interest sooner.

Lastly, suppose instead that interest is compounded quarterly rather than semiannually. What is the future value of \$500,000 at 6.8% compounded quarterly for five years? The future value is larger still, \$700,469, for the same reasoning as shown below:

PV = \$500,000
m = 4
i = 0.017 (= 0.068/4)
N = 5
n = 20 (5 × 4)

Plugging this information into the future value expression gives us:

$$FV_{20} = \$500{,}000(1.017)^{20} = \$500{,}000(1.400938) = \$700{,}469$$

Present Value of a Single Cash Flow Using Periodic Interest Rates

We must also adjust our present value expression to account for more frequent compounding. The same two adjustments are required. First, like before, we must convert the annual interest rate into a periodic interest rate. Second, we need to convert the number of years until the cash flow is to be received into the appropriate number of periods that matches the compounding frequency.

The present value of a single cash flow when interest is paid m times per year is written as follows:

$$PV = \frac{FV_n}{(1+i)^n} \tag{1.4}$$

where

i = annual interest rate divided by m
n = number of interest payments (= $N \times m$)

To illustrate this operation, suppose an investor expects to receive \$100,000, 10 years from today and the relevant interest rate is 8% compounded semiannually. What is the present value of this cash flow? The answer is \$45,638.69 as shown below:

FV_{10} = \$100,000
m = 2
i = 0.04 (= 0.08/2)
N = 10
n = 20 (10 × 2)

Plugging this information into the present value expression gives us:

$$PV = \frac{\$100{,}000}{(1.04)^{20}} = \$45{,}638.69$$

This present value is smaller than if interest were compounded annually. With annual compounding, the present value would be \$46,319.35. The lower value when interest is paid semiannually means that for a given annual interest rate we can invest a smaller amount today and still have \$100,000 in 10 years with more frequent compounding.

Moving to quarterly compounding, all else equal, should result in an even smaller present value. What is the present value of \$100,000 to be received 10 years from today at 8% compounded quarterly? The present value is smaller still, \$45,289.04, as shown below:

FV_{40} = \$100,000
m = 4
i = 0.02 (= 0.08/4)
N = 10
n = 40 (10 × 4)

Plugging this information into the present value expression given by equation (1.4) gives

$$PV = \frac{\$100{,}000}{(1.02)^{40}} = \$45{,}289.04$$

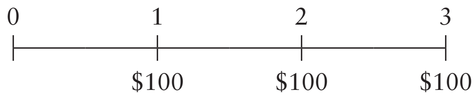
FUTURE AND PRESENT VALUES OF AN ORDINARY ANNUITY

Most securities promise to deliver more than one cash flow. As such, most of the time when we make future/present value calculations, we are working with multiple cash flows. The simplest package of cash flows is called an

annuity. An annuity is a series of payments of fixed amounts for a specified number of periods. The specific type of annuity we are dealing with in our applications is an ***ordinary annuity***. The adjective "ordinary" tells us that the annuity payments come at the end of the period and the first payment is one period from now.

Future Value of an Ordinary Annuity

Suppose an investor expects to receive \$100 at the end of each of the next three years and the relevant interest rate is 5% compounded annually. This annuity can be visualized on the time line presented below:



What is the future value of this annuity at the end of year 3? Of course, one way to determine this amount is to find the future value of each payment as of the end of year 3 and simply add them up. The first \$100 payment will earn 5% interest for two years while the second \$100 payment will earn 5% for one year. The third \$100 payment is already at the end of the year (i.e., denominated in year 3 dollars) so no adjustment is necessary. Mathematically, the summation of the future values of these three cash flows can be written as:

$$\begin{array}{ll} \$100(1.05)^2 = \$100(1.1025) = & \$110.25 \\ \$100(1.05)^1 = \$100(1.0500) = & \$105.00 \\ \$100(1.05)^0 = \$100(1.0000) = & \underline{\$100.00} \\ \text{Total future value} & \$315.25 \end{array}$$

So, if the investor receives \$100 at the end of each of the next three years and can reinvest the cash flows at 5% compounded annually, then at the end of three years the investment will have grown to \$315.25.

The procedure for computing the future value of an annuity presented above is perfectly correct. However, there is a formula that can be used to speed up this computation. Let us return to the example above and rewrite the future value of the annuity as follows:

$$\$100(1.05)^2 + \$100(1.05)^1 + \$100(1.05)^0 = \$315.25$$

This expression can be rewritten as follows by factoring out the \$100 annuity payment:

$$\$100[(1.05)^2 + (1.05)^1 + (1.05)^0]$$

Since $[(1.05)^2 + (1.05)^1 + (1.05)^0] = 3.1525$,

$$\$100[3.1525] = \$315.25$$

The term in brackets is the ***future value of an ordinary annuity of \$1 per year***. Multiplying the future value of an ordinary annuity of \$1 by the annuity payment produces the future value of an ordinary annuity.

The general formula for the future value of an ordinary annuity of \$1 per year is given by

$$FV_N = A\left[\frac{(1+i)^N - 1}{i}\right] \tag{1.5}$$

where

A = amount of the annuity (\$)
i = annual interest rate (in decimal form)

Let us rework the previous example with the general formula where

A = \$100
i = 0.05
N = 3

therefore,

$$FV_N = \$100\left[\frac{(1.05)^3 - 1}{0.05}\right] = \$100(3.1525) = \$315.25$$

This value agrees with our earlier calculation.

Future Value of an Ordinary Annuity when Payments Occur More Than Once per Year

The future value of an ordinary annuity can be easily generalized to handle situations in which payments are made more than one time per year. For example, instead of assuming an investor receives and then reinvests \$100 per year for three years, starting one year from now, suppose that the investor receives \$50 every six months for three years, starting six months from now.

The general formula for the future value of an ordinary annuity when payments occur m times per year is