Extreme Events
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Extreme Events

Robust Portfolio Construction in the Presence of Fat Tails

Malcolm H. D. Kemp
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Preface

There are two reasons for writing this book. The first is timing. This book has been written during a worldwide credit and economic crisis. The crisis has reminded us that extreme events can occur more frequently than we might like. It followed an extended period of relative stability and economic growth that, with hindsight, was the calm before a storm. Throughout this book, this crisis is referred to as the ‘2007–09 credit crisis’.

During the preceding relatively stable times, people had, maybe, paid less attention than they should have to the possibility of extreme events occurring. Perhaps you were one of them. Perhaps I was too. Adversity is always a good spur to careful articulation of underlying reality.

Extreme events were evident even right at the start of the 2007–09 credit crisis, at the end of July 2007 and in the first two weeks of August 2007. This marked the time when previously ruling relationships between interbank money market rates for different deposit terms first started to unravel. It also coincided with some sudden and unexpectedly large losses being incurred by some high profile (quantitatively run) hedge funds that unwound some of their positions when banks cut funding and liquidity lines to them. Other investors suffered unusually large losses on positions they held that had similar economic sensitivities to the ones that these hedge funds were liquidating.

During 2007 investors were also becoming more wary of US sub-prime debt. But, even as late as April 2008, central banks were still expecting markets gradually to regain their poise. It was not to be. In late 2008 the credit crisis erupted into a full-blown global banking crisis with the collapse of Lehman Brothers, the need for Western governments to shore up their banking systems and economies across the globe entering recessions. Market volatility during these troubled times was exceptional, reaching levels not seen since the Great Depression in the early 1930s. Extreme events had well and truly returned to the financial landscape.

The second reason for writing this book is that it naturally follows on from my earlier book on Market Consistency; see Kemp (2009). Market consistency was defined there as the activity of taking account of ‘what the market has to say’ in financial practice. In Market Consistency I argued that portfolio construction should be viewed as the third strand in a single overarching branch of financial theory and practice, the other two strands being valuation methodology and risk management processes. However, I also noted that the application of market consistency to portfolio construction was in some sense simultaneously both core and peripheral. It was ‘core’, in the sense that ultimately there is little point in merely valuing and measuring positions or risk; the focus should also be on managing them. It was also ‘peripheral’, because portfolio management generally also involves taking views about when the market is right and when it
is wrong, and then acting accordingly, i.e., it necessarily does not always agree with ‘what the market has to say’.

These different drivers meant that Market Consistency naturally focused more on valuation methodology and risk management and less on portfolio construction. So, this book seeks to balance the exposition in Market Consistency by exploring in more depth the portfolio construction problem. It is particularly aimed at those practitioners, students and others who, like me, find portfolio construction a fascinating topic to study or a useful discipline to apply in practice.

The style I have adopted when writing this book is similar to the one used for Market Consistency. This involves aiming simultaneously to write both a discursive text and a reference book. Readers will no doubt want a reference book to help them navigate through the many techniques that can be applied to portfolio construction. However, undue focus on the ‘how to’ rather than also on the ‘why should I want to’ or ‘what are the strengths and weaknesses of doing so’ is typically not the right way to help people in practice. Instead, readers ideally also expect authors to express opinions about the intrinsic usefulness of different techniques.

As in Market Consistency I have tried to find a suitable balance between mathematical depth and readability, to avoid some readers being overly daunted by unduly complicated mathematics. The book focuses on core principles and on illuminating them where appropriate with suitably pitched mathematics. Readers wanting a more detailed articulation of the underlying mathematics are directed towards the portfolio construction pages of the www.nematrian.com website, referred to throughout this book as Kemp (2010). Kemp (2010) also makes available a wide range of online tools that, inter alia, can be applied to the portfolio construction problem. Most of the charts in this book and the analyses on which they are based use these tools, and where copyrighted are reproduced with kind permission from Nematrian Limited. A few of the charts and principles quoted in this book are copied from ones already contained in Market Consistency and are reproduced with permission from John Wiley & Sons, Ltd.
Acknowledgements

I would like to thank Pete Baker, Aimee Dibbens and their colleagues at Wiley for encouraging me to embark on writing this book. Thanks are also due to Colin Wilson and others who have read parts of this manuscript and provided helpful comments on how it might be improved. A special appreciation goes to my wife and family for supporting me as this book took shape.

However, while I am very grateful for the support I have received from various sources when writing this book, I still take sole responsibility for any errors and omissions that it contains.
## Abbreviations

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<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>ALM</td>
<td>asset-liability management</td>
</tr>
<tr>
<td>APT</td>
<td>Arbitrage Pricing Theory</td>
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<tr>
<td>AR</td>
<td>autoregressive</td>
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<tr>
<td>ARMA</td>
<td>autoregressive moving average</td>
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<tr>
<td>BL</td>
<td>Black-Litterman</td>
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<tr>
<td>bp</td>
<td>basis point</td>
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<tr>
<td>CAPM</td>
<td>Capital Asset Pricing Model</td>
</tr>
<tr>
<td>CDO</td>
<td>collateralised debt obligation</td>
</tr>
<tr>
<td>cdf</td>
<td>cumulative distribution function</td>
</tr>
<tr>
<td>CDS</td>
<td>credit default swap</td>
</tr>
<tr>
<td>CML</td>
<td>capital market line</td>
</tr>
<tr>
<td>COSO</td>
<td>Committee of Sponsoring Organisations of the Treadway Commission</td>
</tr>
<tr>
<td>CPPI</td>
<td>constant proportional portfolio insurance</td>
</tr>
<tr>
<td>CRO</td>
<td>chief risk officer</td>
</tr>
<tr>
<td>CRRA</td>
<td>constant relative risk aversion</td>
</tr>
<tr>
<td>CVaR</td>
<td>conditional Value-at-Risk (aka TVaR and Expected Shortfall)</td>
</tr>
<tr>
<td>EM</td>
<td>expectation-maximisation algorithm</td>
</tr>
<tr>
<td>EMH</td>
<td>efficient markets hypothesis</td>
</tr>
<tr>
<td>ERM</td>
<td>enterprise risk management</td>
</tr>
<tr>
<td>ESG</td>
<td>economic scenario generator</td>
</tr>
<tr>
<td>EU</td>
<td>European Union</td>
</tr>
<tr>
<td>EVD</td>
<td>extreme value distribution</td>
</tr>
<tr>
<td>EVT</td>
<td>extreme value theory</td>
</tr>
<tr>
<td>FOC</td>
<td>first order conditions</td>
</tr>
<tr>
<td>FSA</td>
<td>Financial Services Authority (UK)</td>
</tr>
<tr>
<td>GAAP</td>
<td>Generally Accepted Accounting Principles</td>
</tr>
<tr>
<td>GMM</td>
<td>Gaussian (i.e., multivariate Normal) mixture models</td>
</tr>
<tr>
<td>GRS</td>
<td>Gibbons, Ross and Shanken (Econometrica 1989) F-test</td>
</tr>
<tr>
<td>HHI</td>
<td>Herfindahl Hirshman Index</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>independent and identically distributed</td>
</tr>
<tr>
<td>IAS</td>
<td>International Accounting Standards</td>
</tr>
<tr>
<td>IASB</td>
<td>International Accounting Standards Board</td>
</tr>
<tr>
<td>ICA</td>
<td>independent components analysis</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
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<tr>
<td>--------------</td>
<td>-----------------------------------------------</td>
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<tr>
<td>IT</td>
<td>information technology</td>
</tr>
<tr>
<td>lpm</td>
<td>lower partial moment</td>
</tr>
<tr>
<td>LTCM</td>
<td>Long Term Capital Management</td>
</tr>
<tr>
<td>MA</td>
<td>moving average</td>
</tr>
<tr>
<td>MCMC</td>
<td>Markov chain Monte Carlo</td>
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<tr>
<td>MDA</td>
<td>maximum domain of attraction</td>
</tr>
<tr>
<td>MRP</td>
<td>minimum risk portfolio</td>
</tr>
<tr>
<td>OLS</td>
<td>ordinary least squares</td>
</tr>
<tr>
<td>ORSA</td>
<td>Own Risk and Solvency Assessment</td>
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<tr>
<td>OSLL</td>
<td>out-of-sample log-likelihood</td>
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<tr>
<td>PCA</td>
<td>principal components analysis</td>
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<tr>
<td>pdf</td>
<td>probability density function</td>
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<tr>
<td>QQ</td>
<td>quantile–quantile</td>
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<tr>
<td>RE</td>
<td>resampled efficiency</td>
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<tr>
<td>RS</td>
<td>regime switching</td>
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<tr>
<td>S&amp;Ls</td>
<td>(US) savings and loans associations</td>
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<tr>
<td>SCR</td>
<td>Solvency Capital Requirement</td>
</tr>
<tr>
<td>SETAR</td>
<td>self-exciting threshold autoregressive</td>
</tr>
<tr>
<td>SIV</td>
<td>structured investment vehicle</td>
</tr>
<tr>
<td>SYSC</td>
<td>Senior Management Arrangements, Systems and Controls</td>
</tr>
<tr>
<td>TAR</td>
<td>threshold autoregressive</td>
</tr>
<tr>
<td>TVaR</td>
<td>tail Value-at-Risk (aka CVaR and Expected Shortfall)</td>
</tr>
<tr>
<td>UK</td>
<td>United Kingdom of Great Britain and Northern Ireland</td>
</tr>
<tr>
<td>US and USA</td>
<td>United States of America</td>
</tr>
<tr>
<td>VaR</td>
<td>Value-at-Risk</td>
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</tbody>
</table>
### Notation

**\( \mathbf{0} \)** = vector of zeros  
**\( \mathbf{1} \)** = vector of ones  
**\( a, \mathbf{a} \)** = active positions, (linear combination) signal mixing coefficients  
**\( \hat{a} \)** = tail index parameter estimate  
**\( b, \mathbf{b} \)** = benchmark, minimum risk portfolio, (distributional mixture) mixing coefficients  
**\( \beta \)** = portfolio or individual security beta  
**\( \gamma, \gamma_i \)** = cumulant of a distribution (if \( i = 1 \) then skew and if \( i = 2 \) then 'excess' kurtosis)  
**\( \Gamma_0 \)** = option gamma  
**\( \Delta \)** = amount to invest in the underlying in a hedging algorithm  
**\( \Delta_0 \)** = option delta  
**\( E \)** = analogue of energy in simulated annealing algorithm  
**\( E (X) \)** = expected value of \( X \)  
**\( E (X | \theta) \)** = expected value of \( X \) given \( \theta \)  
**\( \varepsilon_{ij} \)** = error terms in a regression analysis  
**\( f (x) \)** = probability density function  
**\( F (x), F^{-1} (x) \)** = cumulative distribution function (or more generally some specified distributional form) and its inverse function  
**\( \langle f \rangle \)** = Monte Carlo estimate of the average of a function \( f \)  
**\( \langle \langle f \rangle \rangle \)** = true average of a function \( f \)  
**\( h \)** = time period length  
**\( i, j, k \)** = counting indexes  
**\( I, \mathbf{I} \)** = Identity matrix  
**\( I_t \)** = index series  
**\( I_a (t) \)** = ‘hit’ series when backtesting  
**\( K \)** = option strike price  
**\( L \)** = lottery, Lagrange multiplier  
**\( \lambda \)** = risk-reward trade-off parameter, also eigenvalue  
**\( m \)** = number of assets (or liabilities, or both) in the portfolio optimisation problem  
**\( \mu, \mu, \hat{\mu} \)** = mean of a univariate distribution, vector of population means of a multivariate distribution, vector of sample or estimated means (likewise use of ‘hat’ symbol for other variables)  
**\( n \)** = number of time periods, observations or simulations
Notation

\( N(z), N^{-1}(z) \) = cumulative distribution function for the Normal distribution, and its inverse function

\( o(x) = \) tends to zero more rapidly than \( x \) as \( x \to 0 \)

\( O(n) = \) of order (magnitude) a constant times \( n \) (in analysis of algorithm run times)

\( p(X) = \) probability of \( X \) occurring

\( P(X \leq x) = \) probability that a random variable \( X \) is less than some value \( x \)

\( p(X | \theta), P(X | \theta) = \) probability of \( X \) given \( \theta \)

\( r = \) return on an asset, liability or index

\( \bar{r} = \) mean of \( r \)

\( r_{rf}, r_b = \) risk-free rate of return, benchmark return

\( \rho = \) correlation coefficient, risk measure

\( s = \) stress test, regime

\( S = \) spectrum of an autoregressive time series

\( S_t = \) stock index series

\( S(\alpha, \beta, \gamma, \delta; k) = \) Stable distribution with parameters \( (\alpha, \beta, \gamma, \delta_k) \) using parameter definition

\( k = 0 \) or \( 1 \)

\( \sigma = \) standard deviation

\( T = \) matrix transpose (if used as a superscript), also (when not confusing) time at end of analysis/time horizon, also analogue of temperature in simulated annealing algorithm

\( T(z) = \) transfer function

\( V = \) a volume in a multi-dimensional space

\( V = \) covariance matrix

\( w_i, w_t, w_{i,t}, w_{i,j} = \) weight in asset \( i \), innovation at time \( t \) (possibly expressed in vector form for vector autoregressive series), elements of mixing matrix \( W \)

\( W = \) mixing matrix, terminal wealth

\( X_n \triangleq X = \) equality in distributional form

\( Z_n \xrightarrow{D} F = \) the sequence of random variables, \( Z_n \), converges in distribution to \( F \) as \( n \to \infty \), i.e., in the limit takes the distributional form characterised by \( F \)
1

Introduction

1.1 EXTREME EVENTS

This book is about how best to construct investment portfolios if *a priori* it is reasonable to assume that markets *might* exhibit *fat-tailed* behaviour. It is designed to appeal to a wide variety of practitioners, students, researchers and general readers who may be interested in extreme events or portfolio construction theory either in isolation or in combination. It achieves this aim by

(a) Exploring extreme events, why they might arise in a financial context and how we might best analyse them.

(b) Separately exploring all the main topics in portfolio construction theory applicable even in the absence of fat tails. A special case of any more general approach capable of effectively handling extreme events is the situation where the extent of fat-tailed behaviour is too small to be discernible.

(c) Blending points (a) and (b) together to identify portfolio construction methodologies better able to cater for possible fat-tailed behaviour in the underlying assets or liabilities.

Given its wide intended audience, the book covers these topics both from a more qualitative perspective (particularly in the earlier and later chapters) and from a more quantitative (i.e., mathematical) perspective (particularly in the middle chapters). Where possible, this book has been segmented so that valuable insights can be gained without necessarily having to read the whole text. Conversely, in the author’s opinion, valuable insights arise throughout the book, including the parts that are more mathematical in nature. More general readers are therefore encouraged not to skip over these parts completely, although they do not need to worry too much about following all the details.

By *fat-tailed* behaviour we mean that the distribution of future returns is expected to involve more extreme events than might be expected to occur were returns to follow the (multivariate) (log-) Normal distributions often assumed to apply to markets in basic portfolio construction texts.¹

Most practitioners believe that most markets are ‘fat-tailed’ given this terminology. There is a wide body of empirical academic literature that supports this stance, based on analysis of past market behaviour. There is also a growing body of academic theory, including some involving behavioural finance, explaining why fat-tailed behaviour seems so widespread. So, we might also characterise this book as exploring how best to construct investment portfolios in the real world.

Of course, practitioners and academics alike are not themselves immune from behavioural biases. It is one thing to agree to pay lip service to the notion that market behaviour can be

¹ By ‘multivariate’ we mean that the returns on different return series have a joint distribution, the characterisation of which includes not only how individual return series in isolation might behave, but also how they behave when considered in tandem, see Chapter 3. By ‘(log-) Normal’ we mean that the natural logarithm of $1 + r$ is Normally distributed, where the return, $r$, is expressed in fractional form, see Section 2.3.1.
fat-tailed, but quite another to take this into account in how portfolios are actually constructed. 
Following the dot.com boom and bust in the late 1990s and early 2000s, markets settled into a 
period of unusually low volatility. Strategies that benefited from stable economic conditions, 
e.g., ones that followed so-called ‘carry’ trades or strategies relying on continuing ready 
access to liquidity, proved successful, for a time. The 2007–09 credit crisis, however, painfully 
reminded the complacent that markets and economies more generally can and do experience 
extreme events.

1.2 THE PORTFOLIO CONSTRUCTION PROBLEM

We do not live in a world in which we have perfect foresight. Instead, portfolio construction 
always involves striking a balance between risk and reward, i.e., the risk that the views implicit 
in our portfolio construction will prove erroneous versus the rewards that will accrue if our 
views prove correct. Everyone involved in the management of portfolios, whether of assets or 
of liabilities, faces a portfolio construction problem. How do we best balance risk and return? 
Indeed, what do we mean by ‘best’?

Given the lack of perfect foresight that all mortals face, it is not reasonable to expect a book 
like this to set out guaranteed ways of profiting from investment conditions come what may. 
Instead, it seeks to achieve the lesser but more realistic goal of exploring the following:

(a) core elements of portfolio construction;
(b) mathematical tools that can be used to assist with the portfolio construction problem, and 
their strengths and weaknesses;
(c) ways of refining these tools to cater better for fat-tailed market behaviour;
(d) mindsets best able to cope well with extreme events, and the pitfalls that can occur if we 
do not adopt these mindsets.

1.3 COPING WITH REALLY EXTREME EVENTS

Lack of perfect foresight is not just limited to a lack of knowledge about exactly what the 
future holds. Typically in an investment context we also do not know how uncertain the future 
will be. Using statistical terminology, we do not even know the precise form of the probability 
distribution characterising how the future might evolve.

The differentiation between ‘risk’ and ‘uncertainty’ is a topic that several popular writers 
have sought to explore in recent times, e.g., Taleb (2004, 2007). In this context ‘risk’ is usually 
taken to mean some measurable assessment of the spread of possible future outcomes, with 
‘uncertainty’ then taken to mean lack of knowledge, even (or particularly) concerning the size 
of this spread.

In this book, we take up this baton particularly in Chapters 8 and 9. Holding such an insight 
in mind is, I think, an important contributor to successful portfolio construction. In particular, 
it reminds us that really extreme events seem to have a nasty habit of occurring more often 
than we might like. Put statistically, if there is a 1 in $10^{10}$ (1 in 10 billion) chance of an event 
occurring given some model we have adopted, and there is a 1 in $10^6$ (1 in a million) chance 
that our model is fundamentally wrong, then any really extreme events are far more likely
to be due to our model being wrong than representing random (if unlikely) draws from our original model.²

Yet such insights can also be overplayed. The portfolio construction problem does not go away merely because the future is uncertain. Given a portfolio of assets, someone, ultimately, needs to choose how to invest these assets. Although it is arguably very sensible for them to bear in mind intrinsic limitations on what might be knowable about the future, they also need some framework for choosing between different ways of structuring the portfolio.

This framework might be qualitatively formulated, perhaps as someone’s ‘gut feel’. Alternatively, it might be quantitatively structured, based on a more mathematical analysis of the problem at hand. It is not really the purpose of this book to argue between these two approaches. Indeed, we shall see later that the outcome of essentially any qualitative judgemental process can be reformulated as if it were coming from a mathematical model (and arguably vice versa).

Perhaps the answer is to hold onto wealth lightly. All of us are mortal. The more religious among us, myself included, might warm to this philosophy. But again, such an answer primarily characterises a mindset to adopt, rather than providing specific analytical tools that we can apply to the problem at hand.

1.4 RISK BUDGETING

Some practitioners point to the merits of risk budgeting. This involves identifying the total risk that we are prepared to run, identifying its decomposition between different parts of the investment process and altering this decomposition to maximise expected value-added for a given level of risk. It is a concept that has wide applicability and is difficult to fault. What business does not plan its evolution via forecasts, budgets and the like? Indeed, put like this risk budgeting can be seen to be common sense.

Again, though, we have here principally a language that we can use to describe how to apply investment principles. Risk budgeting principally inhabits the ‘mindset’ sphere rather than constituting an explicit practical toolset directly applicable to the problem at hand. This should not surprise us. Although sensible businesses clearly do use budgeting techniques to good effect, budgeting per se does not guarantee success. So it is with risk budgeting.³

However, language is the medium through which we exchange ideas and so cannot be ignored. Throughout this book, we aim to explain emerging ideas using terms that can be traced back to risk budgeting concepts. This helps clarify the main aspects of the methodology under discussion. It also helps us understand what assumptions need to be made for the relevant methodology to be valid.

² More precisely, in this situation we need the probability of occurrence of the event that we are considering to be much higher (on average) than 1 in 10,000 in the 1 in a million circumstances when our underlying model is assumed to prove to be fundamentally wrong. This, however, is typically what is implied by use of the term ‘fundamentally wrong’. For example, suppose that our model presupposes that possible outcomes are Normally distributed with zero mean and standard deviation of 1. Then the likelihood of an outcome worse than c. –6.4 is 1 in 10 billion. However, suppose that there is actually a one in a million chance that our model is ‘fundamentally wrong’ and that the standard deviation is not 1 but 10. Roughly 26% of outcomes when the standard deviation is 10 will involve an outcome worse than c. –6.4. So, in this instance an event this extreme is roughly 40 times as likely to be a result of our original model being ‘fundamentally wrong’ as it is to be a fluke draw from the original model.

³ Likewise, no portfolio construction technique is able to guarantee success.
1.5 ELEMENTS DESIGNED TO MAXIMISE BENEFIT TO READERS

As explained in Section 1.1, this book aims to appeal to a wide variety of audiences. To do this, I have, as with my earlier book on Market Consistency, sought a suitable balance between mathematical depth and readability, to avoid some readers being overly daunted by unduly complicated mathematics. The book focuses on core principles and on illuminating them where appropriate with suitably pitched mathematics. Readers wanting a more detailed articulation of the underlying mathematics are directed towards the portfolio construction pages of the www.nematrian.com website, referred to throughout this book as Kemp (2010).

To maximise the benefit that both practitioners and students can gain from this book, I include two sections at the end of each main chapter that provide:

(a) Comments specifically focusing on the practitioner perspective. To navigate successfully around markets typically requires an enquiring yet somewhat sceptical mindset, questioning whether the perceived benefits put forward for some particular technique really are as strong as some might argue. So, these sections either focus on the ways that practitioners might be able to apply insights set out earlier in the relevant chapter in their day-to-day work, or highlight some of the practical strengths and weaknesses of techniques that might be missed in a purely theoretical discussion of their attributes.

(b) A discussion of some of the more important implementation challenges that practitioners may face when trying to apply the techniques introduced in that chapter. Where the same challenge arises more than once, I generally discuss the topic at the first available opportunity, unless consideration of the challenge naturally fits better in a later chapter.

The book also includes an Appendix containing some exercises for use by students and lecturers. Each main chapter of the book has associated exercises that further illustrate the topics discussed in that chapter. The exercises are reproduced with kind permission from Nematrian Limited. Hints and model solutions are available on the www.nematrian.com website, as are any analytical tools needed to solve the exercises.

Throughout the book, I draw out principles (i.e., guidance, mainly for practitioners) that have relatively universal application. Within the text these principles are indented and shown in bold, and are referenced by P1, P2, etc.

1.6 BOOK STRUCTURE

The main title of this book is Extreme Events. It therefore seems appropriate to focus first, in Chapters 2 and 3, on fat tails and extreme events. We explore some of the ways in which fat-tailed behaviour can be analysed and the existence or otherwise of extreme events confirmed or rejected. We differentiate between analysis of fat tails in single return series in Chapter 2 and analysis of fat tails in joint (i.e., multiple) return series in Chapter 3. The shift from ‘one’ to ‘more than one’ significantly extends the nature of the problem.

Before moving on to portfolio construction per se, we consider in Chapter 4 some ways in which we can identify what seems to be driving market behaviour. Without some underlying model of market behaviour, it is essentially impossible to assess the merits of different possible approaches to portfolio construction (or risk modelling). We consider tools such as principal components analysis and independent components analysis, and we highlight their links with other statistical and econometric tools such as multivariate regression.
In Chapters 5–7 we turn our attention to the portfolio construction problem. Chapter 5 summarises the basic elements of portfolio construction, both from a quantitative and from a qualitative (i.e., ‘fundamental’) perspective, if fat tails are not present. At a suitably high level, both perspectives can be viewed as equivalent, apart perhaps from the mindset involved. In Chapter 5 we also explore some of the basic mathematical tools that commentators have developed to analyse the portfolio construction problem from a quantitative perspective. The focus here (and in Chapter 6) is on mean-variance portfolio optimisation (more specifically, mean-variance optimisation assuming time stationarity). We consider its application both in a single-period and in a multi-period world.

In Chapter 6 we highlight the sensitivity of the results of portfolio construction analyses to the input assumptions, and the tendency of portfolio optimisers to maximise ‘model error’ rather than ‘risk-return trade-off’. We explore ways of making the results more robust to errors affecting these input assumptions. The academic literature typically assumes that input assumptions are estimated in part from past data. We might argue that asset allocation is a forward-looking discipline, and that the assumptions we input into portfolio construction algorithms should properly reflect our views about what might happen in the future (rather than about what has happened in the past). However, some reference to past data nearly always arises in such analyses. We pay particular attention to Bayesian approaches in which we have some prior (‘intrinsic’) views about the answers or input parameters that might be ‘reasonable’ for the problem and we give partial weight to these alongside partial weight to external (often past) data. The best-known example of this is probably the Black-Litterman approach. Some Bayesian approaches can also be viewed as justifying heuristic techniques that can be applied to the portfolio construction problem. This again highlights the high-level equivalence that exists between quantitative and qualitative approaches to portfolio construction.

In Chapter 7 we identify how to incorporate fat tails into portfolio construction theory. We start by exploring what happens when we relax the assumption of time stationarity, by introducing the concept of regime shifting. This involves assuming that the world is, at any point in time, in one of several possible states, characterised by different distributions of returns on the different assets and liabilities under consideration. The mixing of distributions introduced in such a model naturally leads to fat-tailed behaviour. We then extend these ideas to encompass more general ways of incorporating fat-tailed behaviour. We focus mainly but not exclusively on situations where the regime is characterised not by a single Normal distribution but by a distributional mixture of Normal distributions (because this type of model is sufficiently general that it can approximate other ways in which fat tails might arise). We also explore approaches that involve continuously varying parameterisations of the different regimes and focus on behaviour in continuous rather than discrete time.

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4 In this context, a ‘heuristic’ technique is one that is akin to a rule of thumb that is not principally proposed on the basis of some formal mathematical justification but more because the approach is relatively convenient to implement.
Chapters 2 to 7 are largely concerned with a probability-theoretic view of portfolio construction. In them, we identify, somehow or other, a distributional form to which we believe future outcomes will adhere (or more than one in the case of regime shifting). At least in principle, this involves specifying a likelihood of occurrence for any given future scenario. However, the belief that we can in all cases actually identify such likelihoods arguably involves an overly rosy view about our ability to understand the future. More to the point, regulators and other bodies who specify how financial services entities should operate may want to ensure that entities do not put too many of their eggs into what may prove an uncertain basket.

In recent years this has led to an increased focus on stress testing, which we explore in Chapter 8. Stress testing, in this context, generally involves placing less emphasis on likelihood and more emphasis on magnitude (if large and adverse) and on what might make the scenario adverse. We can view ‘reverse stress testing’ and ‘testing to destruction’ as being at one extreme of this trend. In the latter, we hypothesise a scenario adverse enough to wipe out the business model of the firm in question (or the equivalent if we are considering a portfolio) irrespective of how likely or unlikely it might be to come to pass. We then focus on what might cause such a scenario to arise and whether there are any strategies that we can adopt that might mitigate these risks.

Chapter 9 extends the ideas implicit in Chapter 8 to consider ‘really extreme’ events. It is more heuristic and ‘mindset-orientated’ than the rest of the book. This is inevitable. Such events will almost certainly be so rare that there will be little if any directly relevant data on them. Market implied portfolio construction techniques potentially take on added importance here. Merely because events are rare does not mean that others are not exposed to them too. The views of others, distilled into the market prices of such risks, may be subject to behavioural biases, but may still help us in our search to handle extreme events better.

Finally, in Chapter 10 we collate and summarise in one place all the principles highlighted elsewhere in the book.