ANALYSIS AND DESIGN OF ELASTIC BEAMS

Computational Methods

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This book treats the analysis and design of beams, with a particular emphasis on computational approaches for thin-walled beams. The underlying formulations are based on the assumption of linear elasticity. Extension, bending, and torsion are discussed. Beams with arbitrary cross sections, loading, and boundary conditions are covered, as well as the determination of displacements, natural frequencies, buckling loads, and the normal and shear stresses due to bending, torsion, direct shear, and restrained warping. The Wiley website (http://www.wiley.com/go/pilkey) provides information on the availability of computer programs that perform the calculations for the formulations of this book.

Most of this book deals with computational methods for finding beam cross-sectional properties and stresses. The computational solutions apply to solid and thin-walled open and closed sections. Some traditional analytical formulas for thin-walled beams are developed here. A systematic and thorough treatment of analytical thin-walled beam theory for both open and closed sections is on the author’s website.

The technology essential for the study of a structural system that is modeled by beam elements is provided here. The cross-sectional properties of the individual beams can be computed using the methodology provided in this book. Then, a general-purpose analysis computer program can be applied to the entire structure to compute the forces and moments in the individual members. Finally, the methodology developed here can be used to find the normal and shear stresses on the members’ cross sections.

Historically, shear stress-related cross-sectional properties have been difficult to obtain analytically. These properties include the torsional constant, shear deformation coefficients, the warping constant, and the shear stresses themselves. The formulations of this book overcome the problems encountered in the calculation of these properties. Computational techniques permit these properties to be obtained ef-
ficiently and accurately. The finite element formulations apply to cross sections with arbitrary shapes, including solid or thin-walled configurations. Thin-walled cross sections can be open or closed. For thin-walled cross sections, it is possible to prepare a computer program of analytical formulas to calculate several of the cross-sectional properties efficiently and with acceptable accuracy.

Shape optimization of beam cross sections is discussed. The cross-sectional shape can be optimized for objectives such as minimum weight or an upper bound on the stress level. Essential to the optimization is the proper calculation of the sensitivity of various cross-sectional properties with respect to the design parameters. Standard optimization algorithms, which are readily available in existing software, can be utilized to perform the computations necessary to achieve an optimal design.

For cross-sectional shape optimization, B-splines, in particular, NURBS can be conveniently used to describe the shape. Using NURBS eases the task of adjusting the shape during the optimization process. Such B-spline characteristics as knots and weights are defined.

Computer programs are available to implement the formulations in this book. See the Wiley website. These include programs that can find the internal net forces and moments along a solid or thin-walled beam with an arbitrary cross-sectional shape with any boundary conditions and any applied loads. Another program can be used to find the cross-sectional properties of bars with arbitrary cross-sectional shapes, as well as the cross-sectional stresses. One version of this program is intended to be used as an “engine” in more comprehensive analysis or optimal design software. That is, the program can be integrated into the reader’s software in order to perform cross-sectional analyses or cross-sectional shape optimization for beams. The input to this program utilizes NURBS, which helps facilitate the interaction with design packages.

The book begins with an introduction to the theory of linear elasticity and of pure bending of a beam. In Chapter 2 we discuss the development of stiffness and mass matrices for a beam element, including matrices based on differential equations and variational principles. Both exact and approximate matrices are derived, the latter utilizing polynomial trial functions. Static, dynamic, and stability analyses of structural systems are set forth in Chapter 3. Initially, the element structural matrices are assembled to form global matrices. The finite element method is introduced in Chapter 4 and applied to find simple non–shear-related cross-sectional properties of beams. In Chapter 5 we present Saint-Venant torsion, with special attention being paid to the accurate calculation of the torsional constant. Shear stresses generated by shear forces on beams are considered in Chapter 6; these require the relatively difficult calculation of shear deformation coefficients for the cross section. In Chapter 7 we present torsional stress calculations when constrained warping is present. Principal stresses and yield theories are discussed in Chapter 8. In Chapters 9 and 10 we introduce definitions and formulations necessary to enable cross-sectional shape optimization. In Chapter 9 we introduce the concept of B-splines and in Chapter 10 provide formulas for sensitivities of the cross-sectional properties. In the two appendices we describe some of the computer programs that have been prepared to accompany the book.
The work of Dr. Levent Kitis, my former student and now colleague, has been crucial in the development of this book. Support for some of the research related to the computational implementation of thin-walled cross-sectional properties and stresses was provided by Ford Motor Company, with the guidance of Mark Zebrowski and Victor Borowski. As indicated by the frequent citations to his papers in this book, a major contributor to the theory here has been Dr. Uwe Schramm, who spent several years as a senior researcher at the University of Virginia. Some of the structural matrices in the book were derived by another University of Virginia researcher, Dr. Weise Kang. Most of the text and figures were skillfully crafted by Wei Wei Ding, Timothy Preston, Check Kam, Adam Ziemba, and Rod Shirbacheh. Annie Frazer performed the calculations for several example problems. Instrumental in the preparation of this book has been the help of B. F. Pilkey.

WALTER PILKEY
This book deals with the extension, bending, and torsion of bars, especially thin-walled members. Although computational approaches for the analysis and design of bars are emphasized, traditional analytical solutions are included.

We begin with a study of the bending of beams, starting with a brief review of some of the fundamental concepts of the theory of linear elasticity. The theory of beams in bending is then treated from a strength-of-materials point of view. Both topics are treated more thoroughly in Pilkey and Wunderlich (1994). Atanackovic and Guran (2000), Boresi and Chong (1987), Gould (1994), Love (1944), and Sokolnikoff (1956) contain a full account of the theory of elasticity. References such as these should be consulted for the derivation of theory-of-elasticity relationships that are not derived in this chapter. Gere (2001), Oden and Ripperger (1981), Rivello (1969), and Uugural and Fenster (1981) may be consulted for a detailed development of beam theory.

1.1 REVIEW OF LINEAR ELASTICITY

The equations of elasticity for a three-dimensional body contain 15 unknown functions: six stresses, six strains, and three displacements. These functions satisfy three equations of equilibrium, six strain–displacement relations, and six stress–strain equations.

1.1.1 Kinematical Strain–Displacement Equations

The displacement vector \( \mathbf{u} \) at a point in a solid has the three components \( u_x(x, y, z) \), \( u_y(x, y, z) \), \( u_z(x, y, z) \) which are mutually orthogonal in a Cartesian coordinate system and are taken to be positive in the direction of the positive coordinate axes. In
vector notation,
\[ \mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = [u_x \\ u_y \\ u_z]^T \] (1.1)

Designate the normal strains by \( \epsilon_x, \epsilon_y, \) and \( \epsilon_z \) and the shear strains are \( \gamma_{xy}, \gamma_{xz}, \gamma_{yz}. \) The shear strains are symmetric (i.e., \( \gamma_{ij} = \gamma_{ji} \)). In matrix notation
\[ \mathbf{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = [\epsilon_x \\ \epsilon_y \\ \epsilon_z \\ 2\epsilon_{xy} \\ 2\epsilon_{xz} \\ 2\epsilon_{yz}]^T \] (1.2)

As indicated, \( \gamma_{ik} = 2\epsilon_{ik} \), where \( \gamma_{ik} \) is sometimes called the engineering shear strain and \( \epsilon_{ik} \) the theory of elasticity shear strain.

The linearized strain–displacement relations, which form the Cauchy strain tensor, are
\[ \epsilon_x = \frac{\partial u_x}{\partial x}, \quad \epsilon_y = \frac{\partial u_y}{\partial y}, \quad \epsilon_z = \frac{\partial u_z}{\partial z} \]
\[ \gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}, \quad \gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}, \quad \gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \] (1.3)

In matrix form Eq. (1.3) can be written as
\[ \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ \partial_y & \partial_x & 0 \\ \partial_z & 0 & \partial_x \\ 0 & \partial_z & \partial_y \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \] (1.4)

or
\[ \mathbf{\epsilon} = \mathbf{D} \mathbf{u} \]

with the differential operator matrix
\[ \mathbf{D} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ \partial_y & \partial_x & 0 \\ \partial_z & 0 & \partial_x \\ 0 & \partial_z & \partial_y \end{bmatrix} \] (1.5)
Six strain components are required to characterize the state of strain at a point and are derived from the three displacement functions $u_x$, $u_y$, $u_z$. The displacement field must be continuous and single valued, because it is being assumed that the body remains continuous after deformations have taken place. The six strain–displacement equations will not possess a single-valued solution for the three displacements if the strains are arbitrarily prescribed. Thus, the calculated displacements could possess tears, cracks, gaps, or overlaps, none of which should occur in practice. It appears as though the strains should not be independent and that they should be required to satisfy special conditions. To find relationships between the strains, differentiate the expression for the shear strain $\gamma_{xy}$ with respect to $x$ and $y$,

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} \frac{\partial u_x}{\partial y} + \frac{\partial^2}{\partial x \partial y} \frac{\partial u_y}{\partial x}$$  \hspace{1cm} (1.6)$$

According to the calculus, a single-valued continuous function $f$ satisfies the condition

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$  \hspace{1cm} (1.7)$$

With the assistance of Eq. (1.7), Eq. (1.6) may be rewritten, using the strain–displacement relations, as

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2}$$  \hspace{1cm} (1.8)$$

showing that the three strain components $\gamma_{xy}$, $\epsilon_x$, $\epsilon_y$ are not independent functions. Similar considerations that eliminate the displacements from the strain–displacement relations lead to five additional relations among the strains. These six relationships,

$$\begin{align*}
2 \frac{\partial^2 \epsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\
2 \frac{\partial^2 \epsilon_y}{\partial x \partial z} &= \frac{\partial}{\partial y} \left( -\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} \right) \\
2 \frac{\partial^2 \epsilon_z}{\partial x \partial y} &= \frac{\partial}{\partial z} \left( -\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial y} + \frac{\partial \gamma_{xz}}{\partial x} \right) \\
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} &= \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} \\
\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} &= \frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} \\
\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} &= \frac{\partial^2 \epsilon_z}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial z^2}
\end{align*}$$  \hspace{1cm} (1.9)
are known as the strain compatibility conditions or integrability conditions. Although there are six conditions, only three are independent.

1.1.2 Material Law

The kinematical conditions of Section 1.1.1 are independent of the material of which the body is made. The material is introduced to the formulation through a material law, which is a relationship between the stresses $\sigma$ and strains $\varepsilon$. Other names are the constitutive relations or the stress–strain equations.

Figure 1.1 shows the stress components that define the state of stress in a three-dimensional continuum. The quantities $\sigma_x, \sigma_y,$ and $\sigma_z$ designate stress components normal to a coordinate plane and $\tau_{xy}, \tau_{xz}, \tau_{yz}, \tau_{yx}, \tau_{zx},$ and $\tau_{zy}$ are the shear stress components. In the case of a normal stress, the single subscript indicates that the stress acts on a plane normal to the axis in the subscript direction. For the shear stresses, the first letter of the double subscript denotes that the plane on which the stress acts is normal to the axis in the subscript direction. The second subscript letter designates the coordinate direction in which the stress acts. As a result of the need to satisfy an equilibrium condition of moments, the shear stress components must be symmetric that is,

$$\tau_{xy} = \tau_{yx} \quad \tau_{xz} = \tau_{zx} \quad \tau_{yz} = \tau_{zy} \quad (1.10)$$

Then the state of stress at a point is characterized by six components. In matrix form,
\[ \sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{xz} & \tau_{yz} \end{bmatrix}^T \tag{1.11} \]

For a solid element as shown in Fig. 1.1, a face with its outward normal along the positive direction of a coordinate axis is defined to be a positive face. A face with its normal in the negative coordinate direction is defined as a negative face. Stress (strain) components on a positive face are positive when acting along a positive coordinate direction. The components shown in Fig. 1.1 are positive. Components on a negative face acting in the negative coordinate direction are defined to be positive.

An isotropic material has the same material properties in all directions. If the properties differ in various directions, such as with wood, the material is said to be anisotropic. A material is homogeneous if it has the same properties at every point. Wood is an example of a homogeneous material that can be anisotropic. A body formed of steel and aluminum portions is an example of a material that is inhomogeneous, but each portion is isotropic.

The stress–strain equations for linearly elastic isotropic materials are

\[
\begin{align*}
\epsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \\
\epsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z) \\
\epsilon_z &= \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{xz} &= \frac{\tau_{xz}}{G} \\
\gamma_{yz} &= \frac{\tau_{yz}}{G} \tag{1.12}
\end{align*}
\]

where \( E \) is the elastic or Young’s modulus, \( \nu \) is Poisson’s ratio, and \( G \) is the shear modulus. Only two of these three material properties are independent. The shear modulus is given in terms of \( E \) and \( \nu \) as

\[ G = \frac{E}{2(1 + \nu)} \tag{1.13} \]
Stresses may be written as a function of the strains by inverting the six relationships of Eq. (1.12) that express strains in terms of stresses. The result is

\[
\sigma_x = \lambda e + 2G\epsilon_x \\
\sigma_y = \lambda e + 2G\epsilon_y \\
\sigma_z = \lambda e + 2G\epsilon_z \\
\tau_{xy} = G\gamma_{xy} \\
\tau_{xz} = G\gamma_{xz} \\
\tau_{yz} = G\gamma_{yz}
\]

where \( e \) is the change in volume per unit volume, also called the dilatation,

\[
e = \epsilon_x + \epsilon_y + \epsilon_z
\]

and \( \lambda \) is Lamé’s constant,

\[
\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}
\]
For uniaxial tension, with the normal stress in the \( x \) direction given a constant positive value \( \sigma_0 \), and all other stresses set equal to zero,

\[
\sigma_x = \sigma_0 > 0 \quad \sigma_y = \sigma_z = \tau_{yx} = \tau_{yz} = \tau_{xz} = 0 \quad (1.19a)
\]

The normal strains are given by Hooke’s law as

\[
\epsilon_x = \frac{\sigma_0}{E} \quad \epsilon_y = \epsilon_z = -\frac{\nu \sigma_0}{E} \quad (1.19b)
\]

and the shear strains are all zero. Under this loading condition, the material undergoes extension in the axial direction \( x \) and contraction in the transverse directions \( y \) and \( z \). This shows that the material constants \( \nu \) and \( E \) are both positive:

\[
E > 0 \quad \nu > 0 \quad (1.20)
\]

In hydrostatic compression \( p \), the material is subjected to identical compressive stresses in all three coordinate directions:

\[
\sigma_x = \sigma_y = \sigma_z = -p \quad p > 0 \quad (1.21)
\]

while all shear stresses are zero. The dilatation under this loading condition is

\[
e = \frac{-3p}{3\lambda + 2G} = \frac{-3p(1 - 2\nu)}{E} \quad (1.22)
\]

Since the volume change in hydrostatic compression is negative, this expression for \( e \) implies that Poisson’s ratio must be less than \( \frac{1}{2} \):

\[
\nu < \frac{1}{2} \quad (1.23)
\]

and the following properties of the elastic constants are established:

\[
E > 0 \quad G > 0 \quad \lambda > 0 \quad 0 < \nu < \frac{1}{2} \quad (1.24)
\]

Materials for which \( \nu \approx 0 \) and \( \nu \approx \frac{1}{2} \) are very compressible or very incompressible, respectively. Cork is an example of a very compressible material, whereas rubber is very incompressible.

### 1.1.3 Equations of Equilibrium

Equilibrium at a point in a solid is characterized by a relationship between internal (volume or body) forces \( \overline{p}_V \), such as those generated by gravity or acceleration, and differential equations involving stress. Prescribed forces are designated with a bar placed over a letter. These equilibrium or static relations appear as

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \overline{p}_V = 0
\]
\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \overline{p}_{Vy} = 0 \quad (1.25)
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \overline{p}_{Vz} = 0
\]

where \(\overline{p}_{Vx}, \overline{p}_{Vy}, \overline{p}_{Vz}\) are the body forces per unit volume. In matrix form,

\[
\begin{bmatrix}
\partial_x & 0 & 0 & \cdot & \partial_y & \partial_z & 0 \\
0 & \partial_y & 0 & \cdot & \partial_x & 0 & \partial_z \\
0 & 0 & \partial_z & 0 & \partial_x & \partial_y & 0 \\
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yz} \\
\end{bmatrix}
+ \begin{bmatrix}
\overline{p}_{Vx} \\
\overline{p}_{Vy} \\
\overline{p}_{Vz} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
(1.26)
\]

where the matrix of differential operators \(D^T\) is the transpose of the \(D\) of Eq. (1.5). These relationships are derived in books dealing with the theory of elasticity and, also, in many basic strength-of-materials textbooks.

### 1.1.4 Surface Forces and Boundary Conditions

The forces applied to a surface (i.e., the boundary) of a body must be in equilibrium with the stress components on the surface. Let \(S_p\) denote the part of the surface of the body on which forces are prescribed, and let displacements be specified on the remaining surface \(S_u\). The surface conditions on \(S_p\) are

\[
\begin{align*}
p_x &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\
p_y &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\
p_z &= \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z
\end{align*}
(1.27)
\]

where \(n_x, n_y, n_z\) are the components of the unit vector \(\mathbf{n}\) normal to the surface and \(p_x, p_y, p_z\) are the surface forces per unit area.

In matrix form,

\[
\begin{bmatrix}
p_x \\
p_y \\
p_z \\
\end{bmatrix}
= \begin{bmatrix}
n_x & 0 & 0 & \cdot & n_y & n_z & 0 \\
0 & n_y & 0 & \cdot & n_x & 0 & n_z \\
0 & 0 & n_z & 0 & n_x & n_y & 0 \\
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yz} \\
\end{bmatrix}
= \begin{bmatrix}
\overline{p}_{Vx} \\
\overline{p}_{Vy} \\
\overline{p}_{Vz} \\
\end{bmatrix}
(1.28)
\]
Note that $\mathbf{N}^T$ is similar in form to $\mathbf{D}^T$ of Eq. (1.26) in that the components of $\mathbf{N}^T$ correspond to the derivatives of $\mathbf{D}^T$. The relations of Eq. (1.27) are referred to as Cauchy’s formula.

Surface forces (per unit area) $\mathbf{p}$ applied externally are called prescribed surface tractions $\mathbf{p}$. Equilibrium demands that the resultant stress be equal to the applied surface tractions $\mathbf{p}$ on $S_p$:

$$\mathbf{p} = \mathbf{p} \quad \text{on} \quad S_p$$

(1.29)

These are the static (force, stress, or mechanical) boundary conditions. Continuity requires that on the surface $S_u$, the displacements $\mathbf{u}$ be equal to the specified displacements $\mathbf{u}$:

$$\mathbf{u} = \mathbf{u} \quad \text{on} \quad S_u$$

(1.30)

These are the displacement (kinematic) boundary conditions.

**Unit Vectors on a Boundary Curve** It is helpful to identify several useful relationships between vectors on a boundary curve. Consider a boundary curve lying in the $yz$ plane as shown in Fig. 1.2a. The vector $\mathbf{n}$ is the unit outward normal $\mathbf{n} = n_y \mathbf{j} + n_z \mathbf{k}$ and $\mathbf{t}$ is the unit tangent vector $\mathbf{t} = t_y \mathbf{j} + t_z \mathbf{k}$, where $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the $y$ and $z$ axes. The quantity $s$, the coordinate along the arc of the boundary, is chosen to increase in the counterclockwise sense. As shown in Fig. 1.2a, the unit tangent vector $\mathbf{t}$ is directed along increasing $s$. Since $\mathbf{n}$ and $\mathbf{t}$ are unit vectors, $n_y^2 + n_z^2 = 1$ and $t_y^2 + t_z^2 = 1$. The components of $\mathbf{n}$ are its direction cosines, that is, from Fig. 1.2b,

$$n_y = \cos \theta_y \quad \text{and} \quad n_z = \cos \theta_z$$

(1.31)

since, for example, $\cos \theta_y = n_y / \sqrt{n_y^2 + n_z^2} = n_y$.

From Fig. 1.2c it can be observed that

$$\cos \varphi = n_y \quad \sin \varphi = n_z$$

$$\sin \varphi = -t_y \quad \cos \varphi = t_z$$

(1.32)

As a consequence,

$$n_y = t_z \quad n_z = -t_y$$

(1.33)

and the unit outward normal is defined in terms of the components $t_y$ and $t_z$ of the unit tangent as

$$\mathbf{n} = t_z \mathbf{j} - t_y \mathbf{k} = \mathbf{t} \times \mathbf{i}$$

(1.34)
From Fig. 1.2d it is apparent that

\[ \sin \varphi = -\frac{dy}{ds} \quad \text{and} \quad \cos \varphi = \frac{dz}{ds} \]  

(1.35)

Thus,

\[ n_y = t_z = \frac{dz}{ds} \quad n_z = -t_y = -\frac{dy}{ds} \]  

(1.36)

The vector \( \mathbf{r} \) to any point on the boundary is

\[ \mathbf{r} = y \mathbf{j} + z \mathbf{k} \]

Then

\[ d\mathbf{r} = dy \mathbf{j} + dz \mathbf{k} = \frac{d\mathbf{r}}{ds} \, ds = \left( \frac{dy}{ds} \mathbf{j} + \frac{dz}{ds} \mathbf{k} \right) ds = t \, ds \]  

(1.37)
1.1.5 Other Forms of the Governing Differential Equations

The general problem of the theory of elasticity is to calculate the stresses, strains, and displacements throughout a solid. The kinematic equations $\mathbf{e} = \mathbf{Du}$ (Eq. 1.4) are written in terms of six strains and three displacements, while the static equations $\mathbf{D}^T \mathbf{\sigma} + \mathbf{p}_V = \mathbf{0}$ (Eq. 1.26) are expressed as functions of the six stress components. The constitutive equations $\mathbf{\sigma} = \mathbf{Ee}$ (Eq. 1.18) are relations between the stresses and strains. The boundary conditions of Eqs. (1.29) and (1.30) need to be satisfied by the solution for the 15 unknowns.

In terms of achieving solutions, it is useful to derive alternative forms of the governing equations. The elasticity problem can be formulated in terms of the displacement functions $u_x, u_y, u_z$. The stress–strain equations allow the equilibrium equations to be written in terms of the strains. When the strains are replaced in the resulting equations by the expressions given by the strain–displacement relations, the equilibrium equations become a set of partial differential equations for the displacements. Thus, substitute $\mathbf{e} = \mathbf{Du}$ into $\mathbf{\sigma} = \mathbf{Ee}$ to give the stress–displacement relations $\mathbf{\sigma} = \mathbf{EDu}$. The conditions of equilibrium become

\[
\mathbf{D}^T \mathbf{\sigma} + \mathbf{p}_V = \mathbf{D}^T \mathbf{EDu} + \mathbf{p}_V = \mathbf{0}
\] (1.38)

or, in scalar form,

\[
(\lambda + G) \frac{\partial e}{\partial x} + G\nabla^2 u_x + \mathbf{p}_{Vx} = 0
\]

\[
(\lambda + G) \frac{\partial e}{\partial y} + G\nabla^2 u_y + \mathbf{p}_{Vy} = 0
\] (1.39)

\[
(\lambda + G) \frac{\partial e}{\partial z} + G\nabla^2 u_z + \mathbf{p}_{Vz} = 0
\]

where $\nabla^2$ is the Laplacian operator

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\] (1.40)

The dilatation $e$ is a function of displacements

\[
e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla \cdot \mathbf{u}
\] (1.41)

where $\mathbf{u}$ is the displacement vector, whose components along the $x, y, z$ axes are $u_x, u_y, u_z$, and $\nabla$ is the gradient operator. The displacement vector is expressed as $\mathbf{u} = u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit base vectors along the coordinates $x, y, z$, respectively. The gradient operator appears as

\[
\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}
\] (1.42)
To complete the displacement formulation, the surface conditions on \( S_p \) must also be written in terms of the displacements. This is done by first writing these surface conditions of Eq. (1.27) in terms of strains using the material laws, and then expressing the strains in terms of the displacements, using the strain–displacement relations. The resulting conditions are

\[
\lambda en_x + G n \cdot \nabla u_x + G n \cdot \frac{\partial u}{\partial x} = p_x \\
\lambda en_y + G n \cdot \nabla u_y + G n \cdot \frac{\partial u}{\partial y} = p_y \\
\lambda en_z + G n \cdot \nabla u_z + G n \cdot \frac{\partial u}{\partial z} = p_z
\] (1.43)

where \( n = n_x i + n_y j + n_z k \). If boundary conditions exist for both \( S_p \) and \( S_u \), the boundary value problem is called mixed. The equations of equilibrium written in terms of the displacements together with boundary conditions on \( S_p \) and \( S_u \) constitute the displacement formulation of the elasticity problem. In this formulation, the displacement functions are found first. The strain–displacement relations then give the strains, and the material laws give the stresses.

### 1.2 BENDING STRESSES IN A BEAM IN PURE BENDING

A beam is said to be in pure bending if the force–couple equivalent of the stresses over any cross section is a couple \( M \) in the plane of the section

\[
M = M_y j + M_z k
\] (1.44)

![Figure 1.3 Beam in pure bending.](image-url)