Interest Rate, Term Structure, and valuation modeling

FRANK J. FABOZZI

EDITOR

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Interest Rate, Term Structure, and Valuation Modeling

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preface

The valuation of fixed-income securities and interest rate derivatives, from the most simple structures to the complex structures found in the structured finance and interest rate derivatives markets, depends on the interest rate model and term structure model used by the investor. *Interest Rate, Term Structure, and Valuation Modeling* provides a comprehensive practitioner-oriented treatment of the various interest rate models, term structure models, and valuation models.

The book is divided into three sections. Section One covers interest rate and term structure modeling. In Chapter 1, Oren Cheyette provides an overview of the principles of valuation algorithms and the characteristics that distinguish the various interest rate models. He then describes the empirical evidence on interest rate dynamics, comparing a family of interest rate models that closely match those in common use. The coverage emphasizes those issues that are of principal interest to practitioners in applying interest rate models. As Cheyette states: "There is little point in having the theoretically ideal model if it can't actually be implemented as part of a valuation algorithm."

In Chapter 2, Peter Fitton and James McNatt clarify some of the commonly misunderstood issues associated with interest rate models. Specifically, they focus on (1) the choice between an arbitrage-free and an equilibrium model and (2) the choice between risk neutral and realistic parameterizations of a model. Based on these choices, they classify interest rate models into four categories and then explain the proper use of each category of interest rate model.

Stochastic differential equations (SDE) are typically used to model interest rates. In a one-factor model, an SDE is used to represent the short rate; in two-factor models an SDE is used for both the short rate and the long rate. In Chapter 3 Gerald Buetow, James Sochacki, and I review no-arbitrage interest rate models highlighting some significant differences across models. The most significant differences are those due to the underlying distribution and, as we stress in the chapter, indicates the need to calibrate models to the market prior to their use. The models covered are the Ho-Lee model, the Hull-White model, the Kalotay-

Williams-Fabozzi model, and the Black-Derman-Toy model. The binomial and trinomial formulations of these models are presented.

Moorad Choudhry presents in Chapter 4 an accessible account of the various term structure theories that have been advanced to explain the shape of the yield curve at any time. While no one theory explains the term structure at all times, a combination of two of these serve to explain the yield curve for most applications.

In Chapter 5, David Audley, Richard Chin, and Shrikant Ramamurthy review the approaches to term structure modeling and then present an eclectic mixture of ideas for term structure modeling. After describing some fundamental concepts of the term structure of interest rates and developing a useful set of static term structure models, they describe the approaches to extending these into dynamic models. They begin with the discrete-time modeling approach and then build on the discussion by introducing the continuous-time analogies to the concepts developed for discrete-time modeling. Finally, Audley, Chin, and Ramamurthy describe the dynamic term structure model.

The swap term structure is a key benchmark for pricing and hedging purposes. In Chapter 6, Uri Ron details all the issues associated with the swap term structure derivation procedure. The approach presented by Ron leaves the user with enough flexibility to adjust the constructed term structure to the specific micro requirements and constraints of each primary swap market.

There have been several techniques proposed for fitting the term structure with the technique selected being determined by the requirements specified by the user. In general, curve fitting techniques can be classified into two types. The first type models the yield curve using a parametric function and is therefore referred to as a parametric technique. The second type uses a spline technique, a technique for approximating the market discount function. In Chapter 7, Rod Pienaar and Moorad Choudhry discuss the spline technique, focussing on cubic splines and how to implement the technique in practice.

Critical to an interest rate model is the assumed yield volatility or term structure of yield volatility. Volatility is measured in terms of the standard deviation or variance. In Chapter 8, Wai Lee and I look at how to measure and forecast yield volatility and the implementation issues related to estimating yield volatility using observed daily percentage changes in yield. We then turn to models for forecasting volatility, reviewing the latest statistical techniques that can be employed.

The three chapters in Section Two explain how to quantify fixed-income risk. Factor models are used for this purpose. Empirical evidence indicates that the change in the level and shape of the yield curve are the major source of risk for a fixed-income portfolio. The risk associated with changes in the level and shape of the yield curve are referred to as term structure risk. In Chapter 9, Robert Kuberek reviews some of the leading approaches to term structure factor modeling (arbitrage models, principal component models, and spot rate and functional models), provides the examples of each type of term structure factor model, and explains the advantages and disadvantages of each.

While the major source of risk for a fixed-income portfolio is term structure risk, there are other sources of risk that must be accounted for in order to assess a portfolio's risk profile relative to a benchmark index. These non-term structure risks include sector risk, optionality risk, prepayment risk, quality risk, and volatility risk. Moreover, the risk of a portfolio relative to a benchmark index is measured in terms of tracking risk. In Chapter 10, Lev Dynkin and Jay Hyman present a multi-factor risk model that includes all of these risks and demonstrates how the model can be used to construct a portfolio, rebalance a portfolio, and control a portfolio's risk profile relative to a benchmark.

A common procedure used by portfolio and risk managers to assess the risk of a portfolio is to shift or "shock" the yield curve. The outcome of this analysis is an assessment of a portfolio's exposure to term structure risk. However, there is a wide range of potential yield curve shocks that a manager can analyze. In Chapter 11, Bennet Golub and Leo Tilman provide a framework for defining and measuring the historical plausibility of a given yield curve shock.

Section Three covers the approaches to valuation and the measurement of option-adjusted spread (OAS). Valuation models are often referred to as OAS models. In the first chapter of Section III, Chapter 12, Philip Obazee explains the basic building blocks for a valuation model.

In Chapter 13, Andrew Kalotay, Michael Dorigan, and I demonstrate how an arbitrage-free interest rate lattice is constructed and how the lattice can be used to value an option-free bond. In Chapter 14, we apply the lattice-based valuation approach to the valuation of bonds with embedded options (callable bonds and putable bonds), floaters, options, and caps/floors. In Chapter 15, Gerald Buetow and I apply the lattice-based valuation approach to value forward start swaps and swaptions. A methodology for applying the lattice-based valuation approach to value path-dependent securities is provided by Douglas Howard in Chapter 16.

The Monte Carlo simulation approach to valuing residential mortgage-backed securities—agency products (passthroughs, collateralized mortgage obligations, and mortgage strips), nonagency products, and real-estate backed asset-backed securities (home equity loan and manufactured housing loan-backed deals) is demonstrated by Scott Richard, David Horowitz, and me in Chapter 17. An alternative to the Monte Carlo simulation approach for valuing mortgage products is presented in Chapter 18 by Alexander Levin. The approach he suggests uses low-dimensional grids.

In the last chapter, Chapter 19, the effect of mean reversion on the value of a security and the option-adjusted spread is discussed by David Audley and Richard Chin.

I believe this book will be a valuable reference source for practitioners who need to understand the critical elements in the valuation of fixed-income securities and interest rate derivatives and the measurement of interest rate risk.

I wish to thank the authors of the chapters for their contributions. A book of this type by its very nature requires the input of specialists in a wide range of technical topics and I believe that I have assembled some of the finest in the industry.

Frank J. Fabozzi

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SECTION one

Interest Rate and Term Structure Modeling

CHAPTER 1

Interest Rate Models

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An interest rate model is a probabilistic description of the future evolution of interest rates. Based on today's information, future interest rates are uncertain: An interest rate model is a characterization of that uncertainty. Quantitative analysis of securities with rate dependent cash flows requires application of such a model in order to find the present value of the uncertainty. Since virtually all financial instruments other than default- and option-free bonds have interest rate sensitive cash flows, this matters to most fixed-income portfolio managers and actuaries, as well as to traders and users of interest rate derivatives.

For financial instrument valuation and risk estimation one wants to use only models that are arbitrage free and matched to the currently observed term structure of interest rates. "Arbitrage free" means just that if one values the same cash flows in two different ways, one should get the same result. For example, a 10-year bond putable at par by the holder in 5 years can also be viewed as a 5-year bond with an option of the holder to extend the maturity for another 5 years. An arbitrage-free model will produce the same value for the structure viewed either way. This is also known as the *law of one price*. The term structure matching condition means that when a default-free straight bond is valued according to the model, the result should be the same as if the bond's cash flows are simply discounted according to the current default-free term structure. A model that fails to satisfy either of these conditions cannot be trusted for general problems, though it may be usable in some limited context.

For equity derivatives, lognormality of prices (leading to the Black-Scholes formula for calls and puts) is the standard starting point for option calculations. In the fixed-income market, unfortunately, there is no equally natural and simple assumption. Wall Street dealers routinely use a multiplicity of models based on widely varying assumptions in different markets. For example, an options desk most likely uses a version of the Black formula to value interest rate caps and floors, implying an approximately lognormal distribution of interest rates. A few feet away, the mortgage desk may use a normal interest rate model to evaluate their passthrough and CMO durations. And on the next floor, actuaries may use variants of both types of models to analyze their annuities and insurance policies.

It may seem that one's major concern in choosing an interest rate model should be the accuracy with which it represents the empirical volatility of the term structure of rates, and its ability to fit market prices of vanilla derivatives such as at-the-money caps and swaptions. These are clearly important criteria, but they are not decisive. The first criterion is hard to pin down, depending strongly on what historical period one chooses to examine. The second criterion is easy to satisfy for most commonly used models, by the simple (though unappealing) expedient of permitting predicted future volatility to be time dependent. So, while important, this concern doesn't really do much to narrow the choices.

A critical issue in selecting an interest rate model is, instead, ease of application. For some models it is difficult or impossible to provide efficient valuation algorithms for all financial instruments of interest to a typical investor. Given that one would like to analyze all financial instruments using the same underlying assumptions, this is a significant problem. At the same time, one would prefer not to stray too far from economic reasonableness—such as by using the Black-Scholes formula to value callable bonds. These considerations lead to a fairly narrow menu of choices among the known interest rate models.

The organization of this chapter is as follows. In the next section I provide a (brief) discussion of the principles of valuation algorithms. This will give a context for many of the points made in the third section, which provides an overview of the various characteristics that differentiate interest rate models. Finally, in the fourth section I describe the empirical evidence on interest rate dynamics and provide a quantitative comparison of a family of models that closely match those in common use. I have tried to emphasize those issues that are primarily of interest for application of the models in practical settings. There is little point in having the theoretically ideal model if it can't actually be implemented as part of a valuation algorithm.

VALUATION

Valuation algorithms for rate dependent contingent claims are usually based on a risk neutral formula, which states that the present value of an uncertain cash flow at time T is given by the average over all interest rate scenarios of the scenario cash flow divided by the scenario value at time T of a money market investment of \$1 today.[1] More formally, the value of a security is given by the expectation (average) over interest rate scenarios

$$P = E\left[\sum_i \frac{C_i}{M_i}\right] \tag{1}$$

where C_i is the security's cash flows and M_i is the money market account value at time t_i in each scenario, calculated by assuming continual reinvestment at the prevailing short rate.

The probability weights used in the average are chosen so that the expected rate of return on any security over the next instant is the same, namely the short rate. These are the so-called "risk neutral" probability weights: They would be the true weights if investors were indifferent to bearing interest rate risk. In that case, investors would demand no excess return relative to a (riskless) money market account in order to hold risky positions—hence equation (1).

It is important to emphasize that the valuation formula is not dependent on any *assumption* of risk neutrality. Financial instruments are valued by equation (1) *as if* the market were indifferent to interest rate risk *and* the correct discount factor for a future cash flow were the inverse of the money market return. Both statements are false for the real world, but the errors are offsetting: A valuation formula based on probabilities implying a nonzero market price of interest rate risk and the corresponding scenario discount factors would give the same value.

There are two approaches to computing the average in equation (1): by direct brute force evaluation, or indirectly by solving a related differential equation. The brute force method is usually called the Monte Carlo method. It consists of generating a large number of possible interest rate scenarios based on the interest rate model, computing the cash flows and money market values in each one, and averaging. Properly speaking, only path generation based on random numbers is a Monte Carlo method. There are other scenario methods—e.g., complete sampling of a tree—that do not depend on the use of random numbers.

[1] The money market account is the *numeraire*.

Given sufficient computer resources, the scenario method can tackle essentially any type of financial instrument.[2]

A variety of schemes are known for choosing scenario sample paths efficiently, but none of them are even remotely as fast and accurate as the second technique. In certain cases (discussed in more detail in the next section) the average in equation (1) obeys a partial differential equation—like the one derived by Black and Scholes for equity options—for which there exist fast and accurate numerical solution methods, or in special cases even analytical solutions. This happens only for interest rate models of a particular type, and then only for certain security types, such as caps, floors, swaptions, and options on bonds. For securities such as mortgage passthroughs, CMOs, index amortizing swaps, and for some insurance policies and annuities, simulation methods are the only alternative.

MODEL TAXONOMY

The last two decades have seen the development of a tremendous profusion of models for valuation of interest rate sensitive financial instruments. In order to better understand these models, it is helpful to recognize a number of features that characterize and distinguish them. These are features of particular relevance to practitioners wishing to implement valuation algorithms, as they render some models completely unsuitable for certain types of financial instruments.[3] The following subsections enumerate some of the major dimensions of variation among the different models.

One- versus Multi-Factor

In many cases, the value of an interest rate contingent claim depends, effectively, on the prices of many underlying assets. For example, while the payoff of a caplet depends only on the reset date value of a zero coupon bond maturing at the payment date (valued based on, say, 3-month LIBOR), the payoff to an option on a coupon bond depends on the exercise date values of all of the bond's remaining interest and principal payments. Valuation of such an option is in principle an inherently multidimensional problem.

Fortunately, in practice these values are highly correlated. The degree of correlation can be quantified by examining the covariance matrix of

[2] This is true even for American options. For a review see P. Boyle, M. Broadie, and P. Glasserman, "Monte Carlo Methods for Security Pricing," *Journal of Economic Dynamics and Control* (1997), pp. 1267–1322.

[3] There is, unfortunately, a version of Murphy's law applicable to interest rate models, which states that the computational tractability of a model is inversely proportional to its economic realism.

changes in spot rates of different maturities. A principal component analysis of the covariance matrix decomposes the motion of the spot curve into independent (uncorrelated) components. The largest principal component describes a common shift of all interest rates in the same direction. The next leading components are a twist, with short rates moving one way and long rates the other, and a "butterfly" motion, with short and long rates moving one way, and intermediate rates the other. Based on analysis of weekly data from the Federal Reserve H15 series of benchmark Treasury yields from 1983 through 1995, the shift component accounts for 84% of the total variance of spot rates, while twist and butterfly account for 11% and 4%, leaving about 1% for all remaining principal components.

The shift factor alone explains a large fraction of the overall movement of spot rates. As a result, valuation can be reduced to a one factor problem in many instances with little loss of accuracy. Only securities whose payoffs are primarily sensitive to the shape of the spot curve rather than its overall level (such as dual index floaters, which depend on the difference between a long and a short rate) will not be modeled well with this approach.

In principle it is straightforward to move from a one-factor model to a multi-factor one. In practice, though, implementations of multi-factor valuation models can be complicated and slow, and require estimation of many more volatility and correlation parameters than are needed for one-factor models, so there may be some benefit to using a one-factor model when possible. The remainder of this chapter will focus on one-factor models.[4]

Exogenous versus Endogenous Term Structure

The first interest rate models were not constructed so as to fit an arbitrary initial term structure. Instead, with a view towards analytical simplicity, the Vasicek[5] and Cox-Ingersoll-Ross[6] (CIR) models contain a few constant parameters that define an endogenously specified term structure. That is, the initial spot curve is given by an analytical formula in terms of the model parameters. These are sometimes also called "equilibrium" models, as they posit yield curves derived from an assumption of

[4] For an exposition of two-factor models, see D.F. Babbel and C.B. Merrill, *Valuation of Interest Sensitive Financial Instruments* (New Hope, PA: Frank J. Fabozzi Associates and Society of Actuaries, 1996).

[5] O. Vasicek, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics* (November 1977).

[6] J.C. Cox, J.E. Ingersoll Jr., and S.A. Ross, "A Theory of the Term Structure of Interest Rates," *Econometrica* (March 1985).

economic equilibrium based on a given market price of risk and other parameters governing collective expectations.

For *dynamically* reasonable choices of the parameters—values that give plausible long-run interest rate distributions and option prices—the term structures achievable in these models have far too little curvature to accurately represent typical empirical spot rate curves. This is because the mean reversion parameter, governing the rate at which the short rate reverts towards the long-run mean, also governs the volatility of long-term rates relative to the volatility of the short rate—the "term structure of volatility." To achieve the observed level of long-rate volatility (or to price options on long-term securities well) requires that there be relatively little mean reversion, but this implies low curvature yield curves. This problem can be partially solved by moving to a multi-factor framework—but at a significant cost as discussed earlier. These models are therefore not particularly useful as the basis for valuation algorithms—they simply have too few degrees of freedom to faithfully represent real markets.

To be used for valuation, a model must be calibrated to the initial spot rate curve. That is, the model structure must accommodate an exogenously determined spot rate curve, typically given by fitting to bond prices, or sometimes to futures prices and swap rates. All models in common use are of this type.

There is a "trick" invented by Dybvig that converts an endogenous model to a calibrated exogenous one.[7] The trick can be viewed as splitting the nominal interest rate into two parts: the stochastic part modeled endogenously, and a non-stochastic drift term, which compensates for the mismatch of the endogenous term structure and the observed one. (BARRA has used this technique to calibrate the CIR model in its older fixed-income analytics.) The price of this method is that the volatility function is no longer a simple function of the nominal interest rate.

Short Rate versus Yield Curve

The risk neutral valuation formula requires that one know the sequence of short rates for each scenario, so an interest rate model must provide this information. For this reason, many interest rate models are simply models of the stochastic evolution of the short rate. A second reason for the desirability of such models is that they have the *Markov property*, meaning that the evolution of the short rate at each instant depends only on its current value—not on how it got there. The practical significance of this is that, as alluded to in the previous section, the valuation prob-

[7] P. Dybvig, "Bond and Bond Option Pricing Based on the Current Term Structure," in M. A. H. Dempster and S. Pliska (eds.), *Mathematics of Derivative Securities* (Cambridge, U.K.: Cambridge University Press, 1997).

lem for many types of financial instruments can be reduced to solving a partial differential equation, for which there exist efficient analytical and numerical techniques. To be amenable to this calculation technique, a financial instrument's cash flow at time t must depend only on the state of affairs at that time, not on how the evolution occurred prior to t, or it must be equivalent to a portfolio of such securities (for example, a callable bond is a position long a straight bond and short a call option).

Short-rate models have two parts. One specifies the average rate of change ("drift") of the short rate at each instant; the other specifies the instantaneous volatility of the short rate. The conventional notation for this is

$$dr(t) = \mu(r, t)dt + \sigma(r, t)dz(t) \tag{2}$$

The left-hand side of this equation is the change in the short rate over the next instant. The first term on the right is the drift multiplied by the size of the time step. The second is the volatility multiplied by a normally distributed random increment. For most models, the drift component must be determined through a numerical technique to match the initial spot rate curve, while for a small number of models there exists an analytical relationship. In general, there exists a no-arbitrage relationship linking the initial forward rate curve, the volatility $\sigma(r,t)$, the market price of interest rate risk, and the drift term $\mu(r,t)$. However, since typically one must solve for the drift numerically, this relationship plays no role in model construction. Differences between models arise from different dependences of the drift and volatility terms on the short rate.

For financial instruments whose cash flows don't depend on the interest rate history, the expectation formula (1) for present value obeys the Feynman-Kac equation

$$\frac{1}{2}\sigma^2 P_{rr} + (\mu - \lambda)P_r + P_t - rP + c = 0 \tag{3}$$

where, for example, P_r denotes the partial derivative of P with respect to r, c is the payment rate of the financial instrument, and λ, which can be time and rate dependent, is the market price of interest rate risk.

The terms in this equation can be understood as follows. In the absence of uncertainty ($\sigma = 0$), the equation involves four terms. The last three assert that the value of the security increases at the risk-free rate (rP), and decreases by the amount of any payments (c). The term $(\mu - \lambda)P_r$ accounts for change in value due to the change in the term structure with time, as rates move up the forward curve. In the absence of uncertainty it is easy to express $(\mu - \lambda)$ in terms of the initial forward rates. In the presence of uncertainty this term depends on the volatility as well, and we also have the first term, which is the main source of the complexity of valuation models.

The Vasicek and CIR models are models of the short rate. Both have the same form for the drift term, namely a tendency for the short rate to rise when it is below the long-term mean, and fall when it is above. That is, the short-rate drift has the form $\mu = \kappa(\theta - r)$, where r is the short rate and κ and θ are the mean reversion and long-term rate constants. The two models differ in the rate dependence of the volatility: it is constant (when expressed as points per year) in the Vasicek model, and proportional to the square root of the short rate in the CIR model.

The Dybvig-adjusted Vasicek model is the mean reverting generalization of the Ho-Lee model,[8] also known as the mean reverting Gaussian (MRG) model or the Hull-White model.[9] The MRG model has particularly simple analytical expressions for values of many assets—in particular, bonds and European options on bonds. Like the original Vasicek model, it permits the occurrence of negative interest rates with positive probability. However, for typical initial spot curves and volatility parameters, the probability of negative rates is quite small.

Other popular models of this type are the Black-Derman-Toy (BDT)[10] and Black-Karasinski[11] (BK) models, in which the volatility is proportional to the short rate, so that the ratio of volatility to rate level is constant. For these models, unlike the MRG and Dybvig-adjusted CIR models, the drift term is not simple. These models require numerical fitting to the initial interest rate and volatility term structures. The drift term is therefore not known analytically. In the BDT model, the short-rate volatility is also linked to the mean reversion strength (which is also generally time dependent) in such a way that—in the usual situation where long rates are less volatile than the short rate—the short-rate volatility decreases in the future. This feature is undesirable: One doesn't want to link the observation that the long end of the curve has relatively low volatility to a forecast that in the future the short rate will

[8] T.S.Y. Ho and S.B. Lee, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance* (December 1986); and, J. Hull and A. White, "Pricing Interest Rate Derivative Securities," *The Review of Financial Studies*, 3:4 (1990).

[9] This model was also derived in F. Jamshidian, "The One-Factor Gaussian Interest Rate Model: Theory and Implementation," Merrill Lynch working paper, 1988.

[10] F. Black, E. Derman and W. Toy, "A One Factor Model of Interest Rates and its Application to Treasury Bond Options," *Financial Analysts Journal* (January/February 1990).

[11] F. Black and P. Karasinski, "Bond and Option Prices when Short Rates are Lognormal," *Financial Analysts Journal* (July/August 1992).

become less volatile. This problem motivated the development of the BK model in which mean reversion and volatility are delinked.

All of these models are explicit models of the short rate alone. It happens that in the Vasicek and CIR models (with or without the Dybvig adjustment) it is possible to express the entire forward curve as a function of the current short rate through fairly simple analytical formulas. This is not possible in the BDT and BK models, or generally in other models of short-rate dynamics, other than by highly inefficient numerical techniques. Indeed, it is possible to show that the only short-rate models consistent with an arbitrary initial term structure for which one can find the whole forward curve analytically are in a class that includes the MRG and Dybvig-adjusted CIR models as special cases, namely where the short-rate volatility has the form[12]

$$\sigma(r, t) = \sqrt{\sigma_1(t) + \sigma_2(t) r}\,.$$

While valuation of certain assets (e.g., callable bonds) does not require knowledge of longer rates, there are broad asset classes that do. For example, mortgage prepayment models are typically driven off a long-term Treasury par yield, such as the 10-year rate. Therefore a generic short-rate model such as BDT or BK is unsuitable if one seeks to analyze a variety of assets in a common interest rate framework.

An alternative approach to interest rate modeling is to specify the dynamics of the entire term structure. The volatility of the term structure is then given by some specified function, which most generally could be a function of time, maturity, and spot rates. A special case of this approach (in a discrete time framework) is the Ho-Lee model mentioned earlier, for which the term structure of volatility is a parallel shift of the spot rate curve, whose magnitude is independent of time and the level of rates. A completely general continuous time, multi-factor framework for constructing such models was given by Heath, Jarrow, and Morton (HJM).[13]

It is sometimes said that all interest rate models are HJM models. This is technically true: In principle, every arbitrage-free model of the term structure can be described in their framework. In practice, however, it is impossible to do this analytically for most short-rate Markov models. The only ones for which it is possible are those in the MRG-CIR family described

[12] A. Jeffrey, "Single Factor Heath-Jarrow-Morton Term Structure Models Based on Markov Spot Interest Rate Dynamics," *Journal of Financial and Quantitative Analysis*, 30:4 (December 1995).

[13] D. Heath, R. Jarrow, and A. Morton, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 60:1 (January 1992).

earlier. The BDT and BK models, for instance, cannot be translated to the HJM framework other than by impracticable numerical means. To put a model in HJM form, one must know the term structure of volatility at all times, and this is generally not possible for short-rate Markov models.

If feasible, the HJM approach is clearly very attractive, since one knows now not just the short rate but also all longer rates as well. In addition, HJM models are very "natural," in the sense that the basic inputs to the model are the initial term structure of interest rates and a term structure of interest rate volatility for each independent motion of the yield curve.

The reason for the qualification in the last paragraph is that a generic HJM model requires keeping track of a potentially enormous amount of information. The HJM framework imposes no structure other than the requirement of no-arbitrage on the dynamics of the term structure. Each forward rate of fixed maturity evolves separately, so that one must keep track of each one separately. Since there are an infinite number of distinct forward rates, this can be difficult. This difficulty occurs even in a one factor HJM model, for which there is only one source of random movement of the term structure. A general HJM model does not have the Markov property that leads to valuation formulas expressed as solutions to partial differential equations. This makes it impossible to accurately value interest rate options without using huge amounts of computer time, since one is forced to use simulation methods.

In practice, a simulation algorithm breaks the evolution of the term structure up into discrete time steps, so one need keep track of and simulate only forward rates for the finite set of simulation times. Still, this can be a large number (e.g., 360 or more for a mortgage passthrough), and this computational burden, combined with the inefficiency of simulation methods, has prevented general HJM models from coming into more widespread use.

Some applications require simulation methods because the assets' structures (e.g., mortgage-backed securities) are not compatible with differential equation methods. For applications where one is solely interested in modeling such assets, there exists a class of HJM models that significantly simplify the forward rate calculations.[14] The simplest version of such models, the "two state Markov model," permits an arbitrary dependence of short-rate volatility on both time and the level of interest rates, while the ratio of forward-rate volatility to short-rate volatility is solely a function of term. That is, the volatility of $f(t,T)$, the term T forward rate at time t takes the form

[14] O. Cheyette, "Term Structure Dynamics and Mortgage Valuation," *Journal of Fixed Income* (March 1992). The two state Markov model was also described in P. Ritchken and L. Sankarasubramanian, "Volatility Structure of Forward Rates and the Dynamics of the Term Structure," *Mathematical Finance*, 5(1) (1995), pp. 55–72.

$$\sigma_f(r, t, T) = \sigma(r, t)e^{-\int_t^T k(u)du} \tag{4}$$

where $\sigma(r,t) = \sigma_f(r,t,t)$ is the short-rate volatility and $k(t)$ determines the mean reversion rate or equivalently, the rate of decrease of forward rate volatility with term. The evolution of all forward rates in this model can be described in terms of two state variables: the short rate (or any other forward or spot rate), and the slope of the forward curve at the origin. The second variable can be expressed in terms of the total variance experienced by a forward rate of fixed maturity by the time it has become the short rate. The stochastic evolution equations for the two state variables can be written as

$$\begin{aligned} d\tilde{r}(t) &= (V(t) - k(t)\tilde{r})dt + \sigma(r, t)dz(t) \\ \frac{\partial V}{\partial t} &= \sigma^2(r, t) - 2k(t)V(t) \end{aligned} \tag{5}$$

where $\tilde{r}(t) \equiv r(t) - f(0, t)$ is the deviation of the short rate from the initial forward rate curve. The state variable $V(t)$ has initial value $V(0)=0$; its evolution equation is non-stochastic and can be integrated to give

$$V(t) = \int_0^t \sigma_f^2(r, s, t)ds = \int_0^t \sigma^2(r, s)e^{-2\int_s^t k(u)du} ds \tag{6}$$

In terms of these state variables, the forward curve is given by

$$f(t, T) = f(0, T) + \phi(t, T)\left(\tilde{r} + V(t)\int_t^T \phi(t, s)ds\right) \tag{7}$$

where

$$\phi(t, T) = \sigma_f(r, t, T)/\sigma_f(r, t, t) = e^{-\int_t^T k(s)ds}$$

is a deterministic function.

Instead of having to keep track of hundreds of forward rates, one need only model the evolution of the two state variables. Path independent asset prices also obey a partial differential equation in this model, so it appears possible, at least in principle, to use more efficient numerical methods. The equation, analogous to equation (3), is

$$\frac{1}{2}\sigma^2 P_{\tilde{r}\tilde{r}} + (V - k\tilde{r})P_{\tilde{r}} + (\sigma^2 - 2kV)P_V + P_t - rP + c = 0\,. \tag{8}$$

Unlike equation (3), for which one must use the equation itself applied to bonds to solve for the coefficient μ–λ, here the coefficient functions are all known in terms of the initial data: the short-rate volatility and the initial forward curve. This simplification has come at the price of adding a dimension, as we now have to contend also with a term involving the first derivative with respect to *V*, and so the equation is much more difficult to solve efficiently by standard techniques.

In the special case where $\sigma(r,t)$ is independent of *r*, this model is the MRG model mentioned earlier. In this case, *V* is a deterministic function of *t*, so the P_V term disappears from equation (8), leaving a two-dimensional equation that has analytical solutions for European options on bonds, and straightforward numerical techniques for valuing American bond options. Since bond prices are lognormally distributed in this model, it should be no surprise that the formula for options on pure discount bounds (PDB's) looks much like the Black-Scholes formula. The value of a call with strike price *K*, exercise date *t* on a PDB maturing at time *T* is given by

$$C = P(T)N(h_1) - KP(t)N(h_2)\,, \tag{9}$$

where

$$h_1 = \frac{k}{(1 - e^{-k(T-t)})\sqrt{V(t)}}\ln\frac{P(T)}{KP(t)} + \frac{\sqrt{V(t)}(1 - e^{-k(T-t)})}{2k}\,,$$

$$h_2 = h_1 - \frac{\sqrt{V(t)}(1 - e^{-k(T-t)})}{k}\,,$$

$N(x)$ is the Gaussian distribution, and $P(t)$ and $P(T)$ are prices of PDB's maturing at *t* and *T*. (The put value can be obtained by put-call parity.) Options on coupon bonds can be valued by adding up a portfolio of options on PDBs, one for each coupon or principal payment after the exercise date, with strike prices such that they are all at-the-money at