Investment Guarantees

Modeling and Risk Management for Equity-Linked Life Insurance

MARY HARDY

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M. H.

Contents

Introduction

This book is designed for all practitioners working in equity-linked insurance, whether in product design, marketing, pricing and valuation, or risk management. It is written with actuaries in mind, but it should also be interesting to other investment professionals. The material in this book forms the basis of a one-semester graduate course for students of actuarial science, insurance, and finance. The aim is to provide a comprehensive and self-contained introduction to modeling and risk management for equity-linked life insurance. A feature of the book is the combination of econometric analysis of investment models with their application in pricing and risk management.

The focus is on the stochastic modeling of embedded guarantees that depend on equity performance. In the major part of the book the contracts that are used to illustrate the methods are single premium, separate account products. This class includes variable annuities in the United States, segregated fund contracts in Canada, and unit-linked contracts in the United Kingdom. The investment guarantees associated with this type of product are usually payable contingent on the policyholder's death, and in some cases also apply to survival benefits. For these contracts, the insurer's liability at the expiry of the contract is the excess, if any, of the guaranteed minimum payout and the amount of the policyholder's separate account. Generally, the probability of the guarantee actually resulting in a benefit is small. In the language of finance, we say that the guarantees are usually deep out-of-the-money. In the past this has led to a certain complacency, but it is now recognized that the risk management of these contracts represents a major challenge to insurers, particularly where the investment guarantee applies to maturity benefits, and where separate account products have proved popular with policyholders.

This book took shape as a result of my membership in the Canadian Institute of Actuaries Task Force on Segregated Fund contracts. After that Task Force completed its report, there was a clear demand for some educational material to help actuaries understand the methods that were recommended in the report, and that were subsequently mandated by the regulators. Also, many actuaries and regulators in the United States took a great interest in the report, and the demand for relevant educational material began to come also from across the United States. Meanwhile, in the United

Kingdom, it was becoming clear that investment guarantees associated with annuitization were creating a crisis in the industry.

Much of the material in this book is not new; there are many excellent texts available on time series modeling, on financial engineering, and on the principles of stochastic simulation, for example. There are numerous papers available on the pricing of investment guarantees in insurance, from the financial engineering viewpoint. The objective of this work is to put all the relevant models and methods that are useful in the risk management of equity-linked insurance into a single volume, and to focus specifically on the parts of the theory that are most relevant. This also enables us to develop the theory into practical methods for insurance companies, and to illustrate these with specific reference to equity linked contracts.

There are two common approaches to risk management of equity-linked insurance, particularly separate account products such as variable annuities or segregated funds. The "actuarial" approach uses the distribution of the guarantee liabilities discounted at the risk-free rate of interest. The dynamic-hedging approach uses financial engineering, and assumes that a portfolio of bonds and stocks is used to replicate the guarantee payoff. The replicating portfolio must be rebalanced at frequent intervals, as the underlying stock price changes. The actuarial approach is commonly used for risk management of investment guarantees by insurance companies in North America and in the United Kingdom. The dynamic-hedging approach is used by financial engineers in banks and hedge funds, and occasionally in insurance companies. It has been the case since the earliest equity-linked contracts were issued that many practitioners who use one of these methods harbor a deep distrust of the other method, often based on a lack of understanding of the other side's methodology.

In this book both approaches are presented, discussed, and extensively illustrated with examples. This should help practitioners on either side of the fence talk to each other, at the very least. My own view is that both methods have their merits, and that the best approach is to use both, in appropriate combination.

I have included in Chapter 7 an introduction to the concepts of no-arbitrage pricing, replication, and the risk-neutral measure. I am aware that many people who read this book will be very familiar with this material, but I am also aware of a great deal of misunderstanding surrounding these very fundamental issues. For example, there are many actuaries working with investment guarantees who do not fully comprehend the role of the Q-measure. By focusing solely on the important concepts, I hope to facilitate a better understanding of the financial economics approach. In order to keep the book to a manageable project, I have not generally included the complication of stochastic interest rates, except in Chapter 12, where it is necessary to explain the annuitization liability under the guaranteed annuity option (GAO) contract. This is often dealt with in the more technical literature on equity-linked insurance, such as Persson and Aase (1994) and Lin and Tan (2001).

The book is presented in a progressive, linear structure, starting with models, progressing through modeling, and finally moving on to risk management. In more detail, the structure of the book is as follows.

The first chapter introduces the contracts and some of the basic ideas from financial economics that will be utilized in later chapters. The next four chapters cover some of the econometrics of modeling equity processes.

In Chapter 2, we introduce a number of families of models that have been proposed for equity returns.

In Chapter 3, we discuss parameter estimation for some of the models, using maximum likelihood estimation (MLE). We also discuss ways of using the likelihood to rank the appropriateness of the models for the data.

Because MLE tends to fit the center of the distribution, and may not fit the tails particularly well for some processes, in Chapter 4 we discuss how to adjust the maximum likelihood parameters to improve the fit in other parts of the distribution. This may be important where the far tail of the equity return distribution is critical in the distribution of the investment guarantee payout. This chapter, incidentally, explains how to satisfy the calibration requirements of the Canadian Institute of Actuaries task force report on segregated funds (SFTF 2000).

Chapter 5 describes how to use the Markov chain Monte Carlo (MCMC) method for parameter estimation. This is a Bayesian method for parameter estimation that provides a powerful method for assessing parameter uncertainty.

Having decided on a model for equity returns, and estimated appropriate parameters, we can start to model the investment guarantees. In Chapter 6, we explain how to use stochastic simulation to model the distribution of the liability outgo for an equity-linked contract. This is the basis of the actuarial approach to risk management.

We then move on to the dynamic-hedging approach. This needs some elementary results from financial economics, which are presented in Chapter 7.

Then, in Chapter 8, we apply the methods to investment guarantees. This chapter goes beyond the pure pricing information provided by the Black-Scholes-Merton framework. We also assess the liability that is not covered by the Black-Scholes hedge. The three sources of this unhedged liability are

1. Transactions costs from rebalancing the hedge.
2. Hedging errors arising from discrete hedging intervals.
3. Additional hedging costs arising from the use of realistic equity models, under which the Black-Scholes hedge is no longer self-financing.

In Chapter 9, we discuss how to use risk measures to quantify the tail risk from a distribution; risk measures can also be used for pricing. The most common risk measure in finance is value at risk (VaR). This is a quantile risk measure. More recent theory favors the conditional tail expectation risk measure, also known as Tail-VaR. Both are described in Chapter 9, with examples of application to benefits such as variable annuities and segregated funds.

Chapter 10 describes stochastic emerging cost modeling. This allows us to bring together the actuarial and dynamic-hedging approaches and compare them in a systematic way. Emerging cost modeling is a powerful tool for making decisions about policy design, pricing, and risk management.

Because stochastic simulation is the fundamental tool for analyzing the liabilities for equity-linked insurance, it is useful to discuss the error and uncertainty associated with the method and to consider ways to reduce the variability of results. In Chapter 11, we examine three sources of forecast uncertainty. The first is random sampling variation. It is possible to reduce the effect of this using variance reduction techniques, and these are described with examples where they are useful in modeling embedded investment guarantees. The second is uncertainty in parameter estimation; this is where the Bayesian approach of Chapter 5 is particularly useful. We discuss how to apply Bayesian methods to quantify the effect of parameter uncertainty. Finally, we discuss model uncertainty—that is, how to assess the risk from the possibility that stock returns in the future follow a different model than that used in forecasts.

The final two chapters expand the application of the methods to two different types of equity-linked contracts. The first is the U.K. unit-linked contract with guaranteed annuity option (GAO). This has similarities with the guaranteed minimum income benefit associated with some variable annuity contracts. Issued in the early 1980s, at a time of very high long-term interest rates, the problems of stochastic interest rates and lack of diversification of risk associated with investment guarantees are, unfortunately, exemplified in the serious problems experienced by a number of U.K. insurers arising from maturing GAO contracts. Chapter 12 discusses the actuarial and the dynamic-hedging approaches to risk management of GAOs. In Chapter 13, we discuss equity-indexed annuities (EIA). These offer a combination of minimum return guarantee plus participation in stock appreciation for some equity index. The benefits appear quite similar to the variable annuity with maturity guarantee. However, as we shall demonstrate, the structure of the product is quite different. The actuarial approach is not appropriate for EIA contracts, and a common approach to risk management is a static strategy, effectively using options purchased from a third party to reinsure the investment guarantee liability.

Although many models are presented in the early chapters of the book, most of the examples in later chapters use the regime-switching lognormal model (RSLN) with two regimes. Part of the justification for this is given in Chapter 3, where this model is shown to provide a superior fit to monthly stock return data. Also, the model is easy to understand and is mathematically tractable. However, although I am partial to the RSLN model myself, nothing in the later chapters depends on it, so feel free to use your own favorite model, subject to some quantitative assessment (along the lines of Chapters 3 through 5) of how well it models the stock return process. For those interested in exploring the RSLN model further, the Society of Actuaries intends to make available a Microsoft Excel workbook for fitting the two-regime model to stock return data. The workbook calculates the likelihood for given parameters and data; calculates the maximum likelihood for given data; calculates the distribution function; tests the left tail against a left-tail calibration table (see Chapter 4); and generates random paths for the stock index for a given set of parameters (see Hardy and Hardy 2002).

After I had written the major part of the book, one of the extensively used stock return indices changed its name and composition. The TSE 300 index has been repackaged as the S&P/TSX Composite index. It is still the broad-based Canadian total return index, but is no longer restricted to 300 companies.

Although many people have helped with this work at various stages, all remaining errors are my responsibility. I am receptive to hearing of any; feel free to e-mail me at mrhardy@uwaterloo.ca.

Investment Guarantees

CHAPTER 1

Investment Guarantees

INTRODUCTION

The objective of life insurance is to provide financial security to policyholders and their families. Traditionally, this security has been provided by means of a lump sum payable contingent on the death or survival of the insured life. The sum insured would be fixed and guaranteed. The policyholder would pay one or more premiums during the term of the contract for the right to the sum insured. Traditional actuarial techniques have focused on the assessment and management of life-contingent risks: mortality and morbidity. The investment side of insurance generally has not been regarded as a source of major risk. This was (and still is) a reasonable assumption, where guaranteed benefits can be broadly matched or immunized with fixed-interest instruments.

But insurance markets around the world are changing. The public has become more aware of investment opportunities outside the insurance sector, particularly in mutual fund type investment media. Policyholders want to enjoy the benefits of equity investment in conjunction with mortality protection, and insurers around the world have developed equity-linked contracts to meet this challenge. Although some contract types (such as universal life in North America) pass most of the asset risk to the policyholder and involve little or no investment risk for the insurer, it was natural for insurers to incorporate payment guarantees in these new contracts—this is consistent with the traditional insurance philosophy.

In the United Kingdom, unit-linked insurance rose in popularity in the late 1960s through to the late 1970s, typically combining a guaranteed minimum payment on death or maturity with a mutual fund type investment. These contracts also spread to areas such as Australia and South Africa, where U.K. insurance companies were influential. In the United States, variable annuities and equity-indexed annuities offer different forms of equity-linking guarantees. In Canada, segregated fund contracts became popular in the late 1990s, often incorporating complex guaranteed values on death or maturity. Germany recently introduced equity-linked endowment insurance. Similar contracts are also popular in many other jurisdictions. In this book the term *equity-linked insurance* is used to refer to any contract that incorporates guarantees dependent on the performance of a stock market indicator. We also use the term *separate account insurance* to refer to the group of products that includes variable annuities, segregated funds, and unit-linked insurance. For each of these products, some or all of the premium is invested in an equity fund that resembles a mutual fund. That fund is the separate account and forms the major part of the benefit to the policyholder. Separate account products are the source of some of the most important risk management challenges in modern insurance, and most of the examples in this book come from this class of insurance. The nature of the risk to the insurer tends to be low frequency in that the stock performance must be extremely poor for the investment guarantee to bite, and high severity in that, if the guarantee does bite, the potential liability is very large.

The assessment and management of financial risk is a very different proposition to the management of insurance risk. The management of insurance risk relies heavily on diversification. With many thousands of policies in force on lives that are largely independent, it is clear from the central limit theorem that there will be very little uncertainty about the total claims. Traditional actuarial techniques for pricing and reserving utilize deterministic methodology because the uncertainties involved are relatively minor. Deterministic techniques use “best estimate” values for interest rates, claim amounts, and (usually) claim numbers. Some allowance for uncertainty and random variation may be made implicitly, through an adjustment to the best estimate values. For example, we may use an interest rate that is 100 or 200 basis points less than the true best estimate. Using this rate will place a higher value on the liabilities than will using the best estimate as we assume lower investment income.

Investment guarantees require a different approach. There is generally only limited diversification amongst each cohort of policies. When a market indicator becomes unfavorable, it affects many policies at the same time. For the simplest contracts, either all policies in the cohort will generate claims or none will. We can no longer apply the central limit theorem. This kind of risk is referred to as *systematic, systemic,* or *nondiversifiable* risk. These terms are interchangeable.

Contrast a couple of simple examples:

- An insurer sells 10,000 term insurance contracts to independent lives, each having a probability of claim of 0.05 over the term of the contract. The expected number of claims is 500, and the standard deviation is 22 claims. The probability that more than, say, 600 claims arise is less than 10^{-5}. If the insurer wants to be very cautious not to underprice or underreserve, assuming a mortality rate of 6 percent for each life instead of the best estimate mortality rate of 5 percent for each life will absorb virtually all mortality risk.

- The insurer also sells 10,000 pure endowment equity-linked insurance contracts. The benefit under the insurance is related to an underlying stock price index. If the index value at the end of the term is greater than the starting value, then no benefit is payable. If the stock price index value at the end of the contract term is less than its starting value, then the insurer must pay a benefit. The probability that the stock price index has a value at the end of the term less than its starting value is 5 percent.

  The expected number of claims under the equity-linked insurance is the same as that under the term insurance—that is 500 claims. However, the nature of the risk is that there is a 5 percent chance that all 10,000 contracts will generate claims, and a 95 percent chance that none of them will. It is not possible to capture this risk by adding a margin to the claim probability of 5 percent.

This simple equity-linked example illustrates that, for this kind of risk, the mean value for the number (or amount) of claims is not very useful. We can also see that no simple adjustment to the mean will capture the true risk. We cannot assume that a traditional deterministic valuation with some margin in the assumptions will be adequate. Instead we must utilize a more direct, stochastic approach to the assessment of the risk. This stochastic approach is the subject of this book.

The risks associated with many equity-linked benefits, such as variable-annuity death and maturity guarantees, are inherently associated with fairly extreme stock price movements—that is, we are interested in the tail of the stock price distribution. Traditional deterministic actuarial methodology does not deal with tail risk. We cannot rely on a few deterministic stock return scenarios generally accepted as "feasible." Our subjective assessment of feasibility is not scientific enough to be satisfactory, and experience—from the early 1970s or from October 1987, for example—shows us that those returns we might earlier have regarded as infeasible do, in fact, happen. A stochastic methodology is essential in understanding these contracts and in designing strategies for dealing with them.

In this chapter, we introduce the various types of investment guarantees commonly used in equity-linked insurance and describe some of the contracts that offer investment guarantees as part of the benefit package. We also introduce the two common methods for managing investment guarantees: the actuarial approach and the dynamic-hedging approach. The actuarial approach is commonly used for risk management of investment guarantees by insurance companies in North America and in the United Kingdom. The dynamic-hedging approach is used by financial engineers in banks, in hedge funds, and (occasionally) in insurance companies. In later chapters we will develop both of these methods in relation to some of the major contract types described in the following sections.

MAJOR BENEFIT TYPES

Equity Participation

All equity-linked contracts offer some element of participation in an underlying index or fund or combination of funds, in conjunction with one or more guarantees. Without a guarantee, equity participation involves no risk to the insurer, which merely acts as a steward of the policyholders' funds. It is the combination of equity participation and fixed-sum underpinning that provides the risk for the insurer. These fixed-sum risks generally fall into one of the following major categories.

Guaranteed Minimum Maturity Benefit (GMMB) The guaranteed minimum maturity benefit (GMMB) guarantees the policyholder a specific monetary amount at the maturity of the contract. This guarantee provides downside protection for the policyholder's funds, with the upside being participation in the underlying stock index. A simple GMMB might be a guaranteed return of premium if the stock index falls over the term of the insurance (with an upside return of some proportion of the increase in the index if the index rises over the contract term). The guarantee may be fixed or subject to regular or equity-dependent increases.

Guaranteed Minimum Death Benefit (GMDB) The guaranteed minimum death benefit (GMDB) guarantees the policyholder a specific monetary sum upon death during the term of the contract. Again, the death benefit may simply be the original premium, or may increase at a fixed rate of interest. More complicated or generous death benefit formulae are popular ways of tweaking a policy benefit at relatively low cost.

Guaranteed Minimum Accumulation Benefit (GMAB) With the guaranteed minimum accumulation benefit (GMAB), the policyholder has the option to renew the contract at the end of the original term, at a new guarantee level appropriate to the maturity value of the maturing contract. It is a form of guaranteed lapse and reentry option.

Guaranteed Minimum Surrender Benefit (GMSB) The guaranteed minimum surrender benefit (GMSB) is a variation of the guaranteed minimum maturity benefit. Beyond some fixed date the cash value of the contract, payable on surrender, is guaranteed. A common guaranteed surrender benefit in Canadian segregated fund contracts is a return of the premium.

Guaranteed Minimum Income Benefit (GMIB) The guaranteed minimum income benefit (GMIB) ensures that the lump sum accumulated under a separate account contract may be converted to an annuity at a guaranteed rate. When the GMIB is connected with an equity-linked separate account, it has derivative features of both equities and bonds. In the United Kingdom, the guaranteed-annuity option is a form of GMIB. A GMIB is also commonly associated with variable-annuity contracts in the United States.

CONTRACT TYPES

Introduction

In this section some generic contract types are described. For each of these types, individual insurers' product designs may differ in detail from the basic contract described below. The descriptions given here, however, give the main benefit details.

The first three are all separate account products, and have very similar risk management and modeling issues. These products form the basis of the analysis of Chapters 6 to 11. However, the techniques described in these chapters can be applied to other type of equity-linked insurance. The guaranteed annuity option is discussed in Chapter 12, and equity-indexed annuities are the topic of Chapter 13.

Segregated Fund Contracts—Canada

The segregated fund contract in Canada has proved an extremely popular alternative to mutual fund investment, with around $60 billion in assets in 1999, according to *Risk* magazine. Similar contracts are now issued by Canadian banks, although the regulatory requirements differ.

The basic segregated fund contract is a single premium policy, under which most of the premium is invested in one or more mutual funds on the policyholder's behalf. Monthly administration fees are deducted from the fund. The contracts all offer a GMMB and a GMDB of at least 75 percent of the premium, and 100 percent of premium is common. Some contracts offer enhanced GMDB of more than the original premium. Many contracts offer a GMAB at 100 percent or 75 percent of the maturing value.

The rate-of-administration fee is commonly known as the *management expense ratio* or *MER*. The MER differs by mutual fund type.

The name "segregated fund" refers to the fact that the premium, after deductions, is invested in a fund separate from the insurer's funds. The management of the segregated funds is often independent of the insurer.

A policyholder may withdraw some or all of his or her segregated fund account at any time, though there may be a penalty on early withdrawals.

The insurer usually offers a range of funds, including fixed interest, balanced (a mixture of fixed interest and equity), broad-based equity, and perhaps a higher-risk or specialized equity fund. For policyholders who invest in several funds, the guarantee may apply to each fund separately (a *fund-by-fund* benefit) or may be based on the overall return (the *family-of-funds* approach).

Variable Annuities—United States

The U.S. variable-annuity (VA) contract is a separate account insurance, very similar to the Canadian segregated fund contract. The VA market is very large, with over $100 billion of annual sales each year in recent times.

Premiums net of any deductions are invested in *subaccounts* similar to the mutual funds offered under the segregated fund contracts. GMDBs are a standard contract feature; GMMBs were not standard a few years ago, but are beginning to become so. They are known as VAGLBs or variable-annuity guaranteed living benefits. Death benefit guarantees may be increased periodically.

Unit-Linked Insurance—United Kingdom

Unit-linked insurance resembles segregated funds, with the premium less deductions invested in a separate fund. In the 1960s and early 1970s, these contracts were typically sold with a GMMB of 100 percent of the premium. This benefit fell into disfavor, partly resulting from the equity crisis of 1973 to 1974, and most contracts currently issued offer only a GMDB.

Some unit-linked contracts associated with pensions policies carry a guaranteed annuity option, under which the fund at maturity may be converted to a life annuity at a guaranteed rate. This is a more complex option, of the GMIB variety. This option is discussed in Chapter 12.

Equity-Indexed Annuities—United States

The U.S. equity-indexed annuity (EIA) offers participation at some specified rate in an underlying index. A participation rate of, say, 80 percent of the specified price index means that if the index rises by 10 percent the interest credited to the policyholder will be 8 percent. The contract will offer a guaranteed minimum payment of the original premium accumulated at a fixed rate; a rate of 3 percent per year is common.

Fixed surrender values are a standard feature, with no equity linking. Other contract features vary widely by company. A form of GMAB may be offered in which the guarantee value is set by annual reset according to the participation rate.

Many features of the EIA are flexible at the insurer's option. The MERs, participation rates, and floors may all be adjusted after an initial guarantee period.

The EIAs are not as popular as VA contracts, with less than $10 billion in sales per year. EIA contracts are discussed in more detail in Chapter 13.

Equity-Linked Insurance—Germany

These contracts resemble the U.S. EIAs, with a guaranteed minimum interest rate applied to the premiums, along with a percentage participation in a specified index performance. An unusual feature of the German product is that, for regulatory reasons, annual premium contracts are standard (Nonnemacher and Russ 1997).

EQUITY-LINKED INSURANCE AND OPTIONS

Call and Put Options

Although the risks associated with equity-linked insurance are new to insurers, at least, relative to life-contingent risks, they are very familiar to practitioners and academics in the field of derivative securities. The payoffs under equity-linked insurance contracts can be expressed in terms of *options*.

There are many books on the theory of option pricing and risk management. In this book we will review the relevant fundamental results, but the development of the theory is not covered. It is crucially important for practitioners in equity-linked insurance to understand the theory underpinning option pricing. The book by Boyle et al. (1998) is specifically written with actuaries and actuarial applications in mind. For a general, readable introduction to derivatives without any technical details, Boyle and Boyle (2001) is highly recommended.

The simplest forms of option contracts are:

- A *European call option* on a stock gives the purchaser the right (but not the obligation) to purchase a specified quantity of the underlying stock at a fixed price, called the *strike price,* at a predetermined date, known as the *expiry* or *maturity date* of the contract.
- A *European put option* on a stock gives the purchaser the right to sell a specified quantity of the underlying stock at a fixed strike price at the expiry date.

American options are defined similarly, except that the option holder has the right to exercise the option at any time before expiry. *Asian options* have a payoff based on an average of the stock price over a period, rather than on the final stock price.

To summarize the benefits under the option contracts, we introduce some notation. Let K be the strike price of the option per unit of stock; let S_t be the price of one unit of the underlying stock at time t; and let T be the expiry date of the option. The payoff at time T under the call option will be:

$$(S_T - K)^+ = \max(S_T - K, 0) \tag{1.1}$$

and the payoff under the put option will be

$$(K - S_T)^+ = \max(K - S_T, 0) \tag{1.2}$$

In subsequent chapters we shall see that it is natural to think of the investment guarantee benefits under separate account products as put options on the policyholder's fund. On the other hand, it is more natural to use call options to value the benefits under an equity-indexed annuity.

We often use the terms *in-the-money, at-the-money,* and *out-of-the-money* in relation to options and to equity-linked insurance guarantees. A put option that is in-the-money at time $t < T$ has an underlying stock price $S_t < K$, so that if the stock price at maturity were to be the same as the current stock price, there would be a payment under the guarantee. For a call option, in-the-money means that $S_t > K$, and at-the-money means that the stock and strike prices are roughly equal. Out-of-the-money for a put option means $S_t > K$, and for a call option means $S_t < K$; in either case, if the stock price at maturity is the same as the current stock price, no payment would be required under the guarantee or option contract. We say a contract is deep out-of-the-money or in-the-money if the difference between the stock price and strike price is large, so that it is very likely that a deep out-of-the-money contract will remain out-of-the-money, and similarly for the deep in-the-money contract.

The No-Arbitrage Principle

The *no-arbitrage* principle states that, in well-functioning markets, two assets or portfolios having exactly the same payoffs must have exactly the same price. This concept is also known as the *law of one price;* it is a fundamental assumption of financial economics. The logic is that if prices differ by a fraction, it will be noticed by the market, and traders will move in to buy the cheaper portfolio and sell the more expensive, making an instant risk-free profit or *arbitrage*. This will pressure the price of the cheap portfolio back up, and the price of the expensive portfolio back down, until they return to equality. Therefore, any possible arbitrage opportunity will be eliminated in an instant. Many studies show consistently that the no-arbitrage assumption is empirically indisputable in major stock markets.

This simple and intuitive assumption is actually very powerful, particularly in the valuation of derivative securities. To value a derivative security such as an option, it is sufficient to find a portfolio, with known value, that precisely replicates the payoff of the option. If the option and the replicating portfolio do not have the same price, one could sell the more expensive and buy the cheaper, and make an arbitrage profit. Since this is assumed to be impossible, the value of the option and the value of the replicating portfolio must be identical under the no-arbitrage assumption.

Put-Call Parity

Using the no-arbitrage assumption allows us to derive an important connection between the put option and the call option on a stock.

Let c_t denote the value at t of a European call option on a unit of stock, and p_t the value of a European put option on a unit of the same stock. Both options are assumed to mature at the same date $T > t$ with the same strike price, K. Assume the stock price at t is S_t, then an investor who holds both a unit of stock and a put option on that unit of stock will have a portfolio at time t with value $p_t + S_t$. The payoff at expiry of the portfolio will be

$$p_T + S_T = \max(K, S_T) \tag{1.3}$$

Similarly, consider an investor who holds a call option on a unit of stock together with a pure discount bond maturing at T with face value K. We assume the pure discount bond earns a risk-free rate of interest of r per year, continuously compounded, so that the value at time t of the pure discount bond plus call option is $c_t + Ke^{-r(T-t)}$. The payoff at maturity of the portfolio of the pure discount bond plus call option will be

$$c_T + K = \max(K, S_T) \tag{1.4}$$

In other words, these two portfolios—"put plus stock" and "call plus bond"—have identical payoffs. The no-arbitrage assumption requires that two portfolios offering the same payoffs must have the same price. Hence we find the fundamental relationship between put and call options known as put-call parity, that is,

$$p_t + S_t = c_t + Ke^{-r(T-t)} \tag{1.5}$$

Options and Equity-Linked Insurance

Many benefits under equity-linked insurance contracts can be regarded as put or call options. For example, the liability under the maturity guarantee of a Canadian segregated fund contract can be naturally regarded as an embedded put option. That is, the policyholder who pays a single premium of \$1000 with a 100 percent GMMB is guaranteed to receive at least

$K = \$1000$ at maturity, even if the market value of her or his portfolio is less than \$1000 at that time. It is the responsibility of the insurer to pay $(1000 - S_T)^+$, the excess of the guaranteed amount over the market value of the assets, meaning that the insurer pays the payoff under a put option.

Therefore, the total segregated fund policy benefit is made up of the policyholder's fund plus the payoff from a put option on the fund. From put-call parity we know that the same benefit can be provided using a bond plus a call option, but that route is not sensible when the contract is designed in the separate account format. Put-call parity also means that the U.S. EIA could either be regarded as a combination of fixed-interest security (meeting the minimum interest rate guarantee) and a call option on the underlying stock (meeting the equity participation rate benefit), or as a portfolio of the underlying stock (for equity participation) together with a put option (for the minimum benefit). In fact, the first method is a more convenient approach from the design of the contract.

The fundamental difference between the VA-type guarantee, which we value as a put option to add to the separate account proceeds, and the EIA guarantee, which we value as a call option added to the fixed-interest proceeds, arises from the withdrawal benefits. On withdrawal, the VA policyholder takes the proceeds of the separate account, without the put option payment. The EIA policyholder withdraws with their premium accumulated at some fixed rate, without the call-option payment.

American options may be relevant where equity participation and minimum accumulation guarantees are both offered on early surrender. Asian options are relevant for some EIA contracts where the equity participation can be based on an average of the underlying stock price rather than on the final value.

There is a substantial and rich body of theory on the pricing and financial management of options. Black and Scholes (1973) and Merton (1973) showed that it is possible, under certain assumptions, to set up a portfolio that consists of a long position in the underlying stock together with a short position in a pure discount bond and has an identical payoff to the call option. This is called the replicating portfolio. The theory of no-arbitrage means that the replicating portfolio must have the same value as the call option because they have the same payoff at the expiry date. Thus, the famous Black-Scholes option-pricing formula not only provides the price but also provides a risk management strategy for an option seller—hold the replicating portfolio to hedge the option payoff. A feature of the replicating portfolio is that it changes over time, so the theory also requires the balance of stocks and bonds to be rearranged at frequent intervals over the term of the contract.

The stock price, S_t, is the random variable in the payoff equations for the options (we assume that the risk-free rate of interest is fixed). The