Modeling Derivatives in C++

JUSTIN LONDON

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To the memory of my grandparents
Milton and Evelyn London,
as well as my parents,
Leon and Leslie,
and my sister, Joanna

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Preface

Derivative modeling is at the heart of quantitative research and development on Wall Street. Practitioners (i.e., Wall Street trading desk quants) and academics alike spend much research, money, and time developing efficient models for pricing, hedging, and trading equity and fixed income derivatives. Many of these models involve complicated algorithms and numerical methods that require lots of computational power. For instance, the HJM lattice for pricing fixed income derivatives often requires coding a nonrecombining bushy tree that cannot be easily traversed and grows exponential in time and memory.

C++ is often the programming language of choice for implementing these models due to the language's object-oriented features, speed, and reusability. However, often the implementation "how-to" of these models is quite esoteric to the model creators and developers due to their algorithmic complexity. Most journal articles and white papers that discuss derivative models provide only a theoretical understanding of them as well as their mathematical derivations. While many research papers provide numerical results, few supply the details for how to implement the model, if for no other reason than to allow readers to replicate and validate their results. There are several reasons for this.

It is often the general nature of academics who publish leading research to be pedantic, writing at a level geared for their academic peers, rather than to practitioners. This often leads to papers that spend much time providing mathematical formulas and proofs as opposed to discussions of practical applications and implementations. Few if any of these published papers discuss in detail how these derivative models are to be correctly and efficiently implemented for practical use in the real world. After all, what good is a model if it cannot be used in practice in research and trading environments?

Another reason for the lack of implementation discussions is that many top quant researchers and professors, often with doctorates in mathematics and physics, spend their time developing the mathematical and theoretical underpinnings of the models and leave the actual code implementations to their graduate research students. Graduate research students often are given the task of implementing the models of their advisers as part of collaborative work. Consequently, often only the numerical results, if any, are provided, usually generated from the code implementations of the graduate student.[1]

[1]There are instances where code is provided by the graduate research student. In the paper "Fast Greeks in Forward LIBOR Models" by P. Glasserman and Z. Zhao, the code is given at www-1.gsb.columbia.edu/faculty/pglasserman/Other/get_code.html and is discussed in Chapter 13 of this book.

However, as is more often the case, the code developed by quant researchers and programmers working on Wall Street trading desks is highly valuable and proprietary to the Wall Street institutions just as the Windows operating system code is proprietary to Microsoft and not the developers who work on it. The code is the powerful engine that gives trading desks their competitive advantage over other players in the market. If Wall Street trading desks have a proprietary model that allows them to capture arbitrage opportunities based on "mispricings" between derivative market prices and their theoretical model values, then if this code was readily available to all market participants, the model would be exploited by all those using it, quickly eliminating the profit opportunity and removing the competitive edge of the institution where it was developed.

Similarly, professors and researchers who own the code for the models they develop often are unwilling to release it to the public because keeping it in-house can lead to lucrative consulting contracts with Wall Street institutions and companies that want to contract them to implement and license use of their proprietary model. For example, GFI Group, Inc., states on its web site that two top researchers, John Hull and Alan White, have assisted the company in its development of software for credit derivatives pricing using the Hull-White credit model.

When I was a graduate student in the Financial Engineering Program at the University of Michigan, the theory and mathematical derivations of the models were taught and emphasized. An understanding of stochastic calculus, stochastic processes, partial differential equations, and probability theory was emphasized and was a prerequisite for being able to model, price, and hedge complicated derivatives securities. Since students were assumed to know how to program in C and use Excel, little emphasis was made on efficient coding implementation. At the time, our code was written on Sun Sparc workstations. Upon graduating and completing several other graduate degrees in mathematics and computer science, being able to program became more important than actually understanding the theory behind the models because Wall Street positions for developing code and models to support trading desks require excellent programming skills. However, since one cannot usually program efficient models without an understanding of the theoretical and mathematical intricacies behind them, both an understanding of the theory and being able to program well are necessary. In fact, throughout the book, the theory and mathematical derivations of some of the models are based on the work and lectures of Dr. Vadim Linetsky, who taught the financial engineering courses.[2]

Over time the University of Michigan Financial Engineering Program has been modified to include more practical coding exercises through use of real-time Reuters data feeds. Other well-known financial engineering, mathematical finance, and computational finance programs, such as those at the University of California–Berkley,

[2]Dr. Vadim Linetsky is now an associate professor at Northwestern University in the Department of Industrial Engineering and Management Sciences. He teaches financial engineering courses similar to the ones he taught at the University of Michigan.

the University of Chicago, and Carnegie-Mellon, respectively, may start to adapt their curricula, if they have not done so already, to place more emphasis on the practical implementation and coding of models as many of their graduates head to Wall Street to work in quantitative research and trading development groups.

I felt that since no book bridged the gap between the two and because such a book would have helped me both in school and afterward on the job as a quantitative developer, I should write such a book so as to help others. Such a book was an enormous undertaking and required contacting many of the model developers of some of the more complicated models to try to understand how they implemented them and in some cases to even obtain their code. In those cases where I was not able to get model details or code from an author, I was able to verify the accuracy and robustness of the code I developed by being able to reproduce numerical results of the models in published papers and books.

Modeling Derivatives in C++ is the first book to provide the source code for most models used for pricing equity and fixed income derivatives. The objective of the book is to fill the large gap that has existed between theory and practice of the quantitative finance field. Readers will learn how to correctly code in C++ many derivatives models used by research and trading desks. The book bridges the gap between theory and practice by providing both the theory and mathematical derivations behind the models as well as the actual working code implementations of these models. While there have been other books that have helped bridge this gap such as Clewlow and Strickland's *Implementing Derivatives Models* (John Wiley & Sons, 1998a), they provide only pseudocode and do not emphasize robust and efficient object-oriented code that is reusable. The assumption that readers can easily or correctly translate pseudocode, which may have complex embedded subroutines of numerical computations that is needed, often is mistaken. Sometimes, readers learn by analyzing and reviewing the complete and working code, which is what this book attempts to accomplish. However, *Implementing Derivatives Models* does contain useful model discussions and pseudocode implementations, some of which are implemented and built upon in this book using C++, such as the hedge control variate method discussed in Chapter 2 and the alternating direction implicit method discussed in Chapter 5.

Modeling Derivatives in C++ goes several steps beyond just providing C++ code; it discusses inefficiencies in some of the implementations and how they can be improved with more robust object-oriented implementations by providing code from the QuantLib, an open source quantitative pricing library, as well as by providing alternative implementations. For instance, three separate implementations are given for the Hull-White model to show readers different coding approaches. The book contains hundreds of classes, and there is a complete pricing engine library on the CD-ROM accompanying this book, which includes the code discussed and given in the book. QuantPro, an MFC Windows application, for pricing many equity and fixed income derivatives using the models discussed in the book, as well as for simulating derivatives trades, is also provided on the CD-ROM.

It is the goal of the book that readers will be able to write their own models in

C++ and then be able to adapt some of the coded models in this book to their own pricing libraries and perhaps even use to trade. Most important, the book is intended to guide readers through the complexities and intricacies of the theory and of applying it in practice. The book is aimed at advanced undergraduate students well as graduate (MBA and Ph.D.) students in financial economics, computer science, financial engineering, computational finance, and business as well as Wall Street practitioners working in a quantitative research or trading group who need a comprehensive reference guide for implementing their models.

Readers should have a basic understanding of stochastic calculus, probability theory, linear algebra, partial differential equation (PDEs), and stochastic processes. For those readers who may be lacking the background in some of this material or need to review, the appendixes provide a review of some of this material. Due to the comprehensiveness of the book, it can be used by professors as either a primary text or a supplementary text in their courses.

The chapters are grouped into two main sections: The first focuses on the pricing of equity derivatives and comprises Chapter 1 to Chapter 9, and the second part focuses on the pricing of interest rate derivatives: Chapter 10 to Chapter 14.

Chapter 1 focuses on the derivation and foundations of the Black-Scholes model for asset pricing in the risk-neutral world. The Black-Scholes partial differential equation describes the evolution of all derivatives whose payoff is a function on a single underlying asset following geometric Brownian motion (GBM) and time.

Chapter 2 discusses Monte Carlo methods for valuation of European as well as path-dependent derivatives. Various random number generators for pseudorandom, quasi-random (deterministic), Sobol, and Faure sequences are discussed. Variance reduction techniques using control variates and antithetics are discussed to overcome the computational inefficiency of the Monte Carlo method in its basic form, which typically requires hundreds of thousands of simulations to achieve good accuracy.

Chapter 3 discusses the binomial tree model for pricing European and American equity options. The binomial tree is shown to be a two-state discrete approximation to continuous GBM: The mean and variance of the binomial model match the mean and variance of the lognormal distribution underlying GBM. Furthermore, the binomial model can be adapted to incorporate time-varying volatility, to pricing path-dependent options, and to pricing derivatives depending on more than one asset with two-variable binomial trees.

Chapter 4 generalizes binomial trees to the more flexible and widely used trinomial trees, which approximate GBM diffusion processes with three states. It also discusses implied trees, which are trees constructed to fit observable market prices. Thus, this method builds trees implied by the market.

Chapter 5 discusses finite-difference methods, numerical methods (actually, extensions of the trinomial method) for discretizing PDEs that (path-dependent) derivatives with complex payoffs must satisfy and then solving them over a state-time lattice. The explicit, implicit, and Crank-Nicolson finite-difference methods are discussed as well as the alternating direction implicit method for pricing options that depend on multiple-state variables.

Chapter 6 discusses pricing exotic options including Asian, lookback, and barrier options.

Chapter 7 discusses stochastic volatility models that are used to capture volatility skews and smiles observed in the options markets. Since the constant volatility assumption of Black-Scholes is not valid in the real world, alternative models such as the constant elasticity of variance (CEV), jump diffusion, and multifactor stochastic volatility models can each be used to fit pricing model parameters to observable option market quotes.

Chapter 8 focuses on statistical models for volatility estimation including GARCH models. Chapter 9 deals with stochastic multifactor models for pricing derivatives like basket options.

Chapter 10 begins the second part of the book and focuses on fixed income models. The chapter discusses single-factor short rate models including the Vasicek, Hull-White (HW), Black-Derman-Toy (BDT), and Cox-Ingersoll-Ross (CIR) models.

Chapter 11 focuses on tree-building procedures for the short rate models discussed in Chapter 10. It shows how to calibrate the BDT and HW models initially to the yield curve and then to both the yield and volatility curves, and explains how to price discount bonds, bond options, and swaptions with the models.

Chapter 12 discusses two-factor models as well as the HJM model for pricing fixed income derivatives.

Chapter 13 provides an in-depth discussion of the LIBOR market model (also known as the Brace-Gatarek-Musiela/Jamshidian (BGM/J) model, showing how to calibrate the model to cap and swaption volatilites for pricing. Correlation structures and stochastic extensions of the model are also discussed. The chapter shows the difference and inconsistencies between the LIBOR forward-rate model (LFM) for pricing caps and the Libor swap model (LSM) for pricing swaptions and swaps.

Chapter 14 discusses exotic interest rate derivatives. Bermudan swaptions, range notes, index-amortizing swaps, trigger swaps, and quantos are discussed along with pricing models and implementations for them. Gaussian quadrature is also discussed as a useful tool for evaluating certain numerical integrals used in derivatives pricing such as those for spread options and quantos.

Appendix A contains a probability review of important probability concepts used throughout the book. Appendix B contains a stochastic calculus review of Brownian motion, stochastic integrals, and Ito's formula. Appendix C contains a discussion of the fast Fourier transform (FFT) method, a powerful numerical technique for valuation of higher-dimensional integrals. Appendix D discusses building models, pricing engines, and calibrating models in practice with a focus on building robust models. Appendix E contains some useful code routines including the random number generator for generating uniform deviates for Monte Carlo simulation from Press et al., *Numerical Recipes in* C (1992). Appendix F shows the mathematical details for solving the Black-Scholes PDE using Green's function.

(Appendixes A and B can be found at the end of the book; Appendixes C through F are available as PDFs on the CD-ROM.)

It is my hope and intention that readers will get a lot of value from this book and that it will help them in both their academic studies as well as at work on their jobs. I hope that readers enjoy it as much as I enjoyed writing it. Finally, while I have attempted to be quite complete in the topics covered, the book does not cover everything. In particular, mortgage-backed securities and credit derivatives are not discussed. They will, however, be included in my next undertaking.

JUSTIN LONDON

Chicago, Illinois
October 2004

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"If I have seen farther than others, it is because I was standing on the shoulders of giants."

—Isaac Newton

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J. L.

CHAPTER 1

Black-Scholes and Pricing Fundamentals

This chapter discusses the most important concepts in derivatives models, including risk-neutral pricing and no-arbitrage pricing. We derive the renowned Black-Scholes formula using these concepts. We also discuss fundamental formulas and techniques used for pricing derivatives in general, as well as those needed for the remainder of this book. In section 1.1, we discuss forward contracts, the most basic and fundamental derivative contract. In section 1.2, we derive the Black-Scholes partial differential equation (PDE). In section 1.3, we discuss the concept of risk-neutral pricing and derive Black-Scholes equations for European calls and puts using risk-neutral pricing. In section 1.4, we provide a simple implementation for pricing these European calls and puts. In section 1.5, we discuss the pricing of American options. In section 1.6, we discuss fundamental pricing formulas for derivatives in general. In section 1.7, we discuss the important change of numeraire technique—useful for changing asset dynamics and changing drifts. In section 1.8, Girsanov's theorem and the Radon-Nikodym derivative are discussed for changing probability measures to equivalent martingale measures. In section 1.9, we discuss the T-forward measure, a useful measure for pricing many derivatives; and finally, in section 1.10, we discuss considerations for choosing a numeraire in pricing. (A probability review is provided in Appendix A at the back of the book and a stochastic calculus review is provided in Appendix B.)

1.1 FORWARD CONTRACTS

A security whose value is contingent on the value of an underlying security or macroeconomic variable such as an interest rate or commodity like oil is known as a derivative since the security "derives" its value and is contingent on the value of the underlying asset. Derivatives are known as contingent claims. The simplest derivative and most fundamental financial transaction is a *forward contract*, which is an agreement between two parties to buy or sell an asset, such as a foreign currency, at a certain time $T > 0$ for a certain delivery price, K, set at the contract inception t_0.

Forward contracts are traded over-the-counter (OTC). Standardized exchange-traded contracts, such as those on the Chicago Mercantile Exchange, are known as *futures*.

In a forward contract, there are two parties, usually two financial institutions or a financial institution and its customer: One party agrees to buy the asset in the forward contract at maturity, time T, and is said to be long, and the counterparty agrees to sell the asset to the buyer at T and is said to be short. The contract is settled at maturity T: The short delivers the asset to the long in return for a cash amount K.

If the price of the asset in the spot market at T is S_T, then the payoff, f_T, from the long position at T is:

$$f_T = S_T - K \tag{1.1}$$

since the long receives an asset worth S_T and pays the delivery price K. Conversely, the payoff from the short position is:

$$f_T = K - S_T \tag{1.2}$$

since the short receives the amount K and delivers an asset worth S_T in exchange.

Let's use some notation to help in the pricing analysis over time. Let S_t, $0 \leq t \leq T$ be the current underlying price at time t, let $f_{t,T}$ be the present value of a forward contract at time t maturing at time T, let $F_{t,T}$ be the forward price at time t, and let r be the risk-free rate per annum (with continuous compounding). The forward price is such a delivery price K that makes the present value of a forward contract equal to zero, $f_{0,T} = 0$:

$$K = F_{0,T} = S_0 e^{r(T-t_0)} \tag{1.3}$$

We can show that this must be the forward price using an absence of arbitrage argument: If $F_{0,T} > S_0 e^{r(T-t_0)}$, we can create a synthetic forward position and arbitrage an actual forward contract against this synthetic forward. At time t_0, we can borrow S_0 dollars for a period of $T - t_0$ at the risk-free rate r; we can then use these dollars to buy the asset at the spot price S_0; and finally, we take a short position in the forward contract with delivery price $F_{0,T}$. At time T, we (1) sell the asset for the forward price $F_{0,T}$ and (2) use an amount $e^{r(T-t_0)}S_0$ of the proceeds to repay the loan with accrued interest. This yields an arbitrage profit of $F_{0,T} - S_0 e^{r(T-t_0)}$. Similarly, assuming $F_{0,T} < S_0 e^{r(T-t_0)}$, we do the reverse transaction: At time t, we go long the forward contract and short the synthetic forward position—we invest the proceeds S_0 at rate r, and at time T buy the spot asset at $F_{0,T}$, earning an arbitrage profit of $S_0 e^{r(T-t_0)} - F_{0,T}$. Thus, in the absence of arbitrage we have shown that equation (1.3) must hold. The absence of arbitrage is equivalent to the impossibility of investing zero dollars today and receiving a nonnegative amount tomorrow that is positive with positive probability. Thus, two portfolios having the same payoff at a given future date T must have the same price today. Moreover, by constructing a portfolio of securities having the same instantaneous return as that of a riskless investment—that is, a money market account (MMA)—the portfolio instantaneous return must be the risk-free rate. Investors are then said to be risk-neutral: They expect that all investments with no risk (i.e., uncertainty) should earn the risk-free rate. Investors can always remove systematic (market) risk from the portfolio by holding securities that can be hedged against one another.

We can also show that $F_{0,T} = S_0 e^{r(T-t_0)}$ by using risk-neutral pricing and calculating the present value (PV) directly:

$$f_{0,T} = e^{-r(T-t_0)} E_{t_0}[S_T - K] = e^{-r(T-t_0)}(e^{r(T-t_0)} S_0 - K) = 0 \tag{1.4}$$

where E_{t_0} is the expectation operator at time t_0. Thus, $K = F_{0,T} = e^{r(T-t_0)} S_0$. The risk-free rate is used as both an expected growth rate of the asset $E_{t_0}[S_T] = e^{r(T-t_0)} S_0$ and the discount rate.

We can also calculate the present value of a seasoned forward position at some time t after inception, known as marking to market. At some time t after inception, $0 < t < T$, the PV is generally different from zero:

$$f_{t,T} = e^{-r(T-t)} E_t [S_T - K] = S_t - e^{-r(T-t)} K = S_t - e^{r(T-t_0)} S_0 \tag{1.5}$$

$$= S_t - F_{0,t} = e^{-r(T-t)}[F_{t,T} - F_{0,T}] \tag{1.6}$$

Thus, the present value of a seasoned forward contract can be valued by taking the difference between the forward price at time t and the forward price at time 0 and discounting back to get the PV. If $t = 0$ (i.e., today), then the present value of the forward contract is 0, which is what we would expect. It is important to note that the arbitrage-free and risk-neutral arguments are valid only for traded assets. Forwards on commodities that are held for consumption purposes cannot be valued by these arguments.

These arguments can be used to value a forward on an asset providing a known cash income such as coupon bonds or stocks with discrete dividend payments. Let I_0 be the PV at time t_0 of all income to be received from the asset between times t_0 and T (discounting at the risk-free rate). It is left as an exercise for the reader to show that $K = F_{0,T} = e^{r(T-t_0)}(S_0 - I_0)$ and that at $0 < t < T$ the present value is $f_{t,T} = e^{-r(T-t)} E_t[(S_T - I_T) - K] = S_t - I_t - e^{-r(T-t)} K$. If the asset pays a continuous known dividend yield q, then the growth and discount rates are $e^{(r-q)(T-t)}$ and $e^{-(r-q)(T-t)}$, respectively. If the underlying asset is a foreign currency, then we can view the yield q as the foreign risk-free rate r_f so that the growth and discount rates of the underlying currency S_0 are $e^{(r-rf)(T-t)}$ and $e^{-(r-rf)(T-t)}$, respectively, and the price of a forward contract on S_0 (i.e., British pounds) at time 0 is $F_{0,T} = S_0 e^{(r-rf)(T-t)}$.

Forward contracts and futures contracts are relatively straightforward to value given that the underlying is a traded asset and all variables are known at time t_0: the price of the underlying, the risk-free rate, the time to contract expiration, T, and any cash flows that will occur between t_0 and T. Most derivatives are not easy to value because of the stochastic nature of the underlying variables. In most cases, the underlying factors of a derivative contract are not even traded assets (i.e., volatility and interest rates). Interest rates in a simple model are assumed constant. In actuality, rates fluctuate and one must estimate and consider the evolution of the term structure of rates. Moreover, underlying assets such as stocks, bonds, and foreign currencies follow stochastic (diffusion) processes that must be considered in any realistic financial model.

Throughout this book, we incorporate the stochastic nature of financial variables into all of our models, and our implementations incorporate time evolution. Initially, we assume time-homogenous variables (i.e., constant interest rates), but eventually we relax this assumption and assume variables are a function not only of time, but also of other underlying factors. We begin our examination of derivative models by examining and deriving the most fundamental and ubiquitous pricing model, Black-Scholes.

1.2 BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION

Consider a riskless asset (a money market account or bank account), A_t, started at time 0 that grows with the constant continuously compounded risk-free rate of return r. The value of our money market account (MMA) at time t is:

$$A_t = e^{r(T-t)} \tag{1.7}$$

and it is a solution to a stochastic differential equation (SDE) with zero diffusion coefficient:

$$dA_t = rA_t dt \tag{1.8}$$

subject to $A_0 = \$1$. Equation (1.8) states that an infinitesimal change in our MMA value, dA_t, must be equal to the risk-free rate earned over the change in time, dt. If we know that value of our MMA at $t > 0$, then at time $T > t$, the value is:

$$A_t = A_0 e^{r(T-t)} \tag{1.9}$$

As will be shown, the MMA serves as a good *numeraire*, any positive non-dividend-paying asset, when we need to change measures to get an equivalent martingale measure for risk-neutral pricing of many derivatives (as we discuss in section 1.10).

Now suppose that S_t is the price at time t of a risky stock that pays no dividends (we extend to the case with dividends later). We model its time evolution by some diffusion process with Brownian motion (see Appendix B for a discussion of Brownian motion). But which one to select? The price process we select must satisfy three requirements:

1. The price should always be greater than or equal to zero. That is, our diffusion must have a natural boundary at zero. This immediately rules out arithmetic Brownian motion as a candidate for the realistic stock price process since arithmetic Brownian motion can have negative values.
2. If the stock price hits zero, corporate bankruptcy takes place. Once bankruptcy occurs, $S = 0$; the price can never rise above zero again. So zero should be an *absorbing* (cemetery) boundary.
3. The expected percentage return required by investors from a stock should be independent of the stock's price. Indeed, risk-averse investors will require some rate of return $m = r + r_e$ on the stock, where r_e is the required excess return over and above the risk-free rate r that investors require to compensate for taking the risk of holding the stock (risk premium). We will assume initially that this excess return is constant over time.

These restrictions limit the choice of our stochastic model to:

$$dS_t = mS_t dt + b(S_t, t)dz_t \tag{1.10}$$

where m is the drift coefficient, which in this case is the constant expected rate of return on the stock (in the real world) and $b(S_t, t)$ is some diffusion coefficient, and z_t is a Wiener process—that is, $z_t \sim N(0,1)$. If $b = 0$, then it is the SDE for the risk-free asset. For any risky asset, b cannot be zero. Since we require that zero is an absorbing boundary for the stock price process, we impose an extra restriction on b: $b(0, t) = 0$. Thus, if the stock ever hits zero, it will never rise above zero again (both the drift and diffusion terms are equal to zero in this state, and there is nothing to lift it out of zero). Thus, we can parameterize our diffusion coefficient as follows:

$$b(S, t) = \sigma(S, t)S \tag{1.11}$$

where σ is any positive function of S and t, or possibly some other stochastic variables influencing the stock. It is referred to as the *volatility* of the stock price and is a measure of variability or uncertainty in stock price movements. Clearly, the simplest choice is a constant volatility process:

$$dS_t = mS_t dt + \sigma S_t dz_t$$

or:

$$dS = mSdt + \sigma Sdz \tag{1.12}$$

where we have dropped the time subscript for ease of notation. Here, m and σ are the constant instantaneous expected rate of return on the stock (drift rate) and volatility of the stock price, respectively.

It turns out that this choice of constant volatility, although not entirely realistic, as we will see, is very robust and leads to a tractable model. The process is called *geometric Brownian motion* (geometric refers to the multiplicative nature of fluctuations). The assumption of constant volatility is reasonable as a first approximation. It means that the variance of the percentage return in a short period of time, dt, is the same regardless of the stock price. Then $\sigma^2 dt$ is the variance of the proportional change in the stock price in time dt, and $\sigma^2 S^2 dt$ is the variance of the actual change in the stock price, S, during dt.

The SDE in equation (1.12) can be integrated in closed form. Indeed, suppose we know the stock price S at time t, S_t, and we are interested in the price S_T at time T. We will solve the SDE subject to this initial condition by first introducing a new variable, x:

$$x = f(S) = \ln S \tag{1.13}$$

Ito's lemma (see Appendix B) tells us that any function f of S follows a diffusion process:

$$df = \left(\frac{df}{dt} + mS\frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} \right) dt + \sigma S \frac{df}{dS} dz \tag{1.14}$$

In the case of the logarithmic function we have:

$$dx = \left(m - \frac{\sigma^2}{2} \right) dt + \sigma dz \tag{1.15}$$

or

$$dx = \mu dt + \sigma dz$$

where $\mu = m - \sigma^2/2$.

Hence, a logarithm of the stock price follows an arithmetic Brownian motion with the drift rate $\mu = m - \sigma^2/2$ and diffusion coefficient σ. This SDE can be immediately integrated to yield:

$$x_T = x + \left(m - \frac{\sigma^2}{2} \right) \tau + \sigma\sqrt{\tau}\varepsilon_T \tag{1.16}$$

where we have made use of the fact that

$$dz = \varepsilon\sqrt{dt},\ \tau = T - t$$

and ε is a standard normal deviate. Thus, since $x = \ln S$, then:

$$\ln\frac{S_T}{S} = \left(m - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\varepsilon_T \tag{1.17}$$

or

$$S_T = S\left\{\left(m - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\varepsilon_T\right\} \tag{1.18}$$

This is a closed-form solution to the Brownian motion SDE. We can now find the transition probability density (the probability distribution function of S_T at T given S at t). Given x at t, x_T is normally distributed:

$$x_T \sim N\left(x + \left(m - \frac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right) \tag{1.19}$$

or:

$$p(x_T,\ T \mid x,\ t)dx_T = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left\{-\frac{(x_T - x - \mu\tau)^2}{2\sigma^2\tau}\right\}dx_T \tag{1.20}$$

where $\mu = m - \sigma^2/2$. Then $\ln S_T$ is also normally distributed:

$$\ln S_T \sim N\left(\ln S + \mu(T-t), \sigma\sqrt{T-t}\right)$$

or:

$$p(x_T,\ T \mid x,\ t)dx_T = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\left\{-\frac{\left(\ln\left(\frac{S_T}{S}\right) - \mu\tau\right)^2}{2\sigma^2\tau}\right\}\frac{dS_T}{dS} \tag{1.21}$$

(Note that $dx_T = dS_T/dS$.) This is the *lognormal distribution*.

We can now calculate the moments of the lognormal distribution around zero. We need to calculate the mean and variance by taking expectations:

$$M_n(0) = E_{t,S}\left[S_T^n\right] = \int_0^\infty S_T^n p(S_T,\ T \mid S,\ t) dS_T \tag{1.22}$$

where $E_{t,S}$ is the expectation operator taken over S at time t. However, we can actually calculate the moments without calculating the integral. Since $S_T = \exp(x_T)$, we need to calculate the expectation:

$$M_n(0) = E_{t,x}[e^{nx_T}] \tag{1.23}$$

Since x_T is normally distributed, we can use the characteristic function of the normal distribution to help calculate expectations:

$$\phi(\lambda) = E[e^{i\lambda x_T}] = \exp\left(i\lambda(x + \mu\tau) - \frac{\sigma^2\tau\lambda^2}{2}\right) \tag{1.24}$$

Substituting $i\lambda \to n$ and recalling that $x = \ln S$ and $\mu = m - \sigma^2/2$, we have:

$$M_n(0) = E_{t,S}[e^{nx_T}] = S^n \exp\left(n\mu\tau + \frac{n^2\sigma^2\tau}{2}\right) \tag{1.25}$$

$$= S^n \exp\left(nm\tau + \frac{n(n-1)}{2}\sigma^2\tau\right) \tag{1.26}$$

In particular, the mean:

$$E_{t,S}[S_T] = e^{m\tau}S \tag{1.27}$$

and the variance is:

$$\mathrm{Var}_{t,S} = S^2\ [e^{2mr+\sigma^2\tau} - e^{2mr}] = S^2 e^{2mr}[e^{\sigma^2\tau} - 1]$$

We will use these moments when we need to match moments to the binomial distribution when we value options using binomial and trinomial trees (lattices). We al-