MODELING AND HIGH-PERFORMANCE CONTROL OF ELECTRIC MACHINES

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MODELING AND HIGH-PERFORMANCE CONTROL OF ELECTRIC MACHINES

JOHN CHIASSON

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To My Parents

and to

Marc Bodson

Pour son soutien et son aide jour après jour durant une période très difficile de ma carrière

Amro El-Jaroudi
Mahmoud El Nokali

إِلَى اصدقَائِي الذينَ وَقَفُوا بِجَانِبِي في تِلكَ الأَوقَاتِ العَصيبَةِ مِن حَيَاتي

James B. Lieber

על שתמיד האמנת בי ועל עזרתך הרבה בשעתי הקשה בקריירה האקדמית שלי

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Contents

[1] Sections marked with an asterisk (*) may be skipped without loss of continuity.

Preface

This book is intended to be an exposition of the modeling and control of electric machines, specifically, the direct current (DC) machine and the alternating current (AC) machines consisting of the induction motor, the permanent magnet (PM) synchronous motor, and the brushless DC motor. The particular emphasis here is on techniques used for high-performance applications, that is, applications that require both rapid and precise control of position, speed, and/or torque. Traditionally, DC motors were reserved for high-performance applications (positioning systems, rolling mills, traction drives, etc.) because of their relative ease of control compared to AC machines. However, with the advances in control methods, computing capability, and power electronics, AC motors continue to replace DC motors in high-performance applications. The intent here is to carefully derive the mathematical models of the AC machines and show how these mathematical models are used to design control algorithms that achieve high performance.

Electric machines are a particularly fascinating application of basic electricity and magnetism. The presentation here relies heavily on these basic concepts from Physics to develop the models of the motors. Specifically, Faraday's law ($\xi = -d\phi/dt$, where $\phi = \int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}}$), the magnetic force law ($\vec{\mathbf{F}} = i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}$ or, $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$), Gauss's law ($\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} = 0$), Ampère's law ($\oint \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{\ell}} = i_{\text{free}}$), the relationship between $\vec{\mathbf{B}}$ and $\vec{\mathbf{H}}$, properties of magnetic materials, and so on are reviewed in detail and used extensively to derive the currently accepted nonlinear differential equation models of the various AC motors. The author made his best attempt to make the modeling assumptions as clear as possible and to consistently show that the magnetic and electric fields satisfy Maxwell's equations (as, of course, they must). The classical approach to teaching electric machinery is to present their equivalent circuit models and to analyze these circuit models ad nauseam. Further, the use of the basic Physics of electricity and magnetism to explain their operation is minimized if not omitted. However, the equivalent circuit is a result of assuming constant-speed operation of the machine and computing the sinusoidal steady-state solution of the nonlinear differential equation model of the machine. Here, the emphasis is on explaining how the machines work using fundamental concepts from electricity and magnetism, and on the derivation of their nonlinear differential equation models. The derivation of the corresponding equivalent circuit assuming steady-state conditions is then straightforward.

Electric machines also provide fascinating examples to illustrate concepts from electromagnetic field theory (in contrast to electricity and magnetism). In particular, the way the electric and magnetic fields change as one goes between reference frames that are in relative motion are vividly illustrated using AC machines. For this reason, optional sections are included to show how the electric and magnetic fields change as one goes between a coordinate system attached to the stator to a coordinate system that rotates with the rotating magnetic field produced by the stator currents or a frame attached to the rotor. Also given in an optional section is the derivation of the axial electric and azimuthal magnetic fields in the air gap.

This is also a book on the control of electric machines based on their differential equation models. With the notable exception of the sinusoidal steady-state analysis of the induction motor in Chapter 7, very little attention is given to the classical equivalent circuits as these models are valid only in steady state. Rather, the differential equation models are used as the basis to develop the notions of field-oriented control, input-output linearization, flux observers, least-squares identification methods, state feedback trajectory tracking, and so on. This is a natural result of the emphasis here on high-performance control methods (e.g., field-oriented control) as opposed to classical methods (e.g., V/f, slip control, etc.).

There are of course many good books in the area of electric machines and their control. The author owes a debt of gratitude to Professor W. Leonhard for his book [1] (see the most recent edition [2]), from which he was educated in the modeling and control of electric drives. The present book is narrower in focus with an emphasis on the modeling and operation of electric machines based on elementary classical physics and an emphasis on high-performance control methods using a state-space formulation. The books by P. C. Krause [3] and P. C. Krause et al. [4] are complete in their derivation of the mathematical models of electric machines while C. B. Gray [5] presents electromagnetic theory in the context of electric machines. A comprehensive treatment using SIMULINK to simulate electric machinery is given in C-M. Ong's book [6]. The graduate level books by D. W. Novotny and T. A. Lipo [7], P. Vas [8], J. M. D. Murphy and F. G. Turnbull [9], I. Boldea and S. A. Nasar [10], B. Adkins and R. G. Harley [11], A. M. Trzynadlowski [12], M. P. Kazmierkowski and H. Tunia [13], B. K. Bose [14], and R. Krishnan [15] all cover the modeling and control of electric machines while the books by R. Ortega et al. [16], D. M. Dawson et al. [17], and F. Khorrami et al. [18] emphasize advanced control methods.

The introductory-level books by S. J. Chapman [19], H. Woodson and J. Melcher [20], L. W. Matsch and J. D. Morgan [21], G. McPherson and R. D. Laramore [22], D. V. Richardson [23], P. C. Krause and O. Wasynczuk [24], N. Mohan [25], G. R. Slemon and A. Straughn [26], J. Sokira and W. Jaffe [27], G. J. Thaler and M. L. Wilcox [28], V. Deltoro [29], M. El-Hawary [30], P. C. Sen [31], and G. R. Slemon [32] are among the many books on electric machines from which this author has benefited.

The beautifully written textbooks *PSSC Physics* by the Physical Science Curriculum Study [33], *Physics* by D. Halliday and R. Resnick [34], *Principles of Electrodynamics* by M. Schwartz [35], and *Electromagnetic Fields* by R. K. Wangness [36] are used as references for the theory of electricity and magnetism.

This book borrows from these above works and hopefully makes its own contribution to the literature on electric machines.

Part I of the book consists of the first three chapters. Chapters 1 and 2 present a detailed review of the basic concepts of electricity and magnetism in the context of DC machines and an introduction to control methods, respectively, which will be used extensively in the remaining chapters. The third chapter on magnetic fields and magnetic materials is intended to be a detailed introduction to the subject. For example, most textbooks assume that the reader understands Ampère's law in the form $\oint \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{free}}$ and that $\vec{\mathbf{B}} = \mu\vec{\mathbf{H}}$ in (soft) magnetic materials, yet it is the experience of the author that students do not have a fundamental understanding of these concepts.

These first three chapters are elementary in nature and were written to be accessible to undergraduates. The reason for this is that often control engineers do not have any background in electric machinery while power/electric-machine engineers often do not have any background in basic state-space concepts of control theory. Consequently, it is hoped that these chapters can bring the reader "up to speed" in these areas.

Chapter 1 reviews the basic ideas of electricity and magnetism that are needed to model electric machines. In particular, the notions of magnetic fields, magnetic force and Faraday's law are reviewed by using them to derive the standard model of a DC motor.

Chapter 2 provides an elementary introduction to the control techniques required for the high-performance control of electric machines. This includes an elementary presentation of state feedback control, observers, and identification theory as applied to DC machines to prepare the reader for the subsequent chapters.

Chapter 3 goes into the modeling of magnetic materials in terms of their use in electric machines. The fundamental result of this chapter is the modification of Ampère's law $\oint_C \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 i$ so that it is valid in the presence of magnetic material. This introduces the magnetic intensity field $\vec{\mathbf{H}}$ and its relationship to magnetic induction field $\vec{\mathbf{B}}$ via the magnetization vector $\vec{\mathbf{M}}$ to obtain the more general version of Ampère's law $\oint \vec{\mathbf{H}} \cdot d\vec{\ell} = i_{\text{free}}$. All of this requires a significant discussion of the modeling of magnetic materials. The approximation $\vec{\mathbf{H}} = 0$ in magnetic materials is discussed, and then it is shown how this approximation along with Ampère's law can be used to find the radial component of $\vec{\mathbf{B}}$ in the air gap of electric machines. Also presented is Gauss's law for $\vec{\mathbf{B}}$; this leads to the notion of conservation of flux, as well as the fact that $\vec{\mathbf{B}}$ is normal to the surface of soft magnetic materials. This chapter should be read, but the reader should not get "bogged down" in the chapter. Rather, the main results should be remembered.

Part II consists of Chapters 4 through 10 and presents the modeling and control of AC machines.

Chapter 4 uses the results of Chapters 1 and 3 to explain how a radially directed rotating magnetic field can be established in the air gap of AC machines. In particular, the notions of distributed windings and of sinusoidally wound turns (phase windings) are explained. Ampère's law is then used to show that a sinusoidal (spatially) distributed radial magnetic field is established in the air gap by the currents in the phase windings. The concept of flux linkage in distributed windings is explained, and the chapter ends with an optional section on the azimuthal magnetic field in the air gap.

Chapter 5 explains the fundamental Physics behind the working of induction and synchronous machines. Specifically, this chapter uses a simplified model of the induction motor and shows how voltages and currents are induced in the rotor loops by the rotating magnetic field established by the stator currents. Then it is shown how torque is produced on these induced currents by the same stator rotating magnetic field that induced them introducing the idea of slip. Similarly, the synchronous machine is analyzed to show how the rotating radial magnetic field established by the stator currents produces torque on a rotor carrying constant current.

An optional section on the microscopic point of view of the Physics of the induction motor is also presented. This includes a discussion of how the electric and magnetic fields change as one goes between coordinate systems that are rotating with respect to each other and how one reinterprets the Physics of the machine's operation. The chapter ends with another optional section of the steady-state behavior of an induction machine with a squirrel cage rotor.

Chapter 6 derives the systems of differential equations that mathematically model the two-phase induction and synchronous machines. The concept of leakage is presented and accounted for in the derived models. These models are the accepted models used throughout the literature and form the basis for high-performance control of these machines. In an optional section it is shown that the stator and rotor *magnetic fields* of an induction motor rotate synchronously together as they do in a synchronous machine. The chapter ends with another optional section on the concepts of field energy and co-energy, and how the expression for the torque of an electric machine can be derived using these notions.

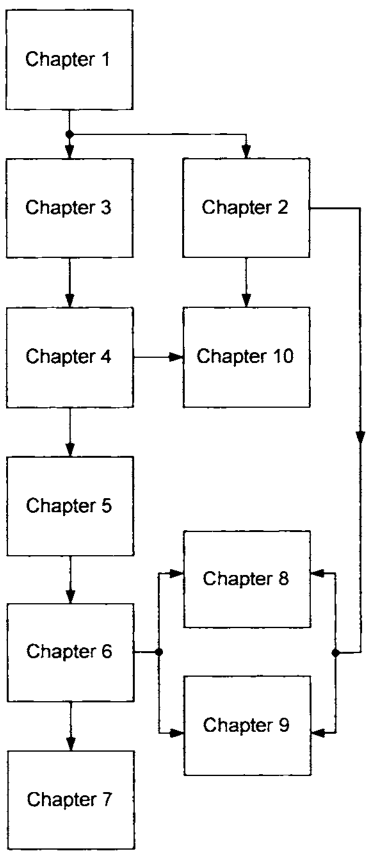
Chapter 7 presents the derivation of the models of three-phase AC machines and their two-phase equivalent models. These derivations readily follow from the results of Chapter 6. The classical steady-state analysis of the induction motor is also presented including its equivalent circuit. The chapter ends with a discussion of why the standard power system is an AC sinusoidal three-phase 60-Hz (or 50-Hz) system.

Chapter 8 covers the control of induction motors presenting both field-oriented control and input-output linearization control. Flux observers, field weakening, and speed observers are also presented along with experimental results. The chapter ends with an optional section on how to identify the induction motor parameters using a nonlinear least-squares technique.

Chapter 9 covers the control of synchronous motors describing field-oriented control, field weakening, speed observers and identification methods. The operation and modeling of permanent magnet stepping motors is also covered.

Chapter 10 covers the modeling and control of PM synchronous motors with trapezoidal back emf, which are also known as brushless DC (BLDC) motors.

The logical dependence of the chapters is shown in the block diagram below assuming that the optional sections are *not* covered.



Logical dependence of the chapters.

Finally, the author's intent for this book was for the reader to understand how electric machines are modeled and to understand the basic techniques in their control. The references at the end of the book are only those directly referenced in the book and are *not* representative of (nor give proper recognition to) the many important contributions made by researchers throughout the world. The reader is referred to Professor Leonhard's book [2] for a much more extensive reference list.

Comments on the Use of the Book

In using this book in a one-semester graduate-level course, the following material was usually covered:

Chapter 1, Sections 1.1−1.7
Chapter 2, Sections 2.1−2.4
Chapter 3, Sections 3.1−3.4
Chapter 4, Sections 4.1−4.5
Chapter 5, Sections 5.1−5.3
Chapter 6, Sections 6.1−6.10
Chapter 7, Sections 7.1−7.3
Chapter 8, Sections 8.1−8.3
Chapter 9, Section 9.1

Sections marked with an asterisk (*) may be omitted without loss of continuity. Some of these optional sections assume familiarity with Maxwell's equations in *differential* form.

Acknowledgments

There are many people that I would to acknowledge for their help in my work in the control of electric machines.

I did my Ph.D. thesis in Algebraic Systems Theory which is quite far from the area of Electric Machines. Nevertheless, I am most grateful to my advisor, Professor E. Bruce Lee, who has always shown me his kindness over the years.

Professor Edward W. Kamen (formerly at the University of Pittsburgh) along with Mr. Stephen Botos (President of Aerotech, Inc.) were instrumental through their enthusiasm and financial support at the University of Pittsburgh, through which many of the results presented here were funded. I am very grateful to Mike Aiello (chief design engineer at Aerotech) for both designing and building our hardware platform, resulting in a successful set of experiments.

I would like to thank The Oak Ridge National Laboratory in particular, Don Adams and Laura Marlino, for funding the recent results on the identification of the induction motor parameters presented in Chapter 8.

Shortly after I arrived at the University of Pittsburgh, I started to work with Marc Bodson (formerly at Carnegie Mellon University) and I want to express my deep gratitude to him for this collaboration, which led to many of the new theoretical and experimental results presented in Chapters 8 and 9.

Also, shortly after I arrived at the University of Tennessee, I began to work with Leon M. Tolbert, and I am very grateful for this collaboration as well.

Mohamed Zribi was the first student I worked with in this area, leading to a paper on feedback linearization control of stepper motors. Ron Rekowski suffered through our first attempts to do some experiments for which I am grateful. My Ph.D. student Bob Novotnak continued this work and did the experiments on the control of the stepper and induction motors presented in Chapters 8 and 9 of this book. Through his skill we were able to obtain experimental results demonstrating very high performance. Jennifer Stephans (Marc Bodson's student) did the early work on identification for the induction motor while my student Kaiyu Wang did the identification experiments for the induction motor given in Chapter 8. Andy Blauch (Marc Bodson's student) did the identification experiments for the PM stepper motor presented in Chapter 9.

I first taught (and learned!) induction motor control using the book by Professor Werner Leonhard, and I am very grateful for his writing that book. I later had the opportunity to visit his institute in Braunschweig Germany from which I left with even more enthusiasm for the field.

I would like to thank the many students who suffered under the early versions of this book, or suffered with me as their advisor, or both. These include Mohamed Zribi, Bob Novotnak, Eric Shook, Ron Rekowski, Walt Barie, Joe Matesa, Chellury Sastry, Gary Campbell, David Schuerer, Atul Chaudhari, Jason Mueller, Samir Mehta, Pete Hammond, Dick Osman, Jim Short, Vincent Allarouse, Marc Aiello, Sean West, Baskar Vairamohan, Yinghui Lu, Zhong Tang, Mengwei Li, Kaiyu Wang, Yan Xu, Madhu Chinthavali, Jianqing Chen, Nivedita Alluri, Zhong Du, Faisal Khan, Pankaj Pandit, Hui Zhang, Wenjuan Zhang, Ben Sooter, Keith McKenzie, Rebin Zhou, Ann Chee Tan, Jesse Richmond, SeongTaek Lee, and all my other students.

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John Wiley & Sons maintains an ftp site at

ftp://ftp.wiley.com/public/sci_tech_med/high_performance_control

for downloading an errata sheet for the book. Instructors, upon obtaining password privileges, will also be able to download the simulation files that go with this textbook. A solutions manual is available to instructors by contacting their local Wiley representative.

Any comments, criticisms, and corrections are most welcome and may be sent to the author at *chiasson@ieee.org.*

John Chiasson

MODELING AND HIGH-PERFORMANCE CONTROL OF ELECTRIC MACHINES

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Part I

DC Machines, Controls, and Magnetics

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1

The Physics of the DC Motor

The principles of operation of a direct current (DC) motor are presented based on fundamental concepts from electricity and magnetism contained in any basic physics course. The DC motor is used as a concrete example for reviewing the concepts of magnetic fields, magnetic force, Faraday's law, and induced electromotive forces (emf) that will be used throughout the remainder of the book for the modeling of electric machines. All of the Physics concepts referred to in this chapter are contained in the book *Physics* by Halliday and Resnick [34].

1.1 Magnetic Force

Motors work on the basic principle that magnetic fields produce forces on wires carrying a current. In fact, this experimental phenomenon is what is used to define the magnetic field. If one places a current carrying wire between the poles of a magnet as in Figure 1.1, a force is exerted on the wire. Experimentally, the magnitude of this force is found to be proportional to both the amount of current in the wire and to the length of the wire that is between the poles of the magnet. That is, F_{magnetic} is proportional to ℓi. The direction of the magnetic field $\vec{\mathbf{B}}$ at any point is defined to be the direction that a small compass needle would point at that location. This direction is indicated by arrows in between the north and south poles in Figure 1.1.

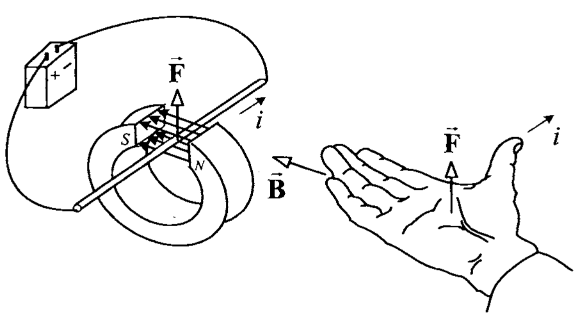

FIGURE 1.1. Magnetic force law. From *PSSC Physics*, 7th edition, by Haber-Schaim, Dodge, Gardner, and Shore, published by Kendall/Hunt, 1991.

With the direction of $\vec{\mathbf{B}}$ perpendicular to the wire, the strength (magnitude) of the *magnetic induction field* $\vec{\mathbf{B}}$ is defined to be

$$B = |\vec{\mathbf{B}}| \triangleq \frac{F_{\text{magnetic}}}{\ell i}$$

where F_{magnetic} is the magnetic force, i is the current, and ℓ is the length of wire perpendicular to the magnetic field carrying the current. That is, B is the proportionality constant so that $F_{\text{magnetic}} = i\ell B$. As illustrated in Figure 1.1, the direction of the force can be determined using the right-hand rule. Specifically, using your right hand, point your fingers in the direction of the magnetic field and point your thumb in the direction of the current. Then the direction of the force is out of your palm.

Further experiments show that if the wire is parallel to the $\vec{\mathbf{B}}$ field rather than perpendicular as in Figure 1.1, then no force is exerted on the wire. If the wire is at some angle θ with respect to $\vec{\mathbf{B}}$ as in Figure 1.2, then the force is proportional to the *component* of $\vec{\mathbf{B}}$ perpendicular to the wire; that is, it is proportional to $B_\perp = B\sin(\theta)$. This is summarized in the *magnetic force law*: Let $\vec{\boldsymbol{\ell}}$ denote a vector whose magnitude is the length ℓ of the wire in the magnetic field and whose direction is defined as the positive direction of current in the bar; then the magnetic force on the bar of length ℓ carrying the current i is given by

$$\vec{\mathbf{F}}_{\text{magnetic}} = i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}}$$

or, in scalar terms, $F_{\text{magnetic}} = i\ell B\sin(\theta) = i\ell B_\perp$. Again, $B_\perp \triangleq B\sin(\theta)$ is the component of $\vec{\mathbf{B}}$ perpendicular to the wire.[1]

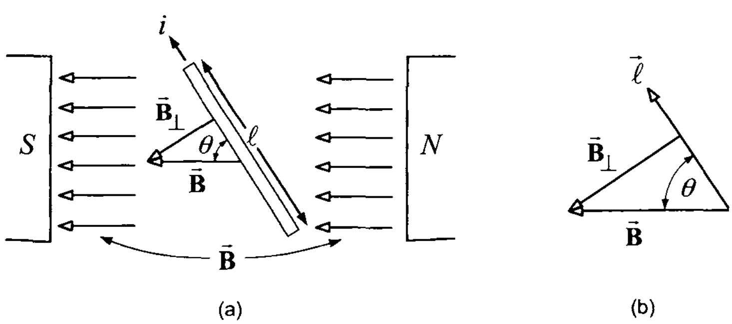

FIGURE 1.2. Only the component $B_\perp$ of the magnetic field which is perpendicular to the wire produces a force on the current.

[1]Motors are designed so that the conductors are perpendicular to the external magnetic field.

Example *A Linear DC Machine* [19]

Consider the simple linear DC machine in Figure 1.3 where a sliding bar rests on a simple circuit consisting of two rails. An external magnetic field is going through the loop of the circuit up out of the page indicated by the $\otimes$ in the plane of the loop. Closing the switch results in a current flowing around the circuit and the external magnetic field produces a force on the bar which is free to move. The force on the bar is now computed.

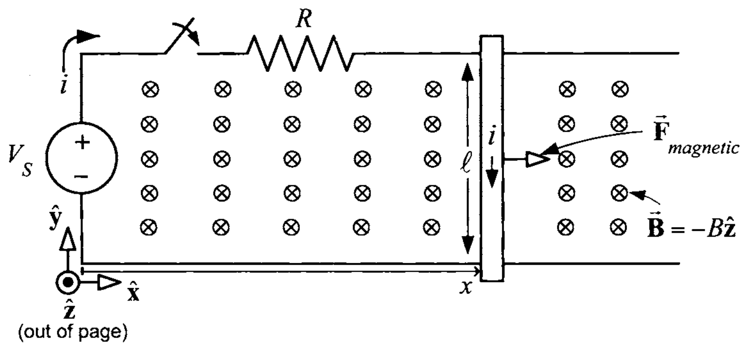

FIGURE 1.3. A linear DC motor.

The magnetic field is constant and points into the page (indicated by $\otimes$) so that written in vector notation, $\vec{\mathbf{B}} = -B\hat{\mathbf{z}}$ with $B > 0$. By the right hand rule, the magnetic force on the sliding bar points to the right. Explicitly, with $\vec{\boldsymbol{\ell}} = -\ell\hat{\mathbf{y}}$, the force is given by

$$\begin{aligned}\vec{\mathbf{F}}_{\text{magnetic}} &= i\vec{\boldsymbol{\ell}} \times \vec{\mathbf{B}} = i(-\ell\hat{\mathbf{y}}) \times (-B\hat{\mathbf{z}}) \\ &= i\ell B\hat{\mathbf{x}}.\end{aligned}$$

To find the equations of motion for the bar, let f be the coefficient of viscous (sliding) friction of the bar so that the friction force is given by $F_f = -f dx/dt$. Then, with m_ℓ denoting the mass of the bar, Newton's law gives

$$i\ell B - f dx/dt = m_\ell d^2x/dt^2.$$

Just after closing the switch at $t = 0$, but before the bar starts to move, the current is $i(0^+) = V_S(0^+)/R$. However, it turns out that as the bar moves the current does *not* stay at this value, but instead decreases due to electromagnetic induction. This will be explained later.

1.2 Single-Loop Motor

As a first step to modeling a DC motor, a simplistic single-loop motor is considered. It is first shown how torque is produced and then how the current in the single loop can be reversed (commutated) every half turn to keep the torque constant.

1.2.1 Torque Production

Consider the magnetic system in Figure 1.4, where a cylindrical core is cut out of a block of a permanent magnet and replaced with a soft iron core. The term "soft" iron refers to the fact that material is easily magnetized (a permanent magnet is referred to as "hard" iron).

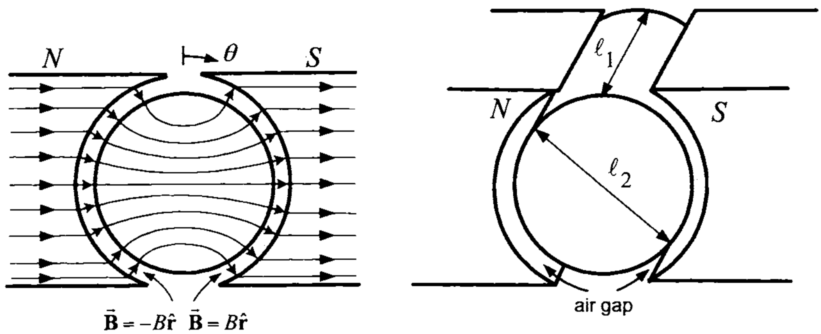

FIGURE 1.4. Soft iron cylindrical core placed inside a hollowed out permanent magnet to produce a radial magnetic field in the air gap.

An important property of soft magnetic materials is that the magnetic field at the surface of such materials tends to be normal (perpendicular) to the surface. Consequently, the cylindrical shape of the surfaces of the soft iron core and the stator permanent magnet has the effect of making the field in the air gap *radially* directed; furthermore, it is reasonably constant (uniform) in magnitude. A mathematical description of the magnetic field in the air gap due to the permanent magnet is simply

$$\vec{\mathbf{B}} = \begin{cases} +B\hat{\mathbf{r}} & \text{for } 0 < \theta < \pi \\ -B\hat{\mathbf{r}} & \text{for } \pi < \theta < 2\pi \end{cases}$$

where $B > 0$ is the magnitude or strength of the magnetic field and θ is an arbitrary location in the air gap.[2]

Figure 1.5 shows a rotor loop wound around the iron core of Figure 1.4. The length of the rotor is ℓ_1 and its diameter is ℓ_2. The torque on this rotor loop is now calculated by considering the magnetic forces on sides a and a' of the loop. On the other two sides of the loop, that is, the front and

[2] Actually it will be shown in a later chapter that the magnetic field must be of the form $\vec{\mathbf{B}} = \pm B(r_0/r)\hat{\mathbf{r}}$ in the air gap, that is, it varies as $1/r$ in the air gap. However, as the air gap is small, the $\vec{\mathbf{B}}$ field is essentially constant across the air gap.