NONLINEAR EFFECTS IN OPTICAL FIBERS
NONLINEAR EFFECTS IN OPTICAL FIBERS

MÁRIO F. S. FERREIRA
CONTENTS

Preface xi

1 Introduction 1

References / 5

2 Electromagnetic Wave Propagation 9

2.1 Wave Equation for Linear Media / 9
2.2 Electromagnetic Waves / 11
2.3 Energy Density and Flow / 13
2.4 Phase Velocity and Group Velocity / 14
2.5 Reflection and Transmission of Waves / 16
   2.5.1 Snell’s Laws / 16
   2.5.2 Fresnel Equations / 17
2.6 The Harmonic Oscillator Model / 21
2.7 The Refractive Index / 23
2.8 The Limit of Geometrical Optics / 24
Problems / 26
References / 27

3 Optical Fibers 29

3.1 Geometrical Optics Description / 30
   3.1.1 Planar Waveguides / 30
   3.1.2 Step-Index Fibers / 33
   3.1.3 Graded-Index Fibers / 36
3.2 Wave Propagation in Fibers / 39
  3.2.1 Fiber Modes / 39
  3.2.2 Single-Mode Fibers / 42
3.3 Fiber Attenuation / 44
3.4 Modulation and Transfer of Information / 45
3.5 Chromatic Dispersion in Single-Mode Fibers / 46
  3.5.1 Unchirped Input Pulses / 48
  3.5.2 Chirped Input Pulses / 52
  3.5.3 Dispersion Compensation / 53
3.6 Polarization Mode Dispersion / 54
  3.6.1 Fiber Birefringence and the Intrinsic PMD / 55
  3.6.2 PMD in Long Fiber Spans / 57
Problems / 60
References / 61

4 The Nonlinear Schrödinger Equation / 63
4.1 The Nonlinear Polarization / 63
  4.1.1 The Nonlinear Wave Equation / 66
4.2 The Nonlinear Refractive Index / 66
4.3 Importance of Nonlinear Effects in Fibers / 68
4.4 Derivation of the Nonlinear Schrödinger Equation / 70
  4.4.1 Propagation in the Absence of Dispersion and Nonlinearity / 73
  4.4.2 Effect of Dispersion Only / 73
  4.4.3 Effect of Nonlinearity Only / 74
  4.4.4 Normalized Form of NLSE / 74
4.5 Soliton Solutions / 75
  4.5.1 The Fundamental Soliton / 76
  4.5.2 Solutions of the Inverse Scattering Theory / 77
  4.5.3 Dark Solitons / 80
4.6 Numerical Solution of the NLSE / 81
Problems / 83
References / 84

5 Nonlinear Phase Modulation / 85
5.1 Self-Phase Modulation / 86
  5.1.1 SPM-Induced Phase Shift / 86
  5.1.2 The Variational Approach / 89
  5.1.3 Impact on Communication Systems / 93
  5.1.4 Modulation Instability / 94
5.2 Cross-Phase Modulation / 97
   5.2.1 XPM-Induced Phase Shift / 97
   5.2.2 Impact on Optical Communication Systems / 100
   5.2.3 Modulation Instability / 103
   5.2.4 XPM-Paired Solitons / 105

Problems / 106
References / 107

6 Four-Wave Mixing / 111
   6.1 Wave Mixing / 112
   6.2 Mathematical Description / 114
   6.3 Phase Matching / 115
   6.4 Impact and Control of FWM / 118
   6.5 Fiber Parametric Amplifiers / 123
      6.5.1 FOPA Gain and Bandwidth / 123
   6.6 Parametric Oscillators / 128
   6.7 Nonlinear Phase Conjugation with FWM / 131
   6.8 Squeezing and Photon Pair Sources / 133

Problems / 135
References / 135

7 Intrachannel Nonlinear Effects / 139
   7.1 Mathematical Description / 140
   7.2 Intrachannel XPM / 142
   7.3 Intrachannel FWM / 147
   7.4 Control of Intrachannel Nonlinear Effects / 149

Problems / 153
References / 153

8 Soliton Lightwave Systems / 155
   8.1 Soliton Properties / 156
      8.1.1 Soliton Interaction / 157
   8.2 Perturbation of Solitons / 159
      8.2.1 Perturbation Theory / 160
      8.2.2 Fiber Losses / 160
   8.3 Path-Averaged Solitons / 162
      8.3.1 Lumped Amplification / 163
      8.3.2 Distributed Amplification / 164
      8.3.3 Timing Jitter / 166
8.4 Soliton Transmission Control / 168
  8.4.1 Fixed-Frequency Filters / 169
  8.4.2 Sliding-Frequency Filters / 170
  8.4.3 Synchronous Modulators / 173
  8.4.4 Amplifier with Nonlinear Gain / 174
8.5 Dissipative Solitons / 176
  8.5.1 Analytical Results of the CGLE / 176
  8.5.2 Numerical Solutions of the CGLE / 180
8.6 Dispersion-Managed Solitons / 183
  8.6.1 The True DM Soliton / 183
  8.6.2 The Variational Approach to DM Solitons / 185
8.7 WDM Soliton Systems / 189
Problems / 192
References / 193

9 Other Applications of Optical Solitons 199
  9.1 Soliton Fiber Lasers / 199
    9.1.1 The First Soliton Laser / 200
    9.1.2 Figure-Eight Fiber Laser / 201
    9.1.3 Nonlinear Loop Mirrors / 201
    9.1.4 Stretched-Pulse Fiber Lasers / 202
    9.1.5 Modeling Fiber Soliton Lasers / 203
  9.2 Pulse Compression / 204
    9.2.1 Grating-Fiber Compressors / 204
    9.2.2 Soliton-Effect Compressors / 207
    9.2.3 Compression of Fundamental Solitons / 210
  9.3 Fiber Bragg Gratings / 213
    9.3.1 Pulse Compression Using Fiber Gratings / 214
    9.3.2 Fiber Bragg Solitons / 216
Problems / 220
References / 220

10 Polarization Effects 225
  10.1 Coupled Nonlinear Schrödinger Equations / 226
  10.2 Nonlinear Phase Shift / 227
  10.3 Solitons in Fibers with Constant Birefringence / 229
  10.4 Solitons in Fibers with Randomly Varying Birefringence / 234
  10.5 PMD-Induced Soliton Pulse Broadening / 236
10.6 Dispersion-Managed Solitons and PMD / 240
Problems / 242
References / 242

11 Stimulated Raman Scattering 245
11.1 Raman Scattering in the Harmonic Oscillator Model / 246
11.2 Raman Gain / 250
11.3 Raman Threshold / 252
11.4 Impact of Raman Scattering on Communication Systems / 255
11.5 Raman Amplification / 258
11.6 Raman Fiber Lasers / 264
Problems / 269
References / 270

12 Stimulated Brillouin Scattering 273
12.1 Light Scattering at Acoustic Waves / 274
12.2 The Coupled Equations for Stimulated Brillouin Scattering / 277
12.3 Brillouin Gain and Bandwidth / 278
12.4 Threshold of Stimulated Brillouin Scattering / 280
12.5 SBS in Active Fibers / 282
12.6 Impact of SBS on Communication Systems / 284
12.7 Fiber Brillouin Amplifiers / 286
  12.7.1 Amplifier Gain / 287
  12.7.2 Amplifier Noise / 289
  12.7.3 Other Applications of the SBS Gain / 290
12.8 SBS Slow Light / 293
12.9 Fiber Brillouin Lasers / 296
Problems / 300
References / 301

13 Highly Nonlinear and Microstructured Fibers 305
13.1 The Nonlinear Parameter in Silica Fibers / 306
13.2 Microstructured Fibers / 309
13.3 Non-Silica Fibers / 314
13.4 Soliton Self-Frequency Shift / 317
13.5 Four-Wave Mixing / 320
PREFACE

The first generation of fiber-optic communication systems was introduced early in the 1980s and operated at modest values of both the bit rate and the link length. In such circumstances, the nonlinear effects were found to be irrelevant. However, the situation changed dramatically during the 1990s with the advent and commercialization of wideband optical amplifiers, wavelength division multiplexing, and high-speed optoelectronic devices. By the end of that decade, the capacity of lightwave systems had already exceeded 1 Tb/s, as a result of the combination of larger number of WDM channels and increased channel data rates, together with denser channel spacings. Significant performance improvements were achieved in the following years, which paved the way for today’s systems with rates approaching 100 Gb/s per channel (wavelength) and wavelength counts of 80–100. On the other hand, higher channel powers are being used in long-haul landline and submarine links in order to increase the distances between amplifiers or repeaters. As a result of all these advances, the nonlinear effects in optical fibers became of paramount importance, since they adversely affect the system performance.

Paradoxically, the same nonlinear phenomena that have several important limitations also offer the promise of addressing the bandwidth bottleneck for signal processing for future ultrahigh-speed optical networks. Electronic devices are not suitable for such systems, due to their cost, complexity, and practical speed limits. Nonlinear optical signal processing, making use of the third-order optical nonlinearity in single-mode fibers, appears as a key and promising technology for improving the transparency and increasing the capacity of future full “photonic networks.”

Starting in 1996, new types of fibers, known as photonic crystal fibers, holey fibers, or microstructured fibers, were developed. These fibers have a relatively narrow core, surrounded by a cladding that contains an array of embedded air holes. Structural changes in such fibers profoundly affect their dispersive and nonlinear properties. The efficiency of the nonlinear effects can be further increased if some highly nonlinear materials are used to make the fibers, instead of silica. Using such highly nonlinear fibers, the required fiber length for nonlinear processing could be reduced to the order of centimeters, instead of the several kilometers long conventional silica fibers. All these advances have led to considerable growth in the field of nonlinear fiber optics during the last decade.
This book provides an introduction to the fascinating world of nonlinear phenomena occurring inside the optical fibers. Though the main emphasis is placed on the physical background of the different nonlinear effects, the technical aspects associated with their impact on optical communication systems, as well as their potential applications, particularly for signal processing, pulse generation, and amplification, are also discussed. An attempt has been made to include the latest and most significant research results in this area. Moreover, several problems are included at the end of each chapter. These aspects contribute to make this book of potential interest to senior undergraduate and graduate students enrolled in M.S. and Ph.D. degree programs, engineers and technicians involved with the fiber-optics industry, and researchers working in the field of nonlinear fiber optics.

I am deeply grateful to the many students and colleagues with whom I have interacted over the years. All of them have contributed to this book either directly or indirectly. In particular, I thank especially Sofia Latas for providing many figures, as well as Margarida Facão, Armando Pinto, and Nelson Muga for several discussions concerning different parts of this text. Last but not least, I thank my family for understanding why I needed to remain working during many of our weekends and vacations.

MÁRIO F. S. FERREIRA

Aveiro, Portugal
February 2011
INTRODUCTION

The propagation of light in optical fibers is based on the phenomenon of total internal reflection, which is known since 1854, when John Tyndall demonstrated the transmission of light along a stream of water emerging from a hole in the side of a tank [1]. Glass fibers were fabricated since the 1920s, but their use remained restricted to medical applications until the 1960s. The use of such fibers for optical communications was not practical, due to their high losses (≈1000 dB/km). However, Kao and Hockham [2] suggested in 1966 that optical fibers could be used in communication systems if their losses were reduced below 20 dB/km. Following an intense activity on the purification of fused silica, such goal was achieved in 1970 by Corning Glass Works, in the United States. Further technological progress allowed the reduction of fiber loss to 0.2 dB/km near the 1550 nm spectral region by 1979 [3]. This achievement led to a revolution in the field of optical fiber communications.

Besides the loss, the fiber dispersion constitutes actually another main problem affecting the performance of an optical communication system. An example of this is the mechanism of group velocity dispersion (GVD), which arises as the frequency components of the signal pulse propagate with different velocities, determining the broadening of the pulse. Dispersive pulse broadening and loss both increase in direct proportion to the length of the link. Traditionally, repeater stations have been used at appropriate intervals over long links for detecting, electrically amplifying, filtering, and then regenerating the optical signal. However, such repeaters are complicated and become expensive to use in large quantities. Fiber amplifiers appear in most cases as an attractive alternative to the electronic repeaters. A single amplifier is able to boost the power in multiple wavelengths simultaneously, whereas a separate electronic repeater is needed for each wavelength. This simple fact made feasible the development and deployment of dense wavelength division multiplexed (DWDM) systems, which have revolutionized network communication systems since the 1990s.

The expansion of fiber networks to encompass larger areas coupled with the use of longer distances between amplifiers or repeaters means that higher optical power levels are needed. In addition, the ever-increasing bit rates imply the use of shorter
pulses having higher intensities. Both these changes increase the likelihood of various nonlinear processes in the fibers. In fact, the nonlinearities of fused silica, from which optical fibers are made, are weak compared to those of many other materials. However, nonlinear effects can be readily observed in optical fibers due to both their rather small field cross sections, which results in high field intensities, and the long interaction lengths provided by them, which significantly enhances the efficiency of the nonlinear processes. These nonlinear processes can impose significant limitations in high-capacity fiber transmission systems.

It seems paradoxical that the same nonlinear phenomena that impose several important limitations also offer the promise of addressing the bandwidth bottleneck for signal processing for future ultrahigh-speed optical networks. In fact, electronic devices are not suitable for such systems, due to their cost, complexity, and practical speed limits. All-optical signal processing appears, therefore, as a key and promising technology for improving the transparency and increasing the capacity of future full “photonic networks” [4].

Nonlinear optical signal processing appears as a potential solution to this demand. In particular, the third-order \( \chi^{(3)} \) optical nonlinearity in silica-based single-mode fibers offers a significant promise in this regard [5]. This happens not only because the third-order nonlinearity is nearly instantaneous—having a response time typically <10 fs—but also because it is responsible for a wide range of phenomena, which can be used to construct a great variety of all-optical signal processing devices.

Silica fiber nonlinearities can be classified into two main categories: stimulated scattering effects (Raman and Brillouin) and effects arising from the nonlinear index of refraction. Stimulated scattering is manifested as intensity-dependent gain or loss, while the nonlinear index gives rise to an intensity-dependent phase of the optic field. The first experimental demonstration of fiber nonlinearities was Erich Ippen’s CS\(_2\)-core fiber Raman laser in early 1970 [6]. Subsequently, Smith’s theoretical paper on stimulated Raman and Brillouin scattering in silica fibers [7] and the first experimental demonstration of stimulated Raman scattering in a single-mode fiber by Stolen et al. [8] were two landmarks in this field.

Stimulated Raman scattering (SRS) results from the interaction between the photons and the molecules of the medium and leads to the transfer of the light intensity from the shorter to the longer wavelengths. The SRS gain in silica has a wide bandwidth on the order of 12 THz (\( \approx 100 \text{ nm at } 1.5 \mu\text{m} \)) due to its amorphous nature. Thus, SRS can lead to the crosstalk between different WDM channels, becoming the most detrimental of the scattering effects in such systems.

Besides the negative aspect pointed out above, the Raman effect can also find several positive applications. One of the readily apparent advantages of Raman gain in glass fibers was the possibility of constructing wideband amplifiers and tunable oscillators [9]. Indeed, the first SRS work also demonstrated a Raman oscillator using mirrors to provide feedback in a 190 cm fiber [8]. However, the goal of a tunable continuous-wave (CW) fiber Raman laser would have to wait for longer low-loss single-mode fibers. It was not until 1983 that studies of Raman amplification from laboratories around the world began to appear. By the end of the 1980s, the signal-to-noise advantages of Raman amplification appeared to be well understood [10].
However, the lack of efficient high-power fiber-coupled pump lasers prevented the practical use of Raman amplification by that time. After 1988, the lightwave world concentrated on the erbium fiber amplifier, and it was not until 1997 that system experiments using Raman amplifiers started to appear. Following those early demonstrations, the use of Raman amplification in transmission systems has become quite common.

Brillouin scattering originates from the interaction between the pump light and acoustic waves generated in the fiber. In this way, a strong wave traveling in one direction provides narrowband gain, with a linewidth on the order of 20 MHz, for light propagating in the opposite direction. Stimulated Brillouin scattering (SBS) in fibers was observed for the first time in 1972 by Ippen and Stolen [11], who used a pulsed narrowband xenon laser operating at 535.3 nm.

The peak of the Brillouin gain coefficient is over 100 times greater than the Raman gain peak, which makes SBS the dominant nonlinear process in silica fibers under some circumstances. This is particularly the case in fiber transmission systems using narrow-linewidth lasers. SBS can be detrimental to such systems in a number of ways: by originating a severe signal attenuation, by causing multiple frequency shifts in some cases, and by introducing a high-intensity backward coupling into the transmission optics. However, Brillouin gain can also find some useful applications, namely, as an inline fiber amplifier [12,13], for channel selection in a closely spaced wavelength-multiplexed network [14,15], temperature and strain sensing [16,17], all-optical slow-light control [18,19], optical storage [20], and so on.

The intensity-dependent refractive index of silica gives rise to three effects: self-phase modulation (SPM), cross-phase modulation (XPM), and four-wave mixing (FWM). The SPM effect corresponds to a spectral broadening of the pulse determined by its own power temporal variation. The first observation of this phenomenon in silica fibers occurred in a 1975 experiment by Lin and Stolen [21]. Earlier, Hasegawa and Tappert [22] suggested the existence of fiber solitons, resulting from a balance between SPM and anomalous GVD. Such solitons were indeed observed experimentally by Mollenauer et al. [23] in 1980 and subsequently led to a number of advances in the generation and control of ultrashort pulses [24,25]. The advent of fiber amplifiers fueled research on optical solitons and eventually led to new types of solitons, such as dispersion-managed and dissipative solitons [26–33].

The XPM effect is similar to the SPM effect but the spectral broadening of the pulses is now due to the influence of other pulses propagating at the same time in the fiber. This effect becomes especially important in WDM systems, where a large number of pulses with different carrier wavelengths are usually transmitted in one fiber. Since the bit pattern in the different channels is completely random, the cancellation of this effect through an intelligent system design will be impossible in practice. The XPM appears indeed as the fundamental effect that determines the maximum capacity of optical transmission systems. However, XPM can also be used with advantage in several applications in the area of nonlinear optical processing [34–37].

Due to the FWM effect, beating between two channels at their difference frequency modulates the signal phase at that frequency, generating new frequencies as
sidebands. The occurrence of the FWM phenomenon in optical fibers was observed for the first time by Stolen et al. [38] using a 9 cm long multimode fiber pumped by a double-pulsed YAG laser at 532 nm. In a WDM system, if the channels are equally spaced, the new components generated by FWM fall at the original channel frequencies, giving rise to crosstalk [39–41]. In contrast to SPM and XPM, which become more significant for high bit rate systems, the FWM does not depend on the bit rate. The efficiency of this phenomenon depends strongly on phase matching conditions, as well as on the channel spacing, chromatic dispersion, and fiber length.

Besides its obvious application in generating new frequencies over a broad spectral range, FWM can also be used to amplify signals over a broad band around the fiber zero-dispersion wavelength. Moreover, FWM can be used for nonlinear phase conjugation, frequency conversion, optical switching, generation of squeezed states of light, and as a source of entangled photon pairs [42–53].

Starting in 1996, new types of fibers, known as tapered fibers, photonic crystal fibers, and microstructured fibers, were developed [54–58]. Structural changes in such fibers profoundly affect their dispersive and nonlinear properties. As a result, new phenomena were observed, such as the supercontinuum generation, in which the optical spectrum of ultrashort pulses is broadened by a factor of more than 200 over a length of only 1 m or less [59–62]. The efficiency of the nonlinear effects can be further increased using fibers made of materials with a nonlinear refractive index higher than that of the silica glass, namely, lead silicate, tellurite, bismuth glasses, and chalcogenide glasses [63–66]. Using such highly nonlinear fibers (HNLFs), the required fiber length for nonlinear processing can be reduced to the order of centimeters, instead of the several kilometers long conventional fibers.

This book is intended to provide a comprehensive account of the various nonlinear effects occurring in optical fibers. An overview is given of the impact of these effects on communication systems, as well as of their potential in different applications, particularly for signal processing, pulse generation, and amplification. This book can be roughly divided into five parts. The first part, consisting of Chapters 2–4, presents the basic concepts and equations that will be used in the rest of the book. Chapter 2 provides a review of the fundamental concepts and properties related to light propagation in linear dielectric media. The harmonic oscillator model is used to describe the interaction between an optical wave and the matter. Chapter 3 discusses the basic linear properties of optical fibers in the perspective of their use in communication systems, a special attention being paid to the phenomena of chromatic dispersion and polarization mode dispersion. A brief introduction to nonlinear optics, the derivation of the nonlinear Schrödinger equation, and a discussion of its soliton solutions are presented in Chapter 4.

The second part, consisting of Chapters 5–7, is dedicated to the description of nonlinear effects arising from the intensity-dependent refractive index of optical fibers. Chapter 5 describes the phenomena of self-phase modulation and cross-phase modulation, as well as their impact on communication systems. Chapter 6 deals with the four-wave mixing process, including some important applications of this phenomenon, such as parametric amplification, parametric oscillation, optical phase conjugation, and the generation of squeezed states of light. While both XPM and
FWM appear as *interchannel* nonlinear effects, the nonlinear interaction among the pulses of the same channel is discussed in Chapter 7 in which two *intrachannel* effects are considered: the intrachannel cross-phase modulation (IXPM) and the intrachannel four-wave mixing (IFWM). Both IXPM and IFWM can occur only when the pulses overlap in time, at least partly, during their propagation, as happens in dispersion-managed transmission systems.

The third part, consisting of Chapters 8–10, is dedicated to the topic of optical fiber solitons and their applications. Chapter 8 deals with the use of optical solitons in communication systems, considering both constant dispersion and dispersion-managed fiber links. Other applications and phenomena involving optical solitons are discussed in Chapter 9. The polarization effects on soliton propagation, considering the cases of both constant and randomly varying birefringence, are discussed in Chapter 10.

The fourth part, consisting of Chapters 11 and 12, presents a discussion of resonant fiber nonlinear effects. Chapter 11 is dedicated to the stimulated Raman scattering effect, whereas Chapter 12 deals with the stimulated Brillouin scattering effect. The similarities and main differences between these two effects, the limitations that they impose on communication systems, and some important applications are discussed in both chapters.

The fifth and last part, consisting of Chapters 13 and 14, is dedicated to the description of the more relevant types of highly nonlinear fibers, together with some of their actual applications in nonlinear optical signal processing. Chapter 13 describes silica-based conventional highly nonlinear fibers, microstructured fibers, and fibers made of highly nonlinear materials, as well as some novel nonlinear phenomena that can be observed with them. Chapter 14 highlights the importance of highly nonlinear fibers to realize different functions in the area of optical signal processing, namely, multiwavelength sources, pulse generation, all-optical regeneration, wavelength conversion, and optical switching.

**REFERENCES**

Light is an electromagnetic phenomenon consisting of electric and magnetic fields that are solutions of Maxwell’s equations. These equations provide the mathematical foundation used to model and evaluate the flow of electromagnetic energy in all situations, of which optical fibers constitute a particular case. The purpose of this chapter is to review the fundamental concepts and properties related to light propagation in linear dielectric media. From Maxwell’s equations, we will derive the linear wave equation and discuss the main properties of electromagnetic waves. Moreover, the harmonic oscillator model will be used to describe the interaction between an optical wave and the matter. Using such a model, the susceptibility, refractive index, and attenuation of an optical material are discussed. Additional information concerning the subject of this chapter can be found in many textbooks [1–6].

2.1 WAVE EQUATION FOR LINEAR MEDIA

The mathematical foundation for the description of electromagnetic wave propagation in a dielectric medium is provided by the Maxwell’s equations. These equations are named after James Maxwell (1831–1879) and can be written as follows:

\[ \nabla \cdot \mathbf{D} = \rho \tag{2.1} \]

\[ \nabla \cdot \mathbf{B} = 0 \tag{2.2} \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3} \]

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2.4} \]
where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic field vectors, respectively, \( \mathbf{D} \) and \( \mathbf{B} \) are the corresponding electric and magnetic flux densities, \( \mathbf{J} \) is the current density vector, and \( \rho \) is the charge density. The electric flux density and the electric field are related in the form

\[
\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}
\]

(2.5)

where \( \varepsilon \) is the permittivity of the medium, \( \varepsilon_0 \) is the vacuum permittivity, and \( \mathbf{P} \) is the induced electric polarization. On the other hand, the relation between the magnetic flux density and the magnetic field is given by

\[
\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mathbf{M}
\]

(2.6)

where \( \mu \) is the permeability of the medium, \( \mu_0 \) is the vacuum permeability, and \( \mathbf{M} \) is the induced magnetic polarization. Since silica, from which optical fibers are made, is a nonmagnetic material, we set \( \mathbf{M} = 0 \) in the following. The constants \( \mu_0 \) and \( \varepsilon_0 \) have the following values:

\[
\mu_0 = 4 \pi \times 10^{-7} \text{ H/m}
\]

(2.7)

\[
\varepsilon_0 = \frac{10^{-9}}{36\pi} \text{ F/m}
\]

(2.8)

Equations (2.1) and (2.2) correspond to Gauss’s law for the electric field and Gauss’s law for the magnetic field, respectively, while Eq. (2.3) is Faraday’s law of induction and Eq. (2.4) is Ampere’s circuital law.

The electric and magnetic fields can be considered as two aspects of a sole physical phenomenon: the electromagnetic field. In the following, we analyze the main characteristics of such a field. We confine our analysis to isotropic, homogeneous, and sourceless materials, so that \( \mathbf{J} = 0 \) and \( \rho = 0 \).

The curl of Eq. (2.3) gives

\[
\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left( -\frac{\partial}{\partial t} \mathbf{B} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})
\]

(2.9)

The left-hand side of Eq. (2.9) can be simplified using the following vector identity:

\[
\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla \cdot \nabla \mathbf{E}
\]

(2.10)

When \( \rho = 0 \), we have from Eqs. (2.1) and (2.5) that \( \nabla \cdot \mathbf{E} = 0 \). In such a case, Eq. (2.10) gives the result

\[
\nabla \times (\nabla \times \mathbf{E}) = -\nabla \cdot \nabla \mathbf{E} = -\nabla^2 \mathbf{E}
\]

(2.11)

and Eq. (2.9) becomes

\[
-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})
\]

(2.12)
Using Eqs. (2.4)–(2.6), Eq. (2.12) can be written in the form

$$\nabla^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2}{\partial t^2} \mathbf{E} = 0$$

(2.13)

A similar procedure can be used to obtain an equation for \( \mathbf{B} \):

$$\nabla^2 \mathbf{B} - \mu \varepsilon \frac{\partial^2}{\partial t^2} \mathbf{B} = 0$$

(2.14)

Equations (2.13) and (2.14) are wave equations, with the wave’s velocity \( v \) given by

$$v = \frac{1}{\sqrt{\mu \varepsilon}}$$

(2.15)

The connection of the velocity of light with the electric and magnetic properties of a material was one of the most important results of Maxwell’s theory. Considering Eqs. (2.7) and (2.8), the following value for the velocity of light in the vacuum is obtained:

$$v = c = 2.997924562 \times 10^8 \text{ m/s}$$

(2.16)

In a material, the velocity of light is less than \( c \). The index of refraction, \( n \), of the material is defined as the ratio of the speed of light in the vacuum, \( c \), to its speed in the material, \( v \):

$$n = \frac{c}{v} = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}}$$

(2.17)

The index of refraction given by Eq. (2.17) corresponds to the real part of the complex refractive index, which will be discussed in Section 2.7. In the case of nonmagnetic materials, we have \( \mu \approx \mu_0 \) and the index of refraction is determined solely by the permittivity of the medium \( \varepsilon \), which depends on the frequency of the incident electromagnetic wave.

### 2.2 ELECTROMAGNETIC WAVES

Equations (2.13) and (2.14) have the following solutions in the form of harmonic plane waves:

$$\mathbf{E} = \text{Re}\{\mathbf{E}_0 e^{i(k \cdot r - \omega t + \phi)}\}$$

(2.18)

$$\mathbf{B} = \text{Re}\{\mathbf{B}_0 e^{i(k \cdot r - \omega t + \phi)}\}$$

(2.19)

where \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \) are constant vectors giving the direction and amplitude of oscillations, \( \omega \) is the angular frequency, \( k \) is the wave vector, and Re indicates the real
part. Hereafter we will drop the “Re”, but it will be understood that the physical fields are given by the real part of the complex field appearing in our equations. Considering Eqs. (2.1) (with $\rho = 0$), (2.5), and (2.18), we have

$$\nabla \cdot \mathbf{E} = i k \cdot \mathbf{E} = 0$$

(2.20)

In a similar way, using Eqs. (2.2) and (2.19), we obtain

$$\nabla \cdot \mathbf{B} = i k \cdot \mathbf{B} = 0$$

(2.21)

Equations (2.20) and (2.21) show that both $\mathbf{E}$ and $\mathbf{B}$ must be perpendicular to the direction of propagation, which is given by $k$.

Assuming that the electric and magnetic fields are given by Eqs. (2.18) and (2.19), Eq. (2.3) becomes

$$i k \times \mathbf{E} = i \omega \mathbf{B}$$

(2.22)

or

$$\mathbf{B} = \frac{1}{\omega} (k \times \mathbf{E}) = \frac{1}{vk} (k \times \mathbf{E})$$

(2.23)

Thus,

$$\mathbf{B} = \frac{1}{v} (\hat{s} \times \mathbf{E})$$

(2.24)

where $\hat{s} = k/k$ is the unit vector in the propagation direction. Equation (2.24) contains three important aspects concerning the electromagnetic waves:

1. $\mathbf{B}$ is perpendicular to $\mathbf{E}$
2. $\mathbf{B}$ is in phase with $\mathbf{E}$
3. the magnitudes of $\mathbf{B}$ and $\mathbf{E}$ are related as $B = E/v$

![Figure 2.1](image-url)  
**Figure 2.1** Propagation of a plane electromagnetic wave.
Figure 2.1 represents the propagation of a plane electromagnetic wave in a direction indicated by the wave vector $\mathbf{k}$.

### 2.3 ENERGY DENSITY AND FLOW

Any text on electromagnetic theory demonstrates that the energy density associated with an electromagnetic wave is given by

$$ U = \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) $$

(2.25)

Using the constitutive relations given by Eqs. (2.5) and (2.6), we obtain

$$ U = \frac{1}{2} \left( \varepsilon |\mathbf{E}|^2 + \frac{|\mathbf{B}|^2}{\mu} \right) = \frac{1}{2} \left( \varepsilon + \frac{1}{\mu \nu^2} \right) |\mathbf{E}|^2 = \varepsilon |\mathbf{E}|^2 $$

(2.26)

In free space, we have

$$ U = \varepsilon_0 |\mathbf{E}|^2 = \frac{|\mathbf{B}|^2}{\mu_0} $$

(2.27)

The presence of both an electric and a magnetic field at the same point in space results in a flow of the field energy. The energy flux density is described by the Poynting vector, $\mathbf{S}$, defined as

$$ \mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{1}{\mu} \mathbf{E} \times \mathbf{B} $$

(2.28)

The energy flux density in a given direction, indicated by the unit vector $\hat{\mathbf{u}}$, is given by the scalar product $\hat{\mathbf{u}} \cdot \mathbf{S}$.

We will use a plane wave to determine some of the properties of the Poynting vector. Since $\mathbf{S}$ involves terms quadratic in $\mathbf{E}$, it is necessary to use the real form of $\mathbf{E}$:

$$ \mathbf{E} = \mathbf{E}_0 \cos \phi, \quad \phi = \mathbf{k} \cdot \mathbf{r} - \omega t + \varphi $$

(2.29)

Also, from Eq. (2.23),

$$ \mathbf{B} = \mathbf{B}_0 \cos \phi = \frac{1}{\nu k} (\mathbf{k} \times \mathbf{E}_0) \cos \phi $$

(2.30)

The Poynting vector becomes

$$ \mathbf{S} = \frac{1}{\mu} \mathbf{E}_0 \times \frac{1}{\nu k} (\mathbf{k} \times \mathbf{E}_0) \cos^2 \phi = \frac{1}{\mu \nu} |\mathbf{E}_0|^2 \hat{s} \cos^2 \phi $$

(2.31)
Since the frequencies associated with light are very high ($10^{14} - 10^{15}$ Hz), we normally do not detect the magnitude of $\mathbf{S}$, but rather its temporal average over a time $T$ determined by the response time of the detector used. Considering that the time average of $\cos^2 \phi$ over many periods is 1/2, the time-averaged value of the magnitude of the Poynting vector is given by

$$I \equiv \langle |\mathbf{S}| \rangle = \frac{1}{2\mu_0} |\mathbf{E}_0|^2$$  \hspace{1cm} (2.32)

$I = \langle |\mathbf{S}| \rangle$ is called the flux density and has units of W/m$^2$.

The energy density is given from Eq. (2.26) by

$$U = \varepsilon |\mathbf{E}_0|^2 \cos^2 \phi$$  \hspace{1cm} (2.33)

with a time average

$$\langle U \rangle = \frac{\varepsilon}{2} |\mathbf{E}_0|^2$$  \hspace{1cm} (2.34)

We may use the definition of the wave velocity, given by Eq. (2.15), to relate the density of flux $I$ to the average energy density $\langle U \rangle$ in the form

$$I = v \langle U \rangle$$  \hspace{1cm} (2.35)

This corresponds to a general result:

$$\text{Energy flux density} = (\text{energy density}) \times (\text{propagation speed})$$

## 2.4 PHASE VELOCITY AND GROUP VELOCITY

Since the refractive index of the medium is frequency dependent, the phase velocity of a wave is in general also a function of the frequency. This fact has important implications when the propagating waves are composed of several frequencies, as is the case in applications using the modulation of light. The velocity of the carrier and the velocity of the modulation will be in general different.

Let us consider the simple situation of a propagating plane wave containing only two frequencies. The total real electric field of such a wave can be written as the sum of fields of the two waves, which we assume to propagate in the $z$-direction and to have the same amplitude $E_{01}$:

$$E(z, t) = E_{01} [\cos(k_1 z - \omega_1 t) + \cos(k_2 z - \omega_2 t)] = 2E_{01} \cos(k_m z - \omega_m t) \cos(k_a z - \omega_a t)$$  \hspace{1cm} (2.36)

where

$$\omega_a = \frac{1}{2} (\omega_1 + \omega_2), \quad \omega_m = \frac{1}{2} (\omega_1 - \omega_2)$$  \hspace{1cm} (2.37)
\[ k_a = \frac{1}{2}(k_1 + k_2), \quad k_m = \frac{1}{2}(k_1 - k_2) \] (2.38)

The quantities \( \omega_a \) and \( k_a \) are the average angular frequency and the average propagation constant, respectively, whereas the quantities \( \omega_m \) and \( k_m \) are designated the modulation frequency and the modulation propagation constant, respectively.

The total field can be regarded as a traveling (carrier) wave of frequency \( \omega_a \) having a time-varying or modulated amplitude \( E_0(z,t) \) such that

\[ E(z,t) = E_0(z,t)\cos(k_az - \omega_at) \] (2.39)

where

\[ E_0(z,t) = 2E_{01}\cos(k_mz - \omega_mt) \] (2.40)

The phase velocity of the carrier wave can be obtained from its phase \( \varphi = (k_az - \omega_at) \) using the relation

\[ v = -\frac{(\partial \varphi / \partial t)_z}{(\partial \varphi / \partial z)_t} \] (2.41)

which gives the result

\[ v = \frac{\omega_a}{k_a} \] (2.42)

Concerning the propagation of the modulation envelope, the rate at which it advances is known as the group velocity, \( v_g \). The group velocity is obtained from Eq. (2.41), considering the phase of the envelope \( (k_mz - \omega_mt) \), and is given by

\[ v_g = \frac{\omega_m}{k_m} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta \omega}{\Delta k} \] (2.43)

The function describing the dependence of \( \omega \) on \( k \), \( \omega = \omega(k) \), is called dispersion relation. When the frequency range \( \Delta \omega \) is small, the ratio \( \Delta \omega / \Delta k \) tends to the derivative of the dispersion relation and the group velocity becomes

\[ v_g = \frac{d\omega}{dk} \] (2.44)

Since \( \omega = kv \), Eq. (2.44) yields

\[ v_g = v + k \frac{dv}{dk} \] (2.45)

2.4 PHASE VELOCITY AND GROUP VELOCITY
The modulation or signal propagates with a velocity $v_g$ that may be greater than, equal to, or less than the phase velocity of the carrier, $v$. In nondispersive media, $v$ does not depend on $k$, and $v_g = v$. In dispersive media, in which $n(k)$ is known, $\omega = kc/n$ and the group velocity can be written in the form

$$v_g = \frac{c}{n} - \frac{kc \, dn}{n^2 \, dk} \quad (2.46)$$

The group velocity can be considered as the propagation velocity of a “group” of waves with frequencies distributed over an infinitesimally small bandwidth centered on $\omega_a$. In the presence of a broad frequency spectrum, the slope of the curve $\omega(k)$ may change over the range of the spectrum. As a consequence, different spectral components propagate at different group velocities, leading to signal distortion. This problem is called group velocity dispersion and will be discussed in Chapter 3 in the context of optical fibers.

### 2.5 REFLECTION AND TRANSMISSION OF WAVES

The phenomenon of reflection and transmission of plane waves at interfaces between dielectrics is useful in exploiting and understanding the behavior of light in dielectric waveguides. Of interest are not only the relations among the angles of incidence, reflection, and refraction, but also the fractions of optical power that are reflected and transmitted at the boundaries, as well as the phase shifts that occur on reflection.

#### 2.5.1 Snell’s Laws

Let us consider a monochromatic plane wave incident on a boundary between two media with refractive indices $n_1$ and $n_2$ (Fig. 2.2). The incident wave is given by

$$E_i = E_{0i} \exp\{i(\mathbf{k}_i \cdot \mathbf{r} - \omega t)\} \quad (2.47)$$

and can be decomposed into two waves with the same frequency—a reflected wave, $E_r$, and a transmitted wave, $E_t$—given by

$$E_r = E_{0r} \exp\{i(\mathbf{k}_r \cdot \mathbf{r} - \omega t)\} \quad (2.48)$$

$$E_t = E_{0t} \exp\{i(\mathbf{k}_t \cdot \mathbf{r} - \omega t)\} \quad (2.49)$$

A relation among the three waves, valid for all points on the interface and for any instant of time, can be verified only if their phases are the same. This condition gives

$$\mathbf{k}_i \cdot \mathbf{r} = \mathbf{k}_r \cdot \mathbf{r} = \mathbf{k}_t \cdot \mathbf{r} \quad (2.50)$$

From Eq. (2.50), we conclude that

$$\mathbf{k}_r - \mathbf{k}_i = b_1 \hat{\mathbf{N}} \quad (2.51)$$

$$\mathbf{k}_t - \mathbf{k}_i = b_2 \hat{\mathbf{N}} \quad (2.52)$$