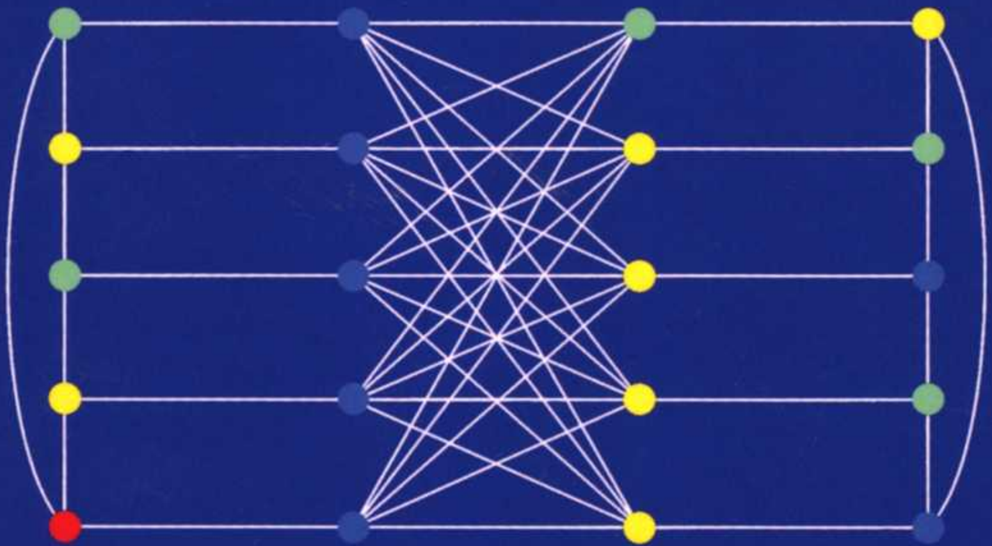
GRAPH COLORING PROBLEMS



TOMMY R. JENSEN

AND

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to Paul Erdős

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Preface

WHY GRAPH COLORING?

Graph coloring problems? The four-color problem has been changed into the four-color theorem, so is there really much more to say or do about coloring? Yes there is, and for several reasons!

First, the last word on the four-color problem has not been said. The ingenious solution by K. Appel, W. Haken, and J. Koch [1, 2], based on the approach of H. Heesch, is a major achievement, but to some mathematicians the solution is unsatisfactory and raises new questions, both mathematical and philosophical.

Second, graph coloring theory has a central position in discrete mathematics. It appears in many places with seemingly no or little connection to coloring. A good example is the Erdős–Stone–Simonovits theorem [3] in extremal graph theory, showing that for a fixed graph G the behavior of the maximum number $f(n,G)$ of edges in a graph on n vertices not containing G as a subgraph depends on the chromatic number $\chi(G)$ of G:

$$\lim_{n\to\infty} \frac{f(n,G)}{n^2} = \frac{\chi(G)-2}{2\chi(G)-2}.$$

Third, graph coloring theory is of interest for its applications. Graph coloring deals with the fundamental problem of partitioning a set of objects into classes, according to certain rules. Time tabling, sequencing, and scheduling problems, in their many forms, are basically of this nature.

Fourth, graph coloring theory continually surprises by producing unexpected new answers. For example, the century old five-color theorem for planar graphs due to P.J. Heawood [4] has recently been furnished with a new proof by C. Thomassen [5], avoiding both the use of Euler's formula and the powerful recoloring technique invented by A.B. Kempe [6], thus making it conceptually simpler than any previous proof.

And finally, even if many deep and interesting results have been obtained during the 100 years of graph coloring, there are very many easily formulated, interesting problems left. This is the most important reason for us, and our book is an attempt to exemplify it. As far as we know it is the first book devoted to unsolved graph coloring problems, but a number of papers sharing the same topic have preceded it—for example, many of the "problems and results" papers by P. Erdős (referred to throughout the book), surveys by W. Klotz [7], Z. Tuza [8], and J. Kahn [9], problem sections in proceedings and newsletters (such as the column by D.B. West [10]), and lists of "problems from the world surrounding perfect graphs" by A. Gyárfás [11] and V. Chvátal [12]. A list of 50 carefully selected problems in graph theory is contained in the book by J.A. Bondy and U.S.R. Murty [13]. Finally, two interesting collections of geometry problems, by H.T. Croft, K.J. Falconer, and R.K. Guy [14], and by W. Moser (McGill University, Canada) and J. Pach [15], share some of our general ideas and contain some coloring problems.

In a delightful paper W.T. Tutte [16] described several difficult coloring conjectures, many of them generalizing the four-color theorem. The paper showed, in Tutte's words, that *"The Four Colour Theorem is the tip of the iceberg, the thin end of the wedge and the first cuckoo of spring."*

THE PROBLEMS

In selecting and presenting the more than 200 problems for this book we had four main objectives in mind:

1. Each problem should be simple to state and understand, and thus problems requiring several or complicated definitions are not included. Only a few of the problems have the character of a broad research program; most of them are specific questions. We have aimed to select for each problem its most attractive formulation, which may not always be the most general or the most specific. But very often we mention more general versions and/or special cases in the comments.
2. The list of problems should tell not only what is *not* known in graph coloring theory. The comments should also provide an exposition of the major known graph coloring results.
3. The history of the problems, and the credit for them and for the results presented, should be as accurate and complete as possible.
4. The list should not consist just of "impossible" problems, but also of questions where progress is definitely possible.

We did not intend to write a textbook to be read from beginning to end, but rather a catalog suitable for browsing. Chapter 1 contains a common basis of graph coloring terminology and a collection of important theorems. The remaining 16 chapters comprise the main body of the book, each containing a list of open problems within a separate area. The necessary background for understanding each problem and the information directly related to it appear together with the statement of the problem. Each chapter is intended to be self-contained and is closed by its own separate list of references. We have paid a price in terms of having to allow some redundancies, but we think that the level is tolerable even for the thorough reader. To make the presentation short and succinct, we have included very few proofs and pictures. Proofs, outlines of arguments, or figures have been added in a few cases when we did not have an appropriate source of reference.

There is one remark we should make concerning the organization of the references. When consulting any given one of the bibliographies it may seem strange that different papers by the same author(s) and published within the same year are not always listed in a consecutively numbered fashion. For example, there is a reference to a paper of Edmonds [1965b] in Chapter 2, but there is no reference to a paper of Edmonds [1965a] preceding it in the bibliography. The explanation is that we have chosen to maintain a consistent numbering of the references throughout the entire book. In other words, the numbering is exactly as it would have been, had the references all been put together into one big list. Thus the same paper is being referred to in the same manner throughout.

UPDATES

The present activity in discrete mathematics is so extensive that a work of this nature is outdated before it is written! Solutions, partial results, and new ideas appear all the time. And there will be interesting questions that we have overlooked, and also, solutions or partial solutions. In some cases we have probably not met objective 3. We apologize for all such cases, and we shall be grateful for corrections, comments, and information.

For easy access to any new and updated information, we have installed an `ftp`-archive at Odense University, Denmark. You can reach this facility via `ftp` using the address

`ftp.imada.ou.dk`

logging in as "`anonymous`" and giving your `e-mail` address as the password. The archive is located in a directory which can be reached by typing the command `cd pub/graphcol`, where a short `README` file is available for further information on how to proceed.

`World Wide Web` access to the archive is also available. You either need to locate the menu of `Danish Information Servers`, and then click successively on the menus for `IMADA`, listed under Odense University, `Research Activities`, and `Graph Theory`. Or you may use the address

`http://www.imada.ou.dk`

to directly reach the `IMADA` Home Page.

The contents of the `ftp`-archive will depend largely on new information (papers, abstracts, questions, solutions, etc.) sent by our readers. Contributions should be `e-mailed` to the address

`graphcol@imada.ou.dk`

to be considered for inclusion.

In addition to the `ftp`-archive, we shall consider writing updates from time to time in the form of articles. Such papers will be submitted to the *Journal of Graph Theory.*

ACKNOWLEDGMENTS

The senior author, Bjarne Toft, learned graph theory from G.A. Dirac, starting in 1966, at the University of Aarhus in Denmark. Dirac's lectures were captivating. He presented the subject as a general and serious mathematical theory, and he did so with rigor and care, to an extent which made some of his colleagues think that graph theory is just a collection of easy facts with trivial proofs. Dirac's thorough style, strongly influenced by the book of Dénes König [17], was, however, a delight for his students. Later, in 1968–1970 and in 1973, Toft spent semesters at the Hungarian Academy of Sciences in Budapest, Hungary, the University of London, England, and the University of Waterloo, Canada. Later, in 1985–1986, while visiting the University of Regina, Canada, and Vanderbilt University, United States, the first steps toward this book were taken.

Tommy Jensen first learned graph theory at Odense University, Denmark, in the first half of the 1980s under the supervision of Bjarne Toft, and later, while studying at the University of Waterloo, Canada, enjoyed the benefit of receiving supervision from D.H. Younger. He owes the beginning of his interest in mathematics to his father, Emil Jensen, who sadly did not get to see the finished version of this book.

We are glad for the opportunities given to us to learn from some of the greatest mathematicians in graph theory, such as P. Erdős, to whom we dedicate this book, and J. Edmonds, T. Gallai, and W.T. Tutte. We are most grateful for the helpfulness and generosity we have met from many sides. In connection with the present work we would like to thank a very large number of people who supplied information—their names may be found in appropriate places in this book. In particular, we wish to thank the following for showing their interest in the project by giving us highly qualified comments and suggestions at various stages: M.O. Albertson, N. Alon, O.V. Borodin, F. Jaeger, M.K. Goldberg, R. Häggkvist, D. Hanson, A.V. Kostochka, L. Lovász, P. Mihók, G. Sabidussi, M. Stiebitz, C. Thomassen, and two anonymous referees.

Of course we alone are responsible for all errors, inaccuracies, and omissions.

Finally, we would like to express our appreciation of the support received from Wiley-Interscience and their editorial staff, in particular Maria Allegra, Elizabeth Murphy, Kimi Sugeno, and Angela Volan, from the Danish Natural Science Research Council, the University of Regina, Canada, and Odense University, Denmark. We also thank Margit Christiansen of the Mathematical Library in Odense—may our readers be as adept in problem solving!

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Graph Coloring Problems

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1

Introduction to Graph Coloring

1.1. BASIC DEFINITIONS

Partitioning a set of objects into classes according to certain rules is a fundamental process in mathematics. A conceptually simple set of rules tells us for each pair of objects whether or not they are allowed in the same class. The theory of graph coloring deals with exactly this situation. The objects form the set of vertices $V(G)$ of a graph G, two vertices being joined by an edge in G whenever they are not allowed in the same class. To distinguish the classes we use a set of colors C, and the division into classes is given by a *coloring* $\varphi : V(G) \rightarrow C$, where $\varphi(x) \neq \varphi(y)$ for all xy belonging to the set of edges $E(G)$ of G. If C has cardinality k, then φ is a *k-coloring*, and when k is finite, we usually assume that $C = \{1, 2, 3, \ldots, k\}$. For $i \in C$ the set $\varphi^{-1}(i)$ is the ith *color class*. Thus each color class forms an *independent* set of vertices; that is, no two of them are joined by an edge. The minimum cardinal k for which G has a k-coloring is the *chromatic number* $\chi(G)$ of G, and G is *$\chi(G)$-chromatic*. The existence of the chromatic number follows from the Well-Ordering Theorem of set theory, and conversely, considering cardinals as special ordinals, the existence of the chromatic number easily implies the Well-Ordering Theorem. However, even if it is not assumed that every set has a well-ordering, but maintaining the property that every set has a cardinality, then the statement "Any finite or infinite graph has a chromatic number" is equivalent to the Axiom of Choice, as proved by Galvin and Komjáth [1991].

If the condition $\varphi(x) \neq \varphi(y)$ for all $xy \in E(G)$ is dropped from the definition of coloring, then φ is called an *improper coloring* of G. Accordingly, the term *proper coloring* is sometimes used when we want to emphasize that this condition holds.

For a hypergraph H with vertex set $V(H)$ and edge set $E(H)$, a *coloring* $\varphi : V(H) \rightarrow C$ must assign at least two different colors to the vertices of every edge in H. That is, no edge is monochromatic. If the edges of H all have the same size r, we say that H is *r-uniform*. Thus the 2-uniform hypergraphs are exactly the graphs. We do not normally allow loops in graphs, nor edges of size at most 1 in hypergraphs; when we do, it will be stated explicitly. We do allow multiple edges. A graph or hypergraph without multiple edges is *simple*. The term *multigraph* is used when we explicitly want to say that multiple edges are allowed in a graph, and the *multiplicity*

$\mu(G)$ will denote the maximum number of edges joining the same pair of vertices in a multigraph G.

The theory of hypergraph coloring is extremely rich, and graph coloring is just one special case. Ramsey theory can be viewed naturally as another special case (see Graham, Rothschild, and Spencer [1990]).

A *homomorphism* of a graph G into a graph H is a mapping $f : V(G) \to V(H)$ such that $f(x)f(y)$ is an edge of H if xy is an edge of G. A k-coloring of G can then be thought of as a homomorphism of G into the complete k-graph K_k. In general, a homomorphism of G into a graph H is called an *H-coloring* of G.

An *edge coloring* of a hypergraph (or graph) H is a mapping $\varphi' : E(H) \to C$, where nondisjoint edges are mapped into distinct elements of the color set C. If C has k elements, then φ' is a *k-edge coloring*. The minimum cardinal k for which H has a k-edge coloring is the *edge-chromatic number* $\chi'(H)$, and H is said to be *$\chi'(H)$-edge-chromatic*.

A *face coloring* of a map M on a surface S (i.e., a bridge-less graph embedded on S) with a set $F(M)$ of faces (or countries) consists of a mapping $\varphi : F(M) \to C$, where neighboring faces (those with a common borderline) are mapped into different elements of the color set C. This corresponds to a vertex coloring of the *dual graph* G, defined by having vertex set $V(G) = F(M)$ and an edge $xy \in E(G)$ for every edge of M on the common borderline of the faces x and y. When the map M is embedded on S, its dual graph can also be embedded on S without crossing edges.

As with face coloring, both hypergraph coloring (with at least three colors) and edge coloring can be translated into vertex-coloring of graphs, as we shall see.

In the following we deal almost exclusively with graphs rather than with maps, even in cases where the results were initially obtained for face coloring. In the time before the papers of Whitney [1932b] and Brooks [1941], coloring theory dealt almost exclusively with maps, even though Kempe [1879] had drawn attention to vertex colorings of graphs: *"If we lay a sheet of tracing paper over a map and mark a point on it over each district and connect the points corresponding to districts which have a common boundary, we have on the tracing paper a diagram of a 'linkage,' and we have as the exact analogue of the question we have been considering, that of lettering the points of the linkage with as few letters as possible, so that no two directly connected points shall be lettered with the same letter. Following this up, we may ask what are the linkages which can be similarly lettered with no less than n letters? The classification of linkages according to the value of n is one of considerable importance."*

Vertex coloring of infinite graphs with a finite number of colors, or more generally H-coloring with a finite graph H, can always be reduced to finite instances. For vertex coloring, this is the content of the following theorem, which may be derived from a theorem of Rado [1949]. Gottschalk [1951] gave a short proof of Rado's theorem using compactness. A similar proof gives an extension of the theorem that includes H-coloring in general.

Theorem 1 (de Bruijn and Erdős [1951]). *If all finite subgraphs of an infinite graph G are k-colorable, where k is finite, then G is k-colorable.*

A short direct graph-theoretic proof of Theorem 1 was obtained by L. Pósa, and may be found for example in the book by Wagner [1970]. It was actually already contained in the Ph.D. thesis of G.A. Dirac at the University of London in 1951. However, as pointed out by G. Sabidussi [personal communication in 1993], this particular proof does not generalize to H-colorings as readily as the proof of Gottschalk. Because of Theorem 1 we shall only deal with finite graphs in the following, except when explicitly stating otherwise.

The reader looking for proofs of the theorems in this chapter may in many cases have to consult the references. However, a well-written general exposition of graph coloring theory, including proofs of several of the theorems we mention, can be found in the classical book on extremal graph theory by Bollobás [1978a]. Another good general source is the forthcoming *Handbook of Combinatorics*, edited by L. Lovász, R.L. Graham, and M. Grötschel, and published by North-Holland.

1.2. GRAPHS ON SURFACES

Many other areas of graph theory besides coloring theory originated from the *four-color problem* of Francis Guthrie: Is every planar graph 4-colorable? Well-written accounts of the problem are contained in the monographs by Ringel [1959], Ore [1967], Biggs, Lloyd, and Wilson [1976], Saaty and Kainen [1977], Barnette [1983], and Aigner [1984].

The four-color problem seems first to have been mentioned in writing in an 1852 letter from A. De Morgan to W.R. Hamilton, written on the same day as De Morgan first heard about the problem from his student Frederick Guthrie, Francis Guthrie's brother. It first appeared in print in an anonymous book review by De Morgan in 1860 (see Wilson [1976]), and later as an open problem raised by Cayley [1878] at a meeting in the London Mathematical Society and in a paper by Cayley [1879]. A proposed solution by Kempe [1879] stood for more than a decade until it was refuted by Heawood [1890] in his first paper. Heawood proved the five-color theorem for planar maps and the best possible twelve-color theorem for the case where each country consists of at most two connected parts. Moreover, he extended the problem to higher surfaces. Dirac [1963] gave an excellent survey of Heawood's achievements.

The higher surfaces (i.e., compact 2-dimensional manifolds) can be classified into three types as follows (see, e.g., Massey [1991]). The sphere with g handles attached is denoted by $\mathbf{S}_g$ (of Euler characteristic $\varepsilon = 2 - 2g$), the projective plane with g handles attached by $\mathbf{P}_g$ (of Euler characteristic $\varepsilon = 1 - 2g$), and the Klein bottle with g handles attached by $\mathbf{K}_g$ (of Euler characteristic $\varepsilon = -2g$). In each case g may assume the value zero. Note that the surfaces $\mathbf{S}_g$ are orientable, whereas $\mathbf{P}_g$ and $\mathbf{K}_g$ are nonorientable.

Theorem 2 (Heawood [1890]). *Let* $\mathbf{S}$ *be a surface of Euler characteristic* ε. *When* $\varepsilon < 2$, *every graph* G *on* $\mathbf{S}$ *can be colored using the Heawood number* $H(\varepsilon)$ *of colors,* *given by*

$$H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{49 - 24\varepsilon}}{2} \right\rfloor.$$

A graph of seven mutually adjacent vertices, the complete 7-graph K_7, embeds on the torus $\mathbf{S}_1$, hence 7 (= $H(0)$) colors are both sufficient and necessary for toroidal graphs.

The topological prerequisite for Heawood's formula is Euler's formula, implying that every graph G embedded on a surface $\mathbf{S}$ of Euler characteristic ε has at most $3|V(G)| - 3\varepsilon$ edges. Since a minimal k-chromatic graph G has minimum degree $\delta(G) \geq k - 1$, such a graph G satisfies

$$(k-1)|V(G)| \leq 2|E(G)| \leq 6|V(G)| - 6\varepsilon.$$

Since $|V(G)| \geq k$, it follows for $k \geq 7$ that $(k-7)k + 6\varepsilon \leq 0$, which in turn implies that $k \leq H(\varepsilon)$.

For the Klein bottle $\mathbf{K}_0$ the Heawood formula gives a seven-color theorem. However, Franklin [1934] proved that six colors suffice to color any graph on the Klein bottle. This is the only case where the Heawood number is not the right answer to the coloring problem for higher surfaces.

Theorem 3 (Heffter [1891], Tietze [1910], Ringel [1954, 1959, 1974], Ringel and Youngs [1968]). *For a surface* $\mathbf{S}$ *of Euler characteristic $\varepsilon < 2$, where* $\mathbf{S}$ *is not the Klein bottle, the Heawood number $H(\varepsilon)$ is the maximum chromatic number of graphs embeddable on* $\mathbf{S}$.

The proof of this major result, completed in 1968, was obtained by embedding the complete $H(\varepsilon)$-graph $K_{H(\varepsilon)}$ on the surface with Euler characteristic ε. This is of course sufficient for a proof of Theorem 3. It is in fact also necessary.

Theorem 4 (P. Ungar and Dirac [1952b], Albertson and Hutchinson [1979]). *For a surface* $\mathbf{S}$ *of Euler characteristic $\varepsilon < 2$, and* $\mathbf{S}$ *different from the Klein bottle, any $H(\varepsilon)$-chromatic graph on* $\mathbf{S}$ *contains $K_{H(\varepsilon)}$ as a subgraph.*

Dirac's arithmetic did not cover the cases $\varepsilon = -1$ and 1, but these cases were later settled by Albertson and Hutchinson [1979]. The idea of the result of Theorem 4 and a proof in the case of the torus were first obtained by P. Ungar, as mentioned by Dirac [1952b].

After various attempts and the achieving of partial results on the four-color problem by many mathematicians, Appel and Haken [1976a] announced a complete proof. The four-color theorem for plane triangulations (i.e., plane graphs in which all faces are triangles), and hence for all planar graphs, follows immediately by induction from

Theorem 5 (Appel and Haken [1977a], Appel, Haken and Koch [1977]). *There exists a set U of 1482 configurations such that*

(a) *Unavoidability: any plane triangulation contains an element of U, and*
(b) *Reducibility: a 4-coloring of a plane triangulation containing an element of U can be obtained from 4-colorings of smaller plane triangulations.*

This is the same basic idea as in Kempe's proof, where U consisted of vertices of degree at most 5. Kempe's only mistake was in his argument for the reducibility of vertices of degree equal to 5. The detailed techniques of Appel, Haken, and Koch are further developments of methods of Heesch [1969], who was the first to emphasize strongly the possibility of a proof of the four-color theorem along these lines (see Bigalke [1988]). The proof of part (a) is based on Euler's formula and an elaborate "discharging procedure." Whereas this part of the proof can in principle be carried out by hand, Appel, Haken, and Koch had to use computer programs to verify that each member of their unavoidable set U of configurations submits to one of two types of reducibility that Heesch had named "C-reducibility" and "D-reducibility." Combining this fact with results of Bernhart [1947], they proved that U satisfies (b) of Theorem 5.

Several surveys of the proof of Theorem 5 exist: for example, Appel and Haken [1977b, 1978] and Woodall and Wilson [1978]. Due to its length, extensive use of verification by computer, some inaccuracies, and omissions of details, the proof of Theorem 5 has been surrounded by some controversy. Appel and Haken [1986, 1989] have themselves addressed the questions raised. Recent accounts of the situation have been given by F. Bernhart [*Math. Reviews* **91m**:05005] in an informative review of the book by Appel and Haken [1989], and by Kainen [1993].

Very recently, N. Robertson, D.P. Sanders, P.D. Seymour, and R. Thomas [personal communication from N. Robertson and P.D. Seymour in 1994] have obtained a new, improved proof of the four-color theorem by using the same general approach as that of Appel, Haken, and Koch. This proof has less than 700 configurations and is based on a simpler discharging procedure. In addition, the proof avoids some of the more problematic details of the proof by Appel, Haken, and Koch (we describe these in Problem 2.1). However, it still relies on extensive computer checking.

An early approach to coloring problems for plane maps and graphs concerned studying the number $P(G,k)$ of all possible different k-colorings of a graph G with colors $1,2,\dots,k$. Birkhoff [1912] noted that $P(G,k)$ as a function of k can be expressed as a polynomial, the so-called *chromatic polynomial* of G, $P(G,k) = a_1k^n + a_2k^{n-1} + \cdots + a_nk$ of degree $n = |V(G)|$. In particular, $\chi(G)$ is the smallest nonnegative integer that is not a zero of $P(G,k)$. Whitney [1932a, 1932b], Birkhoff and Lewis [1946], Tutte [1954, 1970b], and Read [1968] are some of the researchers who have developed the theory of chromatic polynomials. A well-written survey was given by Read and Tutte [1988].

One of Tutte's surprising and beautiful results is the following *golden identity.*

Theorem 6 (Tutte [1970b]). *Let M be a plane triangulation on n vertices. Then*

$$P(M, \tau + 2) = (\tau + 2) \cdot \tau^{3n-10} \cdot (P(M, \tau + 1))^2,$$

where τ is the golden ratio $\frac{1}{2}(1 + \sqrt{5})$, with $\tau + 1 = \tau^2$ and $\tau + 2 = \sqrt{5}\tau$.

Tutte [1970b] noted that $P(M, \tau + 1) \neq 0$. Hence we have the curious consequence that $P(M, \tau + 2)$ is positive, where $\tau + 2 \simeq 3.618\ldots$. Of course, the four-color theorem is equivalent to the statement that $P(M, 4)$ is positive.

As explained by Saaty [1972] and Saaty and Kainen [1977] the four-color theorem has many equivalent formulations. A particularly noteworthy result is

Theorem 7 (Wagner [1937]). *If all planar graphs are 4-colorable, then 4-colorability extends to the class $\mathcal{G}$ of all graphs from which a complete 5-graph K_5 cannot be obtained by deletions (of vertices and/or edges) and contractions of edges (removing possible loops that might arise).*

Thus the four-color theorem is equivalent to the case $k = 5$ of the famous

Hadwiger's Conjecture (Hadwiger [1943]). *Let $\mathcal{G}$ be a class of graphs closed under deletions (of edges and/or vertices) and contractions of edges (removing possible loops that might arise). Then the maximum chromatic number of the graphs in $\mathcal{G}$ equals the number of vertices $(k - 1)$ in a largest complete graph in $\mathcal{G}$.*

For $k = 4$ this was proved by Dirac [1952a]. Recently, Robertson, Seymour, and Thomas [1993a] gave a complete characterization of all 6-colorable graphs from which the complete graph K_6 cannot be obtained by deletions and contractions. As a corollary of the characterization, all such graphs are in fact 5-colorable, assuming the four-color theorem. This proves that Hadwiger's conjecture for $k = 6$ is also equivalent to the four-color theorem. Hadwiger's conjecture is true for $\mathcal{G}$ the class of all graphs embeddable on the same surface S. This follows from Theorems 3, 4, and 5 above, and from a paper by Albertson and Hutchinson [1980a] for the Klein bottle.

A deep extension of the five-color theorem for planar graphs was conjectured by Grünbaum [1973] and proved by Borodin [1979a]. The proof is reminiscent of the four-color proof by Appel, Haken, and Koch; it involves an unavoidable set of some 450 reducible configurations (but no computers).

Theorem 8 (Borodin [1979a]). *Every planar graph has an acyclic 5-coloring, that is, a 5-coloring in which each pair of color classes induces a subgraph without cycles.*

As for 3-colorings of planar graphs, the most important results are

Theorem 9 (Heawood [1898]). *A plane triangulation can be 3-colored if and only if all vertices have even degrees.*