Lebesgue Measure and Integration

An Introduction

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Lebesgue Measure and Integration
PURE AND APPLIED MATHEMATICS

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Lebesgue Measure
and Integration
An Introduction

Frank Burk
For Janet
Contents

Preface xi

Chapter 1. Historical Highlights 1

1.1 Rearrangements / 2
1.2 Eudoxus (408–355 B.C.E.) and the Method of Exhaustion / 3
1.3 The Lune of Hippocrates (430 B.C.E.) / 5
1.4 Archimedes (287–212 B.C.E.) / 7
1.5 Pierre Fermat (1601–1665): \[\int_0^b x^{p/q} \, dx = b^{p/q+1}/(p/q + 1)\] / 10
1.6 Gottfried Leibnitz (1646–1716), Issac Newton (1642–1723) / 12
1.7 Augustin-Louis Cauchy (1789–1857) / 15
1.8 Bernhard Riemann (1826–1866) / 17
1.9 Emile Borel (1871–1956), Camille Jordan (1838–1922), Giuseppe Peano (1858–1932) / 20
1.10 Henri Lebesgue (1875–1941), William Young (1863–1942) / 22
1.11 Historical Summary / 25
1.12 Why Lebesgue / 26
Chapter 2. Preliminaries 32

2.1 Sets / 32
2.2 Sequences of Sets / 34
2.3 Functions / 35
2.4 Real Numbers / 42
2.5 Extended Real Numbers / 49
2.6 Sequences of Real Numbers / 51
2.7 Topological Concepts of $\mathbb{R}$ / 62
2.8 Continuous Functions / 66
2.9 Differentiable Functions / 73
2.10 Sequences of Functions / 75

Chapter 3. Lebesgue Measure 87

3.1 Length of Intervals / 90
3.2 Lebesgue Outer Measure / 93
3.3 Lebesgue Measurable Sets / 100
3.4 Borel Sets / 112
3.5 “Measuring” / 115
3.6 Structure of Lebesgue Measurable Sets / 120

Chapter 4. Lebesgue Measurable Functions 126

4.1 Measurable Functions / 126
4.2 Sequences of Measurable Functions / 135
4.3 Approximating Measurable Functions / 137
4.4 Almost Uniform Convergence / 141

Chapter 5. Lebesgue Integration 147

5.1 The Riemann Integral / 147
5.2 The Lebesgue Integral for Bounded Functions on Sets of Finite Measure / 173
5.3 The Lebesgue Integral for Nonnegative Measurable Functions / 194
5.4 The Lebesgue Integral and Lebesgue Integrability / 224
5.5 Convergence Theorems / 237
Appendix A. Cantor’s Set 252
Appendix B. A Lebesgue Nonmeasurable Set 266
Appendix C. Lebesgue, Not Borel 273
Appendix D. A Space-Filling Curve 276
Appendix E. An Everywhere Continuous, Nowhere Differentiable, Function 279

References 285

Index 288
This book is intended for individuals seeking an understanding of Lebesgue measure and integration. As a consequence, it is not an encyclopedic reference, or a compendium, of the latest developments in this area of mathematics. Only the most fundamental concepts are presented: Lebesgue measure for $R$, Lebesgue integration for extended real-valued functions on $R$. No apologies are made for this approach, after all, it is the proper foundation for any general treatment of measure and integration. In fact, no claim to originality is made for any of the mathematics in this book, but we do accept full responsibility for any mistakes or blunders in its presentation. It is old mathematics after all (standard graduate fare for the last forty or fifty years), but particularly beautiful. It deserves a wider audience. Lebesgue measure and integration, presented properly, reveals mathematical creation in its highest form. Motivation has been the dominant concern, and understanding will be the final measure.

Where to begin? As a concession to understanding the subtleties of measure, and the effort required for such, I have taken the least upper bound axiom as a starting point. (Besides, it would be difficult, if not impossible, to improve on Landau’s (1960) book, *Foundations of Analysis.*) The formal prerequisites are a basic calculus course and a course emphasizing what constitutes a proof, standard methods of proof, and the like. In reality, a curiosity for things mathematical and the “need to understand such,” is both necessary and sufficient.

The arrangement of topics is standard. The historical struggle to give a
rigorous definition of "area" and "area under a curve," resulting in
Lebesgue measure and integration, is the subject of Chapter 1. (Tribute is
paid to our mathematical ancestors by understanding and studying their
results.) Mastery of this material is not necessary for subsequent chapters.
After all, it is an "overview," written with the benefit of hindsight. The
reader may return from time to time as she understands "measurable",
"Borel", "Lebesgue Dominated Convergence," and so on. Mathematical
concepts (undergraduate analysis) that are useful for the understanding
of measure, measurable functions, and integration, are developed in
Chapter 2. Chapter 3, measure theory, is the essence of this book. Here
an elementary, but rigorous, treatment of Lebesgue measure, as a natural
extension of "length of an interval" and as a subject of interest in and of
itself, is presented. Set measurability is via Carathéodory's Condition.
Measurable functions, motivated by the necessity of "measuring" inverse
images of intervals as discussed by Lebesgue [Ma], are defined and
developed in Chapter 4. The last chapter, Chapter 5, begins with the
Riemann integral, developed from step functions. Replacing "step" with
"simple" results in the Lebesgue integral for bounded functions on sets of
finite measure. Some incisive observations and we have the celebrated
convergence theorems that permit the interchange of "limit" and "inte-
gral", and justifies "Lebesgue" for those with such a need. (By the way, if
at any time you are confused or lack a sense of direction, I apologize; for a
solution, reread the master [Ma].) Finally, appendices A-E present other
topics of beauty and inspiration to mathematicians, testament to the
wonderful creativity of the human mind.

This book may be used in many ways: especially as a text for an
undergraduate analysis course, first-year graduate students in statistics or
probability, and other applied areas; a self-study guide to elementary
analysis or as a refresher for comprehensive examinations; a supplement
to the traditional real analysis course taken by beginning graduate
students in mathematics.

I want to thank my good friend and colleague, Gene Meyer, for his
countless hours of discussions and suggestions as to topics, and what
would or would not be appropriate for a book of this nature. Accolades to
Debora Naber. She had the arduous task of translating my handwriting
into the final manuscript. I thank my parents, Glen and Helen Burk,
whose constant encouragement has been a source of strength throughout
my life. Finally, I thank my wonderful wife Janet, who somehow finds the
time to encourage my dreams while rearing our five beautiful children—
(Eric, Angela, Michael, Brandon, and Bryan.).
Even now there is a very wavering grasp of the true position of mathematics as an element in the history of thought. I will not go so far as to say that to construct a history of thought without profound study of the mathematical ideas of successive epochs is like omitting Hamlet from the play which is named after him. That would be claiming too much. But it is certainly analogous to cutting out the part of Ophelia. This simile is singularly exact. For Ophelia is quite essential to the play, she is very charming—and a little mad. Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.

—Alfred North Whitehead

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry. What is best in mathematics deserves not merely to be learned as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement. Real life is, to most men, a long second-best, a perpetual compromise between the real and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs. Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the natural world.

—Bertrand Russell
Lebesgue Measure
and Integration
Some of the major discoveries in quadratures that culminated with the Lebesgue-Young integral are presented in this chapter. Our purpose is twofold:

1. We want to acknowledge our appreciation and gratitude to the thinkers of the past. It is hoped that the student will be motivated to continue these threads that distinguish civilization from barbarism.

Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy.

—Roger Bacon

2. The student will see the process of mathematical creation and generalization as it applies to the development of the Lebesgue integral.

Reason with a capital R = Sweet Reason, the newest and rarest thing in human life, the most delicate child of human history.

—Edward Abbey

If this material is too difficult on the first reading, relax. It will make sense after Chapter 5.
1.1 REARRANGEMENTS

The figures below demonstrate the general idea of "rearranging"; in the first example, a circle rearranged into a parallelogram. This method has been known for hundreds of years.
1.2 EUDOXUS (408–355 B.C.E.) AND THE METHOD OF EXHAUSTION

“Willingly would I burn to death like Phaeton, were this the price for reaching the sun and learning its shape, its size, and its substance.”

—Eudoxus
Eudoxus was responsible for the notion of approximating curved regions with polygonal regions: “truth” for polygonal regions implies “truth” for curved regions. This notion will be used to show that the areas of circles are to each other as the squares of their diameters, an obvious result for regular polygons. “Truth” was to be based on Eudoxus’ Axiom:

Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continuously, there will be left some magnitude which will be less than the lesser magnitude set out.

In modern terminology, let \( M \) and \( \epsilon > 0 \) be given with \( 0 < \epsilon < M \). Then form:

\[
M, M - rM = (1 - r)M, (1 - r)M - r(1 - r)M = (1 - r)^2 M, \ldots,
\]

where \( 1/2 < r \leq 1 \). The axiom tells us that for \( n \) sufficiently large, say \( N \), \( (1 - r)^N M < \epsilon \), a consequence of the set of natural numbers not being bounded above.

Back to what we are trying to show: Let \( c, C \) be circles with areas \( a, A \) and diameters \( d, D \), respectively. We want to show \( a/A = d^2/D^2 \), given that the result is true for polygons and given the Axiom of Eudoxus.

Assume \( a/A > d^2/D^2 \). Then we have \( a^* < a \) so that \( 0 < a - a^* \) and \( a^*/A = d^2/D^2 \). Let \( \epsilon < a - a^* \). Inscribe regular polygons of areas \( p_n, P_n \) in circles \( c, C \) and consider the areas \( a - p_n, A - P_n \):

Now, double the number of sides. What is the relationship between \( a - p_n \) and \( a - p_{2n} \)?
Certainly $a - p_{2^n} < 1/2(a - p_n)$. We are subtracting more than half at each stage of doubling the number of sides. From the Axiom of Eudoxus, we may determine $N$ so that

$$0 < a - p_N < \epsilon < a - a^*,$$

that is,

we have a regular inscribed polygon of $N$ sides, where area $p_N > a^*$. But, $p_N/P_N = d^2/D^2$ and since $a^*/A = d^2/D^2$, we have $p_N/P_N = a^*/A$, that is, $P_N > A$. This cannot be: $P_N$ is the area of an inscribed polygon to the circle $C$ of area $A$.

A similar argument shows that $a/A$ cannot be less than $d^2/D^2$:

*double reductio ad absurdum.*

### 1.3 THE LUNE OF HIPPOCRATES (430 B.C.E.)

Hippocrates, a merchant of Athens, was one of the earliest individuals to find the area of a plane figure (lune) bounded by curves (circular arcs). The crescent-shaped region whose area is to be determined is shown below.

ABC and AFC are circular arcs with centers E and D, respectively. Hippocrates showed that the area of the shaded region bounded by the circular arcs ABC and AFC is exactly the area of the shaded square whose side is the radius of the circle. The argument depends on the following assumption:

(a) The areas of two circles are to each other as the squares of the radii:
From this assumption we conclude that (b) the sectors of two circles with equal central angles are to each other as the squares of the radii:

(c) The segments of two circles with equal central angles are to each other as the squares of the radii:

Hippocrates’ argument proceeds as follows:

From (c), $A_1/A_4 = r^2/((\sqrt{2}r)^2) = 1/2$. Hence $A_1 = 1/2 A_4$ and $A_2 = 1/2 A_4$ and thus $A_1 + A_2 = A_4$.

The area of the lune $= A_1 + A_2 + A_3$
$= A_4 + A_3$
$= \text{area of triangle}$
$= \frac{1}{2}(\sqrt{2}r)(\sqrt{2}r)$
$= r^2$
$= \text{area of the square}$. 
The reader may use similar reasoning on these “lunes”:

1. 
![Diagram of a lune](image1)

2. 
![Diagram of a lune with labeled parts](image2)

He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side.

—Plato

### 1.4 ARCHIMEDES (287–212 B.C.E.)

It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his natural genius, . . .

—Plutarch

This masterpiece of mathematical reasoning is due to one of the greatest intellects of all time, Archimedes of Syracuse. He shows that the area of the parabolic segment is $4/3$ that of the inscribed triangle $ACB$. (The symbol $\triangle$ will denote “area of”.)
The argument proceeds as follows: the combined area of triangle $ADC$ and $BEC$ is one-fourth the area of triangle $ACB$, that is,

$$\triangle ADC + \triangle BEC = \frac{1}{4} \triangle ACB.$$  

Repeating the process, trying to "exhaust" the area between the parabolic curve and the inscribed triangles, we have:

The area of the parabolic segment

$$= \triangle ACB + \frac{1}{4}(\triangle ACB) + \frac{1}{4}\left(\frac{1}{4}(\triangle ACB)\right) + \cdots$$

$$= \triangle ACB \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right)$$

$$= \frac{4}{3} \triangle ACB.$$  

We argue that

$$\triangle ADC + \triangle BEC = \frac{1}{4} \triangle ACB$$

for the parabola $y = ax^2$, $a > 0$. 

The reader should show the tangent line at \( C \) is parallel to \( AB \) and that the vertical line through \( C \) bisects \( AB \) at \( P \). We need to show \( \triangle BEC = \frac{1}{4} \triangle BCP \). Complete the parallelogram:

![Parallelogram Diagram]

We note:

1. \( \triangle CEG = \triangle BEG \) (equal height and base)
2. \( \triangle HGB = \frac{1}{4} \triangle BCP \).

Thus, we must show

\[
\triangle CEG + \triangle BEG = \triangle HGB,
\]

or that

\[
\triangle BEG = \frac{1}{2} \triangle HGB.
\]

This will be accomplished by showing \( FE = \frac{1}{4} FH = \frac{1}{4} QB \). Since

\[
FE = a((X_C + X_B)/2)^2 - \left[aX_C^2 + 2aX_C \times \frac{1}{2}(X_B - X_C)\right]
\]

\[
= \frac{1}{4}a(X_B - X_C)^2,
\]

and

\[
QB = aX_B^2 - [aX_C^2 + 2aX_C(X_B - X_C)]
\]

\[
= a(X_B - X_C)^2,
\]

we are done.
... there was far more imagination in the head of Archimedes than in that of Homer.

— Voltaire

Archimedes will be remembered when Aeschylus is forgotten because languages die and mathematical ideas do not.

— G.H. Hardy

The reader may show that the area of the parabolic segment is $2/3$ the area of the circumscribed triangle $ADB$ formed by the tangent lines to the parabola at $A$ and $B$ with base $AB (EC = CD)$.

1.5 PIERRE FERMAT (1601–1665): $\int_0^b x^{p/q} dx = b^{p/q+1}/(p/q + 1)$

It appears that Fermat, the true inventor of the differential calculus, ...

— Laplace

The Italian mathematician Cavalieri demonstrated (1630's) that

$$\int_0^b x^n dx = \frac{b^{n+1}}{n+1}$$

for $n = 1, 2, \ldots, 9$. But it was Fermat who was able to show

$$\int_0^b x^{p/q} dx = \frac{b^{p/q+1}}{\frac{p}{q} + 1},$$

where $p/q$ is a positive rational number.

Fermat divided the interval $[0, b]$ into an infinite sequence of subintervals with endpoints (herefore a finite number of subintervals of equal
width) \(br^n\), 0 < r < 1, and erected a rectangle of height \((br^n)^{p/q}\) over the subinterval \([br^{n+1}, br^n]\) (see below).

Let \(S_r\) denote the sum of the areas of the exterior rectangles. We have

\[
S_r = (b - br)b^\xi + (br - br^2)(br)^\xi + \cdots + (br^n - br^{n+1})(br^n)^\xi + \cdots
\]

\[
= b^{\xi+1}_r (1 - r) \left[ 1 + r^{\xi+1}_r + r^{(\xi+1)2}_r + \cdots + r^{(\xi+1)n}_r + \cdots \right]
\]

\[
= \frac{b^{\xi+1}_r (1 - r)}{(1 - r^{\xi+1}_r)}
\]

\[
= b^{\xi+1}_r \left[ 1 - (r^{\xi}_r)^q \right] \frac{(1 - r^{\xi}_r)}{(1 - r^{\xi}_r)} \left[ 1 - (r^{\xi}_r)^{p+q} \right]
\]

\[
= b^{\xi+1}_r \left( 1 + r^{\xi}_r + \cdots + r^{\frac{p+q-1}{q}} \right) \frac{1}{1 + r^{\xi}_r + \cdots + r^{\frac{p+q-1}{q}}}
\]

\[
\rightarrow b^{\xi+1}_r \frac{q}{p+q} \quad \text{as} \; r \rightarrow 1
\]

\[
= \frac{b^{\xi+1}_r}{q + 1}.
\]

... a master of masters.

—E.T. Bell
Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half.

—Gottfried Leibnitz

Nature and Nature's laws lay hid in night; God said, "Let Newton be!" and all was light.

—Alexander Pope

The capital discovery that differentiation and integration are inverse operations belongs to Newton and Leibnitz.

—Sophus Lie

During the seventeenth and eighteenth centuries the integral was thought of in a descriptive sense, as an antiderivative, due to the beautiful Fundamental Theorem of Calculus (FTC), as developed by Leibnitz and Newton. The ease of this method for specific functions probably induced a sense of euphoria, as generations of calculus students can attest to after struggling through Riemann sums. A particular function \( f \) on \([a, b]\) was integrated by finding an antiderivative \( F \) so that \( F' = f \) or by finding a power series expansion and using the FTC to integrate termwise. The Leibnitz-Newton integral of \( f \) was \( F(b) - F(a) \), that is,

\[
\int_a^b f(x) \, dx = F(b) - F(a),
\]

where \( F' = f \).

We give an argument of Leibnitz and a result of Newton to illustrate the power of these geniuses.

Leibnitz: \( \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \).