NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

LEON LAPIDUS
GEORGE F. PINDER
University of Vermont

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Library of Congress Cataloging in Publication Data:

Lapidus, Leon.
Numerical solution of partial differential
equations in science and engineering.
"A Wiley-Interscience publication."
Includes index.
1. Science—Mathematics. 2. Engineering.
mathematics. 3. Differential equations, Partial—
Numerical solutions. I. Pinder, George Francis,
1942– II. Title.
Q172.L36 515.3'53 81-16491
ISBN 0-471-09866-3 AACR2
ISBN 0-471-35944-0 (paperback)

10 9 8 7 6 5 4 3

Preface

This book was written to provide a text for graduate and undergraduate students who took our courses in numerical methods. It incorporates the essential elements of all the numerical methods currently used extensively in the solution of partial differential equations encountered regularly in science and engineering. Because our courses were typically populated by students from varied backgrounds and with diverse interests, we attempted to eliminate jargon or nomenclature that would render the work unintelligible to any student. Moreover, in response to student needs, we incorporated not only classical (and not so classical) finite-difference methods but also finite-element, collocation, and boundary-element procedures. After an introduction to the various numerical schemes, each equation type—parabolic, elliptic, and hyperbolic—is allocated a separate chapter. Within each of these chapters the material is presented by numerical method. Thus one can read the book either by equation-type or numerical approach.

After writing much of the finite-difference discussion found herein, Leon Lapidus died suddenly on May 5, 1977, while working in his office in the Department of Chemical Engineering at Princeton University. In completing the manuscript, I have attempted to keep his work intact. I also adopted his nomenclature and editorial style.

The successful completion of this manuscript is, in no small measure, due to the efforts of those who gave generously of their time in reading, criticizing and modifying the early drafts of the book, and those who helped proofread the final copy. Particular recognition is due to M. B. Allen, N. R. Amundson, M. Celia, and D. H. Tang, who read the entire manuscript and to L. Abriola, V. V. Nguyen and R. Page, who helped verify the typesetting. Mrs. L. Lapidus was helpful throughout the preparation of the work, particularly in the final stages of publication. I also wish to thank Dorothy Hannigan, who produced a beautifully typed manuscript under very difficult circumstances. Finally, I would like to express my appreciation to my wife, Phyllis, who provided an environment and the encouragement essential to the completion of the work.

GEORGE F. PINDER

Princeton, New Jersey
November 1981

in memory
typical and past

not for you
never for you

rather

in love
in thanks
in life
special and now

yes

we see your face
we hear your words
we feel your presence

you teach us
we learn

so no
never in memory

now and always
for you, to you, with you

in love
in thanks
in life

MARY LAPIDUS HEWETT

Contents

ONE

Fundamental Concepts

This chapter serves as a detailed introduction to many of the concepts and characteristics of partial differential equations (hereafter abbreviated PDEs). Commonly encountered notation and the classification of PDE are discussed together with some features of analytical and numerical solutions.

1.0 NOTATION

Consider a partial differential equation (PDE) in which the independent variables are denoted by $x, y, z, \ldots$ and the dependent variables by $u, v, w, \ldots$. Direct functionality is often written in the form

$$u = u(x, y, z), \tag{1.0.1}$$

which, in this particular case, designates u as a function of the independent variables x, y, and z. Partial derivatives are often denoted as follows:

$$u_x = \frac{\partial u}{\partial x}; \quad u_y = \frac{\partial u}{\partial y}; \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}; \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}; \quad \ldots. \tag{1.0.2}$$

Employing the definitions of (1.0.1) and (1.0.2), we can thus represent a PDE in the general form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, \ldots) = 0, \tag{1.0.3}$$

where F is a function of the indicated quantities and at least one partial derivative exists.

As examples, consider the following PDEs:

$$u_{xx} + u_{yy} = 0$$

$$u_x = u + x^2 + y^2$$

$$u_{xxx} = u_{yy} + u^2$$

$$(u_x)^2 + (u_y)^2 = \exp(u).$$

The *order* of a PDE is defined by the highest-order derivative in the equation. Therefore,

$$u_x - bu_y = 0$$

is of first order,

$$u_{xx} + u_y = 0$$

is of second order, and

$$u_{xxxx} + u_{yyyy} = 0$$

is of fourth order. When several interdependent PDEs are encountered, the order is established by combining all the equations into a single equation. For example, the following system of equations is of second order although each contains only first-order derivatives; that is,

(1.0.4a) $$u_x + v_y = u_z$$

(1.0.4b) $$u = w_x$$

(1.0.4c) $$v = w_y$$

can alternatively be written

(1.0.5) $$w_{xx} + w_{yy} = w_{xz}.$$

When written in the form of (1.0.5), it is readily apparent that (1.0.4) is of second order.

In the solution of PDEs, the property of linearity plays a particularly important role. Consider, for example, the first-order equation

(1.0.6) $$a(\cdot)u_x + b(\cdot)u_y = c(\cdot).$$

The linearity of this equation is established by the functionality of the coefficients $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$. In the case of (1.0.6), if the coefficients are constant or functions of the independent variables only, $[(\cdot) \equiv (x, y)]$, the PDE is linear; if the coefficients are also functions of the dependent variable $[(\cdot) \equiv (x, y, u)]$, the PDE is quasilinear; if the coefficients are functions of the first derivatives, $[(\cdot) \equiv (x, y, u, u_x, u_y)]$, the PDE is nonlinear. Thus the following PDEs are classified as indicated:

$$u_x + bu_y = 0 \qquad \text{(linear)}$$

$$u_x + uu_y = x^2 \qquad \text{(quasilinear)}$$

$$u_x + (u_y)^2 = 0. \qquad \text{(nonlinear)}$$

In general, when the coefficients of an nth-order PDE depend upon nth-order derivatives, the equation is nonlinear; when they depend upon mth-order derivatives, $m<n$, the equation is quasilinear. These features are important because whereas many analytical properties of linear and even quasilinear PDEs are known, as a general rule, each nonlinear PDE must be considered individually.

The analytical solution of a PDE, which may be written

$$u=u(x,y),$$

denotes a function that, when substituted back into the PDE, generates an identity. Of course, when one discusses the solution of a PDE, it is necessary to consider appropriate auxiliary initial and boundary conditions. For example, the transient temperature distribution in a homogeneous rod of finite length with insulated sides is described by the system

$$u_x=u_{yy}, \qquad x>0,\quad 0<y<1 \qquad \text{(PDE)}$$

$$u(0,y)=f(y), \qquad x=0,\quad 0<y<1 \qquad \text{(initial condition)}$$

$$u(x,0)=\phi(x), \qquad y=0,\quad x\geqslant 0$$

$$u(x,1)=\theta(x), \qquad y=1,\quad x\geqslant 0. \qquad \text{(boundary condition)}$$

Such a specification usually leads to a *well-posed* problem. Almost all reasonable problems are well posed and yield a solution that is unique and depends continuously on the auxiliary conditions (Hadamard, 1923). Alternatively, a well-posed problem can be considered as one for which small perturbations in the auxiliary conditions lead to small changes in the solution.

It is instructive at this point to compare briefly the solution properties of ordinary differential equations, herein denoted as ODEs. The general form of a first-order ODE is

$$\frac{du}{dx}=f(x,u),$$

where f is a function of the indicated quantities. In the case of an ODE, a specification of (x,u) yields a unique value of du/dx; by contrast, a specification of (x,y,u) in a first-order PDE only gives a connection between u_x and u_y but does not uniquely determine each. In the case of a second-order ODE, the solution specifies a point and a tangent line on the solution trajectory in a plane; by contrast, these concepts of a *point*, *plane*, or *tangent line* for the ODE are extended to a *curve*, *three-dimensional space*, and *tangent plane* for the PDE. In other words, for an ODE, there are solution curves in a two-dimensional space that are required to pass through a point, while for a PDE there are solution surfaces in three-dimensional space that are required to pass through a curve or line. These differences are, of course, a direct result of the increase in number of independent variables in the PDE as compared to the ODE.

1.1 FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this section we consider some of the fundamental features of first-order PDEs. The principal objective is to present an overview of the basic concepts in this area; for a definitive analysis we recommend the books by Courant (1962) and Aris and Amundson (1973).

1.1.1 First-Order Quasilinear Partial Differential Equations

Consider the quasilinear PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{1.1.1}$$

in the two independent variables x and y. The extension to more independent variables is rather obvious and thus is not discussed here. Also, the linear PDE is considered as a special case of (1.1.1) and is mentioned specifically when appropriate.

Suppose that we are located at a point $P(x, y, u)$ on the solution surface $u = u(x, y)$ (Figure 1.1) and we move in a direction given by the vector $\{a, b, c\}$. But at any point on the surface, the direction of the normal is given by the vector $\{u_x, u_y, -1\}$. It is obvious from (1.1.1) that a scalar product of these two vectors vanishes (i.e., the two vectors are orthogonal). Thus $\{a, b, c\}$ is perpendicular to the normal and must lie in the tangent plane of the surface $u = u(x, y)$. Thus the PDE is a mathematical statement of the geometrical requirement that any solution surface through the point $P(x, y, u)$ must be tangent to a vector with components $\{a, b, c\}$. Further, since $\{a, b, c\}$ is always tangent to the surface, we never leave the surface. Note also that since

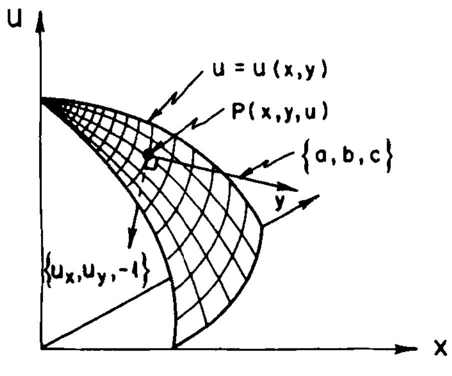

Figure 1.1. Solution surface $u = u(x, y)$ with vector $\{a, b, c\}$ tangent to u and vector $\{u_x, u_y, -1\}$ normal to u at point $P(x, y, u)$.

$u = u(x, y)$

$$du = u_x\,dx + u_y\,dy \tag{1.1.2}$$

and thus $\{a, b, c\} \equiv \{dx, dy, du\}$.

The solution to (1.1.1) is readily obtained using the following theorem.

Theorem 1

The general solution of the quasilinear PDE

$$au_x + bu_y = c$$

is given by

$$G(v, w) = 0,$$

where G is an arbitrary function and where $v(x, y, u) = c_1$ and $w(x, y, u) = c_2$ form a solution of the equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{1.1.3}$$

Note that (1.1.3) comprises a set of two independent ODEs (a two-parameter family of curves in space). Further, one set of these can be written as

$$\frac{dy}{dx} = \frac{b(x, y, u)}{a(x, y, u)} \tag{1.1.4}$$

and is termed a *characteristic curve*. When $a = a(x, y)$ and $b = b(x, y)$ only, (1.1.4) is a function in (x, y) space. In this case we refer to the curve as a characteristic ground or base curve.

When a and b are constant, (1.1.4) defines a set of parallel lines in (x, y) space. In either of these last two cases (1.1.4) may be evaluated without knowing $u(x, y)$; in the quasilinear case (1.1.4) cannot be evaluated until $u(x, y)$ is also known. However, in any three-dimensional (x, y, u) plot, such as that in Figure 1.1, one can project down onto the x-y plane to obtain

$$\frac{dy}{dx} = \frac{b(x, y, u)}{a(x, y, u)}.$$

The characteristic equation (1.1.4) may be obtained directly through an examination of the PDE and (1.1.2). Restating these equations; we have two equations in the values u_x and u_y:

$$(1.1.5a) \qquad au_x + bu_y = c$$

$$(1.1.5b) \qquad (dx)u_x + (dy)u_y = du.$$

Obviously, both equations must hold on the solution surface and yet one can interpret each equation as a plane element; these plane elements intersect on a line along which different values of u_x and u_y may exist. In other words, u_x and u_y are themselves indeterminate along this line, but at the same time they are related or determinate to each other since the equations must hold.

To exploit this feature, we use a well-known principle of linear algebra. If a square coefficient matrix for a set of n linear simultaneous equations has a vanishing determinant, a necessary condition for finite solutions to exist is that when the right-hand side is substituted for any column of the coefficient matrix, the resulting determinants must also vanish. Thus, if we treat (1.1.5) as linear algebraic equations in u_x and u_y, we may write

$$(1.1.6) \qquad \begin{bmatrix} a & b \\ dx & dy \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} c \\ du \end{bmatrix}.$$

From the property above, it then follows that

$$(1.1.7) \qquad \det\begin{bmatrix} a & b \\ dx & dy \end{bmatrix} = 0; \qquad \det\begin{bmatrix} c & b \\ du & dy \end{bmatrix} = 0;$$

$$\det\begin{bmatrix} a & c \\ dx & du \end{bmatrix} = 0,$$

implying linear dependence of u_x and u_y. Evaluating the determinants leads directly to the statement of (1.1.3),

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

1.1.2 Initial Value or Cauchy Problem

Now we raise the question of how initial data (initial or boundary conditions) specified on a prescribed curve or line Γ interact with the equations given by (1.1.3). Suppose that this space curve Γ prescribes the values of x, y, and u as a function of some parameter r. This means that

$$(1.1.8) \qquad x = x(r), \qquad y = y(r), \qquad u = u(r).$$

The characteristic curves passing through Γ can be described using an independent variable, say s, along the characteristic. Thus (1.1.3) can be restated as the set

$$\frac{dx}{ds} = a \tag{1.1.9a}$$

$$\frac{dy}{ds} = b \tag{1.1.9b}$$

and along this curve the PDE merely becomes

$$\frac{du}{ds} = c. \tag{1.1.9c}$$

Combination of (1.1.8) and (1.1.9) provides a solution to this problem which can be expressed in parametric terms as

$$x = x(r, s); \qquad y = y(r, s); \qquad u = u(r, s). \tag{1.1.10}$$

We have now involved the initial curve Γ and the characteristics to yield $u = u(r, s)$. The only problem that can occur is in the inversion of r, s, and u to functions of the independent variables x and y. This can be done (see Aris and Amundson, 1973, p. 9) provided that the Jacobian J, defined as

$$J = x_s y_r - y_s x_r = a y_r - b x_r, \tag{1.1.11}$$

is nonzero. When $J = 0$, the initial curve Γ is itself a characteristic curve and there are infinitely many solutions of the initial value or Cauchy problem.

1.1.3 Application of Characteristic Curves

Example 1

To illustrate some of the features of the abbreviated discussion above, we consider two examples. The first involves the solution to the following form of the transport equation:

$$u_x + v(\cdot)u_y = F(\cdot), \tag{1.1.12}$$

where $v(\cdot)$ is the velocity of propagation of an initial profile. When $v(\cdot) = v(x, y, u)$ the equation is quasilinear and the characteristics are curved and defined by substituting for a and b in (1.1.4):

$$\frac{dy}{dx} = v(x, y, u) \tag{1.1.13}$$

and, from (1.1.3),

$$\frac{du}{dx} = F(\cdot). \tag{1.1.14}$$

When $v(\cdot)$ is constant, the problem of solving (1.1.13) is simplified, because now the characteristic equation is

$$\frac{dy}{dx} = v = \text{constant}$$

and a given profile (see below) or initial condition at $x=0$ is propagated without change of shape in the direction of the x axis with velocity v.

When $v(\cdot)=\text{constant}$ and $F(\cdot)\equiv 0$, we have

(1.1.15) $$u_x + vu_y = 0$$

and the equations of interest are

(1.1.16) $$\frac{dx}{1} = \frac{dy}{v} = \frac{du}{0}.$$

The characteristics are now straight lines inclined to the x axis at an angle $\theta = \tan^{-1} v$ or with slope v. Along these characteristics $du=0$ or $u=\text{constant}$. This leads to a plot such as Figure 1.2, where the parallel straight lines are shown. Each straight line has the equation $y = vx + \text{constant}$ with the constant determined by the particular conditions at $x=0$ (initial conditions) or $y=0$ (boundary conditions). These are the conditions specified along the Γ data line. The solution $u(x, y)$ slides up a characteristic unchanged in its value.

Note that there is no approximation in this solution. The answer obtained is "correct" in the sense that only if $dx/ds = a$ needs to be integrated numerically along the characteristics will any error be involved.

Example 2

As a second example, consider an isothermal plug flow reactor with a first-order reaction. The relevant PDE and boundary conditions are

(1.1.17a) $$u_x + vu_y = -ku$$

(1.1.17b) $$u=0, \qquad x=0, \qquad y \geqslant 0$$

(1.1.17c) $$u=u_0, \qquad x>0, \qquad y=0,$$

where u represents the concentration of material, v is the velocity of flow of material through the tube, and a first-order reaction (sink) is involved. The reactor contains no reactant initially and is then fed with a reactant with a fixed concentration u_0. Defining the dimensionless groups

$$\eta = \frac{u}{u_0}, \qquad \theta = kx, \qquad \tau = \frac{yk}{v},$$

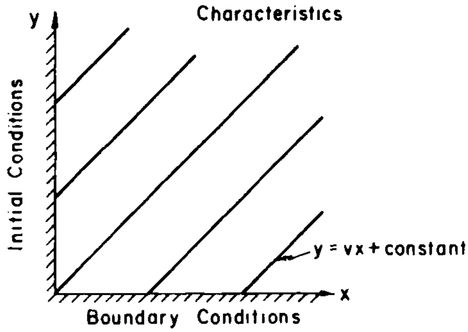

Figure 1.2. Characteristic curves $y = vx$ with boundary and initial conditions indicated.

we may rewrite (1.1.17) as

(1.1.18a) $$\eta_\theta + \eta_\tau = -\eta$$

(1.1.18b) $$\eta = 0, \quad \theta = 0, \quad \tau \geqslant 0$$

(1.1.18c) $$\eta = 1, \quad \theta > 0, \quad \tau = 0.$$

The characteristic equations are

$$\frac{d\theta}{1} = \frac{d\tau}{1} = \frac{-d\eta}{\eta}$$

or

(1.1.19a) $$\frac{d\tau}{d\theta} = 1; \quad \theta \geqslant 0, \quad \tau \geqslant 0$$

and

(1.1.19b) $$\frac{d\eta}{d\tau} = -\eta; \quad \begin{array}{lll} \eta = 0, & \theta = 0, & \tau \geqslant 0 \\ \eta = 1, & \theta > 0, & \tau = 0. \end{array}$$

Because (1.1.19) are linear they are easily integrated to yield

(1.1.20a) $$\eta = 0, \quad \tau \geqslant \theta$$

(1.1.20b) $$\eta = e^{-\tau}, \quad \tau < \theta.$$

Equations 1.1.20 represent the complete solution for the problem. Using the arbitrary numerical values of θ and τ of 2.5, Figures 1.3 and 1.4 can be developed. These are two- and three-dimensional representations of η as a function of τ and θ.

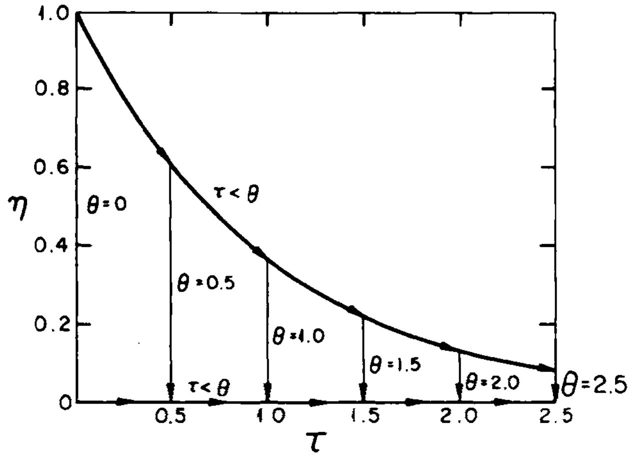

Figure 1.3. Two-dimensional representation of concentration (η) vs. distance (τ) with selected values of the second space variable θ also indicated (see Figure 1.4).

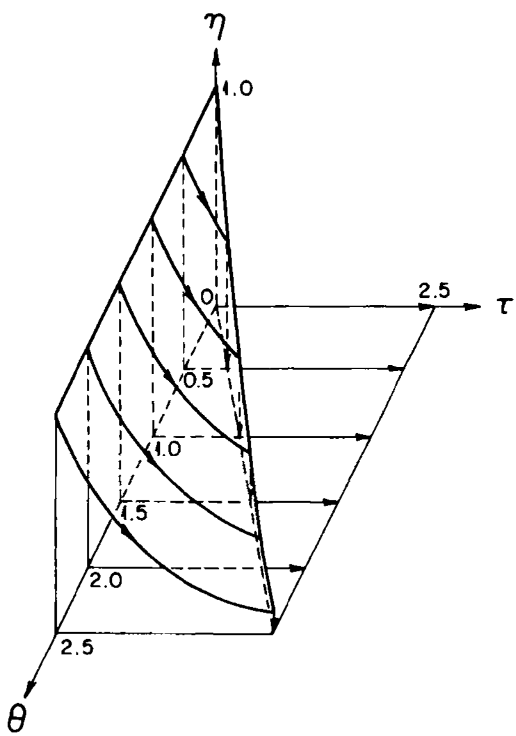

Figure 1.4. Three-dimensional representation of concentration (η) vs. the two space coordinates τ and θ.

Although this is only a small sampling of applications of characteristic lines to the solution of first-order PDE, it serves as an introduction to the scheme we use later in developing a classification for second-order PDEs. Before turning our attention to second-order PDEs, let us briefly extend the concept of characteristic lines to nonlinear first-order PDEs.

1.1.4 Nonlinear First-Order Partial Differential Equations

When the first-order PDE is nonlinear, it can be written (see Section 1.0)

$$F(x, y, u, u_x, u_y) = 0, \tag{1.1.21}$$

where

$$\left(\frac{\partial F}{\partial u_x}\right)^2 + \left(\frac{\partial F}{\partial u_y}\right)^2 \neq 0.$$

A well-known problem described by an equation of the form of (1.1.21) arises in geometric optics. The appropriate expression is

$$u_x^2 + u_y^2 = 1.$$

Much of what we introduced in the discussion of linear first-order PDE is still retained in the nonlinear case but in a more complex form. Now characteristic lines become characteristic strips; the so-called Monge cone in which the tangent to the solution surface must lie is a surface generated by a one-parameter family of straight lines through a fixed point of its vertex. In the quasilinear case, the cone becomes linear or a Monge axis.

Without attempting to present the details of the derivation of the characteristic equations, we indicate here that analogous to (1.1.9) (the initial value or Cauchy problem) there are now five ODEs:

$$\frac{dx}{ds} = F_{u_x} \tag{1.1.22a}$$

$$\frac{dy}{ds} = F_{u_y} \tag{1.1.22b}$$

$$\frac{du}{ds} = u_x F_{u_x} + u_y F_{u_y} \tag{1.1.22c}$$

$$\frac{du_x}{ds} = -F_x - u_x F_u \tag{1.1.22d}$$

$$\frac{du_y}{ds} = -F_y - u_y F_u. \tag{1.1.22e}$$

When $F(x, y, u, u_x, u_y) = a(\cdot)u_x + b(\cdot)u_y - c = 0$, the quasilinear case, (1.1.22), becomes (1.1.9).

1.2 SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Now let us consider some features of second-order PDE that will be useful in the ensuing chapters on numerical solutions. By comparison, the first-order PDE was relatively uncomplicated in the sense that the characteristic curves could be located and $u(x, y)$ determined along those curves. In the second-order case, the characteristics may or may not play a role.

Consider the following second-order PDE written in two independent variables:

(1.2.1)
$$a(\cdot)u_{xx} + 2b(\cdot)u_{xy} + c(\cdot)u_{yy} + d(\cdot)u_x + e(\cdot)u_y + f(\cdot)u + g(\cdot) = 0.$$

As in earlier sections, we denote (1.2.1) as linear if $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ are constant or functions only of x and y; quasilinear if $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ are functions of x, y, u, u_x, and u_y; and nonlinear in all other cases. Typical examples of second-order PDEs are the following well-known equations:

$u_{xx} + u_{yy} = 0$	Laplace's equation
$u_{xx} + u_{yy} = f(x, y)$	Poisson's equation
$u_x = u_{yy}$	heat flow or diffusion equation
$u_x = u_{yy} + u_{zz}$	heat flow or diffusion equation
$u_x + uu_y = ku_{yy}$	Burger's equation
$u_{xx} = u_{yy}.$	wave equation

1.2.1 Linear Second-Order Partial Differential Equations

There exists an extensive body of knowledge regarding linear PDEs. This information is generally cataloged according to the form of the PDE. Every linear second-order PDE in two independent variables can be converted into one of three standard or *canonical* forms which we identify as hyperbolic, parabolic, or elliptic. In this canonical form at least one of the second-order terms in (1.2.1) is not present.

There is a practical reason for identifying the type of PDE in which one is interested. When coupled with initial and boundary conditions, the method and form of solution will be dependent on the type of PDE.

The classification can take many forms. We assume (for now) that if

(1.2.2a) $b^2 - ac > 0$ the PDE is *hyperbolic*

(1.2.2b) $b^2 - ac = 0$ the PDE is *parabolic*

(1.2.2c) $b^2 - ac < 0$ the PDE is *elliptic*.

Let us now examine the canonical forms and their associated transformations. The three canonical forms are written in terms of the new variables ξ and η as:

(1.2.3a) $$u_{\xi\xi} - u_{\eta\eta} + \cdots = 0$$

or *hyperbolic*

$$u_{\xi\eta} + \cdots = 0$$

(1.2.3b) $$u_{\xi\xi} + \cdots = 0$$ *parabolic*

(1.2.3c) $$u_{\xi\xi} + u_{\eta\eta} + \cdots = 0.$$ *elliptic*

We shall see that the hyperbolic PDE has two real characteristic curves, the parabolic PDE has one real characteristic curve, and the elliptic PDE has no real characteristic curves.

From (1.2.2) or (1.2.3) we can see that the heat flow equation $u_x = u_{yy}$ is parabolic and already in canonical form, and the Laplace equation $u_{xx} + u_{yy} = 0$ is elliptic and already in canonical form. There are other cases, however, in which (1.2.2) must be used and the equations and their classifications may change because of coefficients. Thus

$yu_{xx} + u_{yy} = 0$	Tricomi's equation, elliptic for $y > 0$, hyperbolic for $y < 0$
$(1 + y^2)u_{xx} + (1 + y^2)u_{yy} - u_x = 0$	elliptic
$u_{xx} + uu_{yy} = 0$	elliptic for $u > 0$ hyperbolic for $u < 0$
$u_{xx} + (1 - x^2 - y^2)u_{yy} = 0$	elliptic inside unit circle hyperbolic outside
$yu_{xx} + xu_{xy} + yu_{yy} = 0.$	hyperbolic, $x > 2y$ parabolic, $x = 2y$ elliptic, $x < 2y$

With these preliminaries in hand, let us now consider the canonical transformations. We ignore all terms in (1.2.1) except the second derivatives because the lower-order terms do not influence the results. We introduce the change of variables (implicit here) of

(1.2.4) $$\xi = \phi(x, y), \qquad \eta = \psi(x, y)$$

and develop, using the chain rule,

(1.2.5a) $$u_x = u_\xi \phi_x + u_\eta \psi_x$$

(1.2.5b) $$u_y = u_\xi \phi_y + u_\eta \psi_y$$

(1.2.6) $$u_{xx} = u_{\xi\xi}\phi_x^2 + 2u_{\xi\eta}\phi_x\psi_x + u_{\eta\eta}\psi_x^2 + \cdots$$

(1.2.7) $$u_{xy} = u_{\xi\xi}\phi_x\phi_y + u_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + u_{\eta\eta}\psi_x\psi_y + \cdots$$

(1.2.8) $$u_{yy} = u_{\xi\xi}\phi_y^2 + 2u_{\xi\eta}\phi_y\psi_y + u_{\eta\eta}\psi_y^2 + \cdots .$$

Substitution into (1.2.1) yields

(1.2.9) $$au_{xx} + 2bu_{xy} + cu_{yy} = Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + \cdots ,$$

where

(1.2.10) $$A = a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2$$

(1.2.11) $$B = a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y$$

(1.2.12) $$C = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 .$$

From (1.2.10), (1.2.11), and (1.2.12) one can obtain the following relationship between a, b, c and A, B, C:

(1.2.13) $$B^2 - AC = (b^2 - ac)(\phi_x\psi_y - \phi_y\psi_x)^2 .$$

It is apparent that, under this change of variables, the sign of $b^2 - ac$ remains invariant with respect to $B^2 - AC$; moreover, $\phi_x\psi_y - \phi_y\psi_x$, which is the Jacobian of the transformation, must always be kept nonzero. If an explicit change of variables had been used,

$$\xi = \alpha_1 x + \beta_1 y + \gamma_1$$
$$\eta = \alpha_2 x + \beta_2 y + \gamma_2 ,$$

the Jacobian requirement would mean that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$.