CRYPTOGRAPHY, INFORMATION THEORY, AND ERROR-CORRECTION
A Handbook for the 21st Century

AIDEN A. BRUEN
MARIO A. FORCINITO
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Cryptography,
Information Theory,
and Error-Correction
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Cryptography, Information Theory, and Error-Correction

A Handbook for the 21st Century

Aiden A. Bruen
Mario A. Forcinito
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Preface

It is our privilege and pleasure to welcome all our readers to the dynamic world of cryptography, information theory and error correction. Both authors have considerable industrial experience in the field. Also, on the academic side Dr. Bruen has been a long-time editor of leading research journals such as “Designs, Codes and Cryptography”. Prior to his appointment in Calgary he worked in mathematical biology at Los Alamos. The book is an outgrowth both of presentations to industry groups and of a lecture course at the University of Calgary. The course was for undergraduate and graduate students in Computer Science, Engineering and Mathematics.

In addition to the academic topics in that course, we also include material relating to our industrial consulting work and experience in writing patents on the topics in the title of the book. In particular we describe revolutionary new algorithms in chapter 24 for hash functions and symmetric cryptography including quantum cryptography. These have been patented and have already made their way into industry.

This book can be read at many different levels. For example, it can be used as a reference or a text for courses in any of the three subjects or for a combined course. To this end we have included over three hundred worked examples and problems, with answers or solutions as needed. But we were determined to make the work highly accessible to the general reader as well. We hope that the exposition fulfills this goal. Large sections of this book have been written in such a way that little is required in the way of mathematical background. In places this was difficult to do but we believe that the effort has been worthwhile.

The three topics become more and more entwined as science and technology develop. In our opinion, the time when the three topics can be treated in isolation is rapidly drawing to a close. For example, if you search the internet for cryptographic information it is more and more likely that you will run up against terms such as entropy, CRC checksums, random number generators and the like. [Digressing: the main undergraduate course in computer science which is concerned with data structures — and we have all taught it — covers Huffman codes and compression at length but the word entropy is never mentioned. This is a shame].

Thus it seemed quite appropriate to us to try to write a complete but highly accessible
account of the three subjects stressing, above all, their interconnections and their unity. These interconnections can be hidden if one relies only on separate accounts of the three subjects. In addition, as part of information theory, we discuss some potential applications in cell biology. In the last chapter we present some new, exciting algorithms which combine all three of the subjects.

This is not the first time that a book combining the three subjects has been attempted. Several very good recent books, specializing in cryptography, have a few chapters on the other subjects. But our goal was to give a full in-depth account. We should mention that other books have handled nicely two of the three topics. A splendid book, published in 1988 by Dominic Welsh gives an account of all three of the subjects. However, a lot has happened since 1988. Also, our focus, emphasis and level of detail is different.

Let us briefly explain the 3 subjects.

**Cryptography.** This is an ancient subject concerned with the secret transmission of messages between two parties denoted by A and B. This could be done if A, B shared a secret language, say, not known to outsiders. More generally they can communicate in secret by sharing a common secret "key". Then A uses the key to scramble the message to B, who unscrambles the message with a copy of the same key that is owned by A. We may think of military commanders sending secret messages to each other or home movie providers sending movies to authorized customers. Apart from secrecy there are also crucial questions in cryptography involving authentication and identification.

**Information Theory.** This subject, also known as Shannon Theory after Claude Shannon, the late American mathematician and engineer, gives precise mathematical meaning to the term "information". This leads to answers to such questions as the following:

How much compression of data can be carried out without losing any information?

What is the maximum amount of information that can be transmitted over a noisy channel?

This fundamental question is answered precisely in Shannon’s famous channel capacity theorem which was discovered around 1948.

**Error Correction.** We introduce redundancy ["good redundancy"] for the transmission of messages, as opposed to the "bad redundancy" which was banished using compression. In this way we try to ensure that the receiver decodes accurately within the bounds of the Shannon capacity theorem mentioned in the previous section. The wonderful pictures of far-away planets, that have recently been made available, are just one example of what error-correcting codes can do. With a modern modem we can both compress as well as encode and decode to any required degree of accuracy.
Interconnections. These are spelled out in detail in the text but let us give a few short informal connections. How secure is your cryptographic password? It depends on how hard it is to guess it, i.e. it depends on its entropy as measured in Shannon bits. We then need information theory to properly discuss this.

In cryptography, A is sending information in secret to B, but what exactly is information and how is it measured? Again we need information theory.

Suppose that A is sending a secret key $K$ over a channel to B in order to encrypt, at some future date, a secret message $M$ with $K$ and transmit it to B. Now, a basic property of $K$ is this. If the transmission of $K$ is off by even one big then B will end up with a message that is completely different from the intended message $M$. The bottom line is that a transmission error could be catastrophic. The best way to guard against this is for A to use robust error-correction when sending the cryptographic key $K$ to B.

The great Claude Shannon made the following fundamental point. In error-correction, the receiver B is trying to correctly decode what the transmitter A has sent to B over a “noisy channel”. Compare this to the cryptographic situation where A is sending secret messages to B. They must contend with the eavesdropper — the evil Eve — who is listening in. We can think of Eve as receiving a “noisy” version of $M$ and trying to decipher, or decode $M$. We are back to coding theory. [Parenthetically, we mention that Shannon designed an interesting theory of the stock market by regarding the market as a very noisy channel!].

We must point out that this point of view of Shannon is extremely useful and not just as a formal device. We drive the point home with several problems in Chapter 16 where the analogy becomes quite striking. Moreover, in Chapter 24, A and B and Eve may have the same information to start out with, yet A and B have to come up with a way of beating Eve and publicly generating a secret key using a technique known as “Privacy Amplification”.

Here is yet another basic interconnection. Random numbers and pseudo-random numbers are the work-horses of cryptography, especially symmetric cryptography. One of the best ways of generating them is with shift-registers. In fact, as is pointed out in Schneier [Sch96], “stream ciphers based on shift registers have been the workhorse of military cryptography since the beginning of electronics”. But shift registers are central in information theory as they are great proving-grounds [or grave-yards] for questions on entropy. To understand entropy you have to confront shift registers. But — and here is the astonishing part — these shift registers, over any field, correspond exactly to cyclic linear codes which are at the heart of error-correction. For the expert, Reed-Solomon codes, and not just their error-correction, are merely special kinds of shift registers in disguise!

We move on now to a more conventional-type preface and address some standard questions.

Intended Readership. This is a book for everyone and can be used at many different levels. We are writing for many different kinds of readers.
1. All-rounders or renaissance types who have taken some mathematics or computer science or engineering [or none of the above] and who want to find out about these topics and have some fun.

2. Undergraduates or graduate students in mathematics, computer science or engineering.

3. Instructors of algebra and linear algebra who would like some real life practical applications in their courses, such as shift registers.

4. Biologists who may be interested in our discussions of such topics as biological compression and the channel capacity corresponding to the genetic code.

5. IT workers, venture capitalists and others who want an overview of the basics.

6. Academics looking for a good source of important (and doable) research problems.

7. Philosophers and historians of science who want to move on from quantum theory and relativity to a new, practical area which also, incidentally, has strong connections to quantum mechanics.

**Rewards for Readers.** If you make a good effort at understanding this book and working out some of the problems you will be well rewarded. This book covers everything you need. In particular, you will elevate your skills and mathematical maturity to a new level. You will also have an excellent background — better than that of most practitioners — in these areas. You will be ready to think about a career in cryptography or codes or even information theory. The market, especially in such areas as data compression is hot. You will be very well-placed for advanced work in cryptography, error-correction or information theory.

**Our Goals.** We want to help develop your skills and inspire you to new heights. Let this book be your inspiration. Master it and then get out and write those patents!

**Possible Courses Using this Book.** There is more than enough material for a stand-alone course at the undergraduate or graduate levels, in any of the three areas. The extensive list of problems and worked examples will be a big help. For those few chapters that don’t have problems there are opportunities for many fun group-projects geared towards reporting on patents, publications etc. We would recommend some “poaching” among the three parts of the book in such a course. A one year combined course would also work well.

**A Course for Non-Specialists.** Most of Part I, apart from the Chapter on elliptic curves, requires very little mathematical background but covers a lot of ground in cryptography. In information theory, we highly recommend Chapter 10 which gives a panorama of information theory and interesting related topics such as the “MBA problem” on weighings. The chapter
on topics related to the genetic code does not require much background, and should be of considerable interest. We also recommend Chapter 18 introducing coding theory. Chapter 20 tells the amazing story of how the famous perfect Golay code $G_{11}$ was first published in a Finnish soccer magazine in connection with the football pools in that country. Chapter 24 describes what appears to be a breakthrough in symmetric cryptography, error correction, and hash functions. We highly recommend it!

**Level, Mathematical Style, Proofs, Exercises.** We have made a considerable effort to ensure that the chapters are as accessible as possible. In terms of style, our motto, which is the opposite of many mathematicians and engineers, is this: “Never use a symbol if you can get away with a word.”

What about proofs? It really depends. If the proof enhances the ideas we try to present it. Also, some results, such as the Shannon source-coding result are so astonishing that we have to give the details. However, in the case of the noisy channel theorem we have a different approach. From teaching, we found it considerably more effective to give five or six different approaches rather than to just give the standard official proof.

This book was not written just for theoreticians. Much of our time was spent in designing good problems and solution. We urge our readers to take advantage of them.

**Mathematical Prerequisites.** Honestly? We try to cover everything “on the fly” along with one special chapter on specialized topics but here is a short summary of what we need.

Calculus: a small amount having to with the concavity of a graph [second derivative] and function maxima [first derivative, end points].

Linear algebra: Multiplying matrices, subspaces, invertibility and determinants.

Elementary probability and statistics: Mean and variance, Bernoulli trials, the normal curve, law of large numbers.

Algebra: A small amount of material on groups, finite fields, modular arithmetic.

Here and there we go over the top. For example, a bit of Fourier analysis for the Shannon sampling theorem is needed. But generally speaking, the above list covers most of the material and we do discuss the needed background as we go along.

**What’s New.** Most of the Chapter have a “New, Noteworthy” heading where we try to summarize such matters. However, here is a brief summary of “what’s new” in the book. The topics are listed in no particular order.

- An in-depth integrated discussion of cryptography, information theory and error-correction emphasizing their interconnections, including new, clear, accessible proofs of major results, along with new results.
• A discussion of RSA that clears up several issues and shows how, for example, a given encryption index may have several decryption indices: Also, an indication of a possible new attack on RSA.

• A study of potential applications of information theory in cellular biology.

• An overview of important practical considerations in modern cryptography and communication theory.

• A whole new treatment of “perfect secrecy”, including a refutation of the standard assertion concerning the equivalence of perfect secrecy and the one-time pad, together with a proof of the equivalence of perfect secrecy and latin squares.

• A highly accessible summary of information theory and its applications for non-specialists.

• A detailed look at hash functions from the point of view of linear codes.

• A detailed discussion of shift registers in cryptography, information theory and error correction including several new results and their application to the Berlekamp–Massey theory of Reed–Solomon and BCH decoding.

• A clarification of several points of confusion in the literature relating to security.

• A presentation of five different approaches to Shannon’s noisy channel theorem.

• A detailed discussion of the sampling theorem and Shannon’s fundamental band-limited capacity formula to the effect that $C = B \log(1 + \frac{S}{N})$, using precise statistical and geometrical techniques.

• A look at some of the history of cryptography and coding theory including a brief biography of Claude Shannon and an account of the original discovery of the Golay code in a Finnish soccer-pools magazine.

• A description of invariant theory and combinatorics applied to coding theory with particular reference to “the computer algebra theorem of the twentieth century” i.e. the nonexistence of a plane of order 10 and related work of one of the authors.

• Connections between MDS codes, secret-sharing schemes, Bruck Nets and Euler’s “famous problem of the 36 officers”.

• A brief description of research work due to the author and two co-authors solving, in the main, the fifty year old problem of finding the longest MDS code.

• A streamlined approach to Reed–Solomon codes via MDS codes.
• A highly accessible account of the decoding of Reed–Solomon codes.

• A major breakthrough in symmetric (and quantum) cryptography using some new research due to the authors and David Wehlau: the work has been patented and is being used in industry.

Missing Topics. We seem to have covered all the essential topics. We meant to discuss convolutional codes but ran out of space. But they can be covered from the shift register point of view and feed back shift registers have been covered in considerable detail. We also wanted to put in some computer code. We plan on putting some on the website if there is a demand for it.

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Book Website, Corrections. We will maintain a website for the book at


We have done our best to correct the errors but, inevitably, some will remain. We invite our reads to submit errors to mario@SUREngineering.com We will post them, with attribution, on the website.

About the Authors. Aiden A. Bruen was born in Galway, Ireland. He read mathematics for his Undergraduate and Master’s degree in Dublin. He received his doctorate at the University of Toronto, supervised by F. A. Sherk. At Toronto he also worked with H. S. M. Coxeter, E. Ellers and A. Lehman. He is a Professor of Mathematics and Statistics at the University of Calgary.

Mario A. Forcinito was born in Buenos Aires, Argentina where he took his Bachelors degree in Engineering. He obtained his doctorate in Engineering at the University of Calgary under the supervision of M. Epstein. He is an industrial engineering consultant in Calgary.

Update. New results are constantly being obtained. As this book was going to press, it was reported in the Toronto Globe and Mail that “encryption circles are buzzing with news that mathematical functions embedded in common security applications have previously unknown weaknesses”.

In particular, the report cites security vulnerabilities, discovered by E. Biham and R. Chen of the Israel Institute of Technology, in the SHA-1 hash function algorithm. SHA-1 is used in popular programs such as PGP and SSL and is considered the “gold-standard” of such algorithms. Certified by NIST (the National Institute of Standards and Technology) it is the only algorithm approved as a Digital Signature Standard by the US government.

In Chapter 24 we discuss the construction of hash functions from error-correcting codes.
Dedications

Dedicated to my beloved wife Katri and to the memory of my late parents, Edward A Bruen and Bríd Bean de Brún (AAB)

Also dedicated to my beloved wife Claudia and to my parents, Alberto Forcinito and Olga Swystun de Forcinito (MAF)
Part I

Mainly Cryptography
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Chapter 1

Historical Introduction and the Life and Work of Claude E. Shannon

Goals, Discussion  We present here an overview of historical aspects of classical cryptography. Our objective is to give the reader a panoramic view of how the fundamental ideas and important developments fit together. This overview does not pretend to be exhaustive but gives a rough time line of development of the milestones leading to modern cryptographic techniques. The reader interested in a complete historical review is advised to consult the definitive treatise by Kahn [Kah67].

Following this we discuss the life and work of Claude Shannon, the founding father of modern cryptography, information theory and error correction.

1.1 Historical Background

Cryptology is made up of two Greek words: kryptos, meaning "hidden," and ology, meaning "science." It is defined in [Bri97] as the science concerned with communications in secure and usually secret form. It encompasses both cryptography (from the Greek graphia meaning "writing") and cryptanalysis, or the art of extracting the meaning of a cryptogram.

Cryptography has a history that is almost as long as the history of the written word. Some four millennia ago (see [Kah67] p. 71) an Egyptian scribe recorded in stone the first known hieroglyphic symbol substitution in the tomb of Khnumhotep II, a nobleman of the time. Although the intention in this case was to exalt the virtues of the person, rather than to send a secret message, the scribe used for the first time one of the fundamental
elements used by cryptographers throughout the ages, namely, substitution. He used unusual hieroglyphic symbols, known perhaps only to the elite, in place of the more common ones.

In substitution, the sender replaces each letter of a word in a message by a new letter (or sequence of letters or symbols) before sending the message. The recipient, knowing the formula used for the substitution—the secret key—is able to reconstruct the message from the scrambled text that is received. It is assumed that only the recipient and the sender know the secret key.

The other main cryptographic technique used is transposition (or permutation), in which the letters of the message are simply rearranged according to some prescribed formula that would be the secret key in this case.

The Greeks were the inventors of the first transposition cipher. The Spartans [Kah67], in the fifth century B.C. were the first recorded users of cryptography for correspondence. They used a secret device called a scytale consisting of a tapered baton around which was spirally wrapped a strip of either parchment or leather on which the message was written. When unwrapped, the letters were scrambled, and only when the strip was wrapped around an identically sized rod could the message be read.

Today, even with the advent of high-speed computers, the principles of substitution and transposition form the fundamental building blocks of ciphers used in symmetric cryptography.

To put it in a historical perspective, asymmetric or public key cryptography was not invented until the 1970s. Exactly when it was invented, or who should take most of the credit, is an issue still in dispute. Both the NSA\(^1\) and the CESG\(^2\) have claimed priority in the invention of public key cryptography.

Cryptography has had several reincarnations in almost all cultures. Because of the necessity of keeping certain messages secret (i.e. totally unknown to potential enemies) governments, armies, ecclesiastics, and economic powers of all kinds have been associated throughout history with the development of cryptography. This trend continues today.

The Roman general Julius Caesar was the first attested user of substitution ciphers for military purposes ([Kah67] p. 83). Caesar himself recounted this incident in his Gallic Wars. Caesar found out that Cicero’s station was besieged and realized that without help he would not be able to hold out for long. Caesar had a volunteer ride ahead with an encrypted message fastened to a spear, which he hurled into the entrenchment. Basically, Cicero was told to keep up his courage and that Caesar and his legions were on their way.

In the cipher form used by Caesar, the first letter of the alphabet “A” was replaced by the fourth letter “D”, the second letter “B” by the fifth, “E”, and so on. In other words, each original letter was replaced by a letter three steps further along in the alphabet. To

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\(^1\)United States National Security Agency  
\(^2\)Britain’s Communications Electronics Security Group