INSTITUTIONAL FINANCE

Pricing Derivatives by Simulation

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Simulation and Optimization in Finance
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CHAPTER 14

Pricing Derivatives by Simulation

In Chapter 13, we introduced the main ideas behind the pricing of standard financial derivative instruments, or simply derivatives. We saw that the main assumption underlying derivative pricing schemes is the assumption that there is no arbitrage in the markets. When there is no arbitrage, the price of a derivative can be found as the expected value of its discounted payouts when the expected value is taken with respect to a transformation of the original probability distribution of outcomes, called the risk-neutral probability measure.

The same principles that guide the computation of the fair price of standard derivatives extend to the pricing of more complex derivatives. However, the difference is that nice closed-form formulas of the Black-Scholes type cannot necessarily be found for complex derivatives. Such derivatives must be priced with different numerical techniques, and simulation is one such tool.

We begin this chapter by showing how simulation can be used to price some of the simple derivatives we discussed in Chapter 13, such as European call options. Although simulation does not need to be applied in this context, techniques that make the simulation procedures more efficient can be demonstrated in a familiar setting, and benchmarked against a known final price. These examples help us illustrate more advanced simulation techniques, called variance reduction methods, whose goal is to make the simulation process as efficient as possible, and minimize the variance of the estimate. We review several such methods, including antithetic variables, stratified sampling, importance sampling, and control variates. We also review quasirandom (also called quasi–Monte Carlo) methods for simulation that use low discrepancy number sequences to obtain a good representation for the probability distribution being simulated. We then give examples of pricing more complex derivatives, such as barrier options and American options, by simulation, and discuss evaluating the sensitivity of derivative to changes in underlying parameters by crude and pathwise simulation methods.

14.1 COMPUTING OPTION PRICES WITH CRUDE MONTE CARLO SIMULATION

As we mentioned at the beginning of this chapter, the main idea behind computing prices of options (and other financial securities) by simulation is to generate a set of payoffs, and discount them to the present to find the expected value of all discounted payoffs under a probability distribution called the risk-neutral probability measure. The expected value of payoffs is the "fair" price of the derivative. Typically, when pricing financial derivatives, the prices of the underlying securities are assumed to follow specific kinds of random walks.[1] The most straightforward way to price a derivative is to create paths of realizations of the random walks for the derivative's underlying, compute the payoff along each path, discount to the present, and find the appropriate weighted average of the payoffs as an estimator for the expected value of the payoff. This is referred to as using *crude Monte Carlo*. It is not always the most efficient way to find a derivative's price, but it is tangible and easy to implement.

In this section, we give a couple of examples of how crude Monte Carlo can be used for pricing options. Smart ways to simulate the prices of options that exploit knowledge about the simulation process or the underlying distributions are discussed in section 14.2.

14.1.1 Pricing a European Call Option by Simulation

As we explained in section 13.4.2, a widely used formula for European options is the Black-Scholes formula.[2] It provides a closed-form expression for computing the price of the option. In section 13.4.2, we also showed that the underlying assumption used in the derivation of the Black-Scholes formula is that the underlying asset price follows a geometric Brownian motion.[3] The evolution of the asset price can then be described by the equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

where W_t is standard Brownian motion and μ and σ are the drift and the volatility of the process, respectively. For technical reasons (absence of arbitrage), when pricing an option, the drift μ is replaced by the risk-free rate r in the Black-Scholes formula.

Under the assumption for the random process followed by the asset price, the value of the asset price S_T at time T given the asset price S_t at time t can be computed as

$$S_T = S_t\, e^{(r-\frac{1}{2}\sigma^2)\cdot(T-t)+\sigma\cdot\sqrt{(T-t)}\cdot\tilde{\varepsilon}}$$

where $\tilde{\varepsilon}$ is a standard normal random variable.[4]

Hence, the option price obtained from the Black-Scholes formula can be approximated by simulation if a large number of values for the normal random variable $\tilde{\varepsilon}_t$ are generated. By creating scenarios for the stock price S_T at time T, we can compute the discounted payoffs of the option, and find the expected payoff. Suppose we generate N scenarios for $\tilde{\varepsilon}$: $\varepsilon^{(1)}, \ldots, \varepsilon^{(N)}$. Then the price of a European call option with strike price K will be

$$C_t = e^{-r\cdot(T-t)} \cdot \sum_{n=1}^{N} \frac{1}{N} \cdot \max\left\{S_t\, e^{(r-\frac{1}{2}\sigma^2)\cdot(T-t)+\sigma\cdot\sqrt{(T-t)}\cdot\varepsilon^{(n)}} - K, 0\right\}$$

The expression above is the expected value of the option payoffs, that is, the weighted average of the option payoffs. The "weight," or the probability of each scenario, is assumed to be $1/N$ since the scenarios are picked at random, and the frequency of their occurrence already incorporates the probability distribution of $\tilde{\varepsilon}$. (See this chapter's Software Hints, as well as files **Ch14-PricingBySimulation.xlsx**, **Ch14-OptionPricingVBA.xlsm**, and **EuropeanCall.m**, for an actual implementation of the simulation.)

It appears unnecessarily complicated to price the option this way, and indeed, in practice simulation is rarely used for this kind of simple problem. There are more complex derivatives and more sophisticated models for asset price behavior; in such cases, it can be simpler to generate scenarios and evaluate prices by simulation than to derive closed-form analytical formulas mathematically. For example, if the underlying asset follows a mean reversion process, the Black-Scholes formula will not work for a European call option, but simulation can help us evaluate the option price easily. In addition, in the case of portfolios and baskets of multiple assets, generating joint scenarios for multiple securities through simulation can help capture the otherwise complicated effect of interactions among different risk factors influencing the future value of the portfolio or derivatives.

Let us illustrate another advantage of simulating the price of a European call option rather than using the Black-Scholes formula. Recall that one of the assumptions in the Black-Scholes formula is that the interest rate r