Fibonacci and Catalan Numbers: An Introduction

Ralph P. Grimaldi
FIBONACCI AND CATALAN NUMBERS
FIBONACCI AND CATALAN NUMBERS
AN INTRODUCTION

Ralph P. Grimaldi
Rose-Hulman Institute of Technology
Dedicated to the Memory of

*Josephine and Joseph*

*and*

*Mildred and John*

*and*

*Madge*
CONTENTS

PREFACE xi

PART ONE THE FIBONACCI NUMBERS

1. Historical Background 3
2. The Problem of the Rabbits 5
3. The Recursive Definition 7
4. Properties of the Fibonacci Numbers 8
5. Some Introductory Examples 13
6. Compositions and Palindromes 23
7. Tilings: Divisibility Properties of the Fibonacci Numbers 33
8. Chess Pieces on Chessboards 40
9. Optics, Botany, and the Fibonacci Numbers 46
10. Solving Linear Recurrence Relations: The Binet Form for $F_n$ 51
11. More on $\alpha$ and $\beta$: Applications in Trigonometry, Physics, Continued Fractions, Probability, the Associative Law, and Computer Science 65
12. Examples from Graph Theory: An Introduction to the Lucas Numbers 79
13. The Lucas Numbers: Further Properties and Examples 100
14. Matrices, The Inverse Tangent Function, and an Infinite Sum 113
15. The $gcd$ Property for the Fibonacci Numbers 121
16. Alternate Fibonacci Numbers 126
17. One Final Example? 140

PART TWO  THE CATALAN NUMBERS

18. Historical Background 147
19. A First Example: A Formula for the Catalan Numbers 150
20. Some Further Initial Examples 159
21. Dyck Paths, Peaks, and Valleys 169
22. Young Tableaux, Compositions, and Vertices and Arcs 183
23. Triangulating the Interior of a Convex Polygon 192
24. Some Examples from Graph Theory 195
25. Partial Orders, Total Orders, and Topological Sorting 205
26. Sequences and a Generating Tree 211
27. Maximal Cliques, a Computer Science Example, and the Tennis Ball Problem 219
28. The Catalan Numbers at Sporting Events 226
29. A Recurrence Relation for the Catalan Numbers 231
30. Triangulating the Interior of a Convex Polygon for the Second Time 236
31. Rooted Ordered Binary Trees, Pattern Avoidance, and Data Structures 238
32. Staircases, Arrangements of Coins, The Handshaking Problem, and Noncrossing Partitions 250
33. The Narayana Numbers 268
34. Related Number Sequences: The Motzkin Numbers, The Fine Numbers, and The Schröder Numbers 282
35. Generalized Catalan Numbers 290

36. One Final Example? 296

Solutions for the Odd-Numbered Exercises 301

Index 355
In January of 1992, I presented a minicourse at the joint national mathematics meetings held that year in Baltimore, Maryland. The minicourse had been approved by a committee of the Mathematical Association of America—the mission of that committee being the evaluation of proposed minicourses. In this case, the minicourse was especially promoted by Professor Fred Hoffman of Florida Atlantic University. Presented in two two-hour sessions, the first session of the minicourse touched upon examples, properties, and applications of the sequence of Fibonacci numbers. The second part investigated comparable ideas for the sequence of Catalan numbers. The audience was comprised primarily of college and university mathematics professors, along with a substantial number of graduate students and undergraduate students, as well as some mathematics teachers from high schools in the Baltimore and Washington, D.C. areas.

Since its first presentation, the coverage in this minicourse has expanded over the past 19 years, as I delivered the material nine additional times at later joint national mathematics meetings—the latest being the meetings held in January of 2010 in San Francisco. In addition, the topics have also been presented completely, or in part, at more than a dozen state sectional meetings of the Mathematical Association of America and at several workshops, where, on occasion, some high school students were in attendance. Evaluations provided by those who attended the lectures directed me to further relevant material and also helped to improve the presentations.

At all times, the presentations were developed so that everyone in the audience would be able to understand at least some, if not a substantial amount, of the material. Consequently, this resulting book, which has grown out of these experiences, should be looked upon as an introduction to the many interesting properties, examples, and applications that arise in the study of two of the most fascinating sequences of numbers. As we progress through the various chapters, we should soon come to understand why these sequences are often referred to as ubiquitous, especially in courses in discrete mathematics and combinatorics, where they appear so very often. For the Fibonacci numbers, we shall find applications in such diverse areas as set theory, the compositions of integers, graph theory, matrix theory, trigonometry, botany, chemistry, physics, probability, and computational complexity. We shall find the Catalan numbers arise in situations dealing with lattice paths, graph theory, geometry, partial orders, sequences, pattern avoidance, partitions, computer science, and even sporting events.
FEATURES

Following are brief descriptions of four of the major features of this book.

1. **Useful Resources**
   The book can be used in a variety of ways:
   (i) As a textbook for an introductory course on the Fibonacci numbers and/or the Catalan numbers.
   (ii) As a supplement for a course in discrete mathematics or combinatorics.
   (iii) As a source for students seeking a topic for a research paper or some other type of project in a mathematical area they have not covered, or only briefly covered, in a formal mathematics course.
   (iv) As a source for independent study.

2. **Organization**
   The book is divided into 36 chapters. The first 17 chapters constitute Part One of the book and deal with the Fibonacci numbers. Chapters 18 through 36 comprise Part Two, which covers the material on the Catalan numbers. The two parts can be covered in either order. In Part Two, some references are made to material in Part One. These are usually only comparisons. Should the need arise, one can readily find the material from Part One that is mentioned in conjunction with something covered in Part Two.
   Furthermore, each of Parts One and Two ends with a bibliography. These references should prove useful for the reader interested in learning even more about either of these two rather amazing number sequences.

3. **Detailed Explanations**
   Since this book is to be regarded as an introduction, examples and, especially, proofs are presented with detailed explanations. Such examples and proofs are designed to be careful and thorough. Throughout the book, the presentation is focused primarily on improving understanding for the reader who is seeing most, if not all, of this material for the first time.
   In addition, every attempt has been made to provide any necessary background material, whenever needed.

4. **Exercises**
   There are over 300 exercises throughout the book. These exercises are primarily designed to review the basic ideas provided in a given chapter and to introduce additional properties and examples. In some cases, the exercises also extend what is covered in one or more of the chapters. Answers for all the odd-numbered exercises are provided at the back of the book.

ANCILLARY

There is an *Instructor’s Solution Manual* that is available for those instructors who adopt this book. The manual can be obtained from the publisher via written request.
on departmental letterhead. It contains the solutions for all the exercises within both parts of the book.

ACKNOWLEDGMENTS

If space permitted, I should like to thank each of the many participants at the mini-courses, sectional meetings, and workshops, who were so very encouraging over the years. I should also like to acknowledge their many helpful suggestions about the material and the way it was presented.

The work behind this book could not have been possible without the education I received because of the numerous sacrifices made by my parents Carmela and Ralph Grimaldi. Thanks are also due to Helen Calabrese for her constant encouragement. As an undergraduate at the State University of New York at Albany, I was so very fortunate to have professors like Robert C. Luippold, Paul T. Schaefer, and, especially, Violet H. Larney, who first introduced me to the fascinating world of abstract algebra. When I attended New Mexico State University as a graduate student, there I crossed paths, both inside and outside the classroom, with Professors David Arnold, Carol Walker, and Elbert Walker. They had a definite impact on my education. Even more thanks is due to Professor Edward Gaughan and, especially, my ever-patient and encouraging advisor, Professor Ray Mines. Also, it was on a recent sabbatical at New Mexico State University where I was able to put together so much of the material that now makes up this book. Hence, I must thank their mathematics department for providing me with an office, with such a beautiful view, and the resources necessary for researching so much of what is written here.

One cannot attempt to write a book such as this without help and guidance. Consequently, I want to thank John Wiley & Sons, Inc. for publishing this book. On a more individual level, I want to thank Shannon Corliss, Stephen Quigley, and Laurie Rosatone for their initial interest in the book. In particular, I must acknowledge the assistance and guidance provided throughout the development of the project by my editors Susanne Steitz-Filler and Jacqueline Palmieri. Special thanks are due to Dean Gonzalez for all his efforts in developing the figures. Finally, I must gratefully applaud the constant help and encouragement provided by Senior Production Editor Kristen Parrish who managed to get this author over so many hurdles that seemed to pop up.

I also want to acknowledge the helpful comments provided by Charles Anderson and the reviewers Professor Gary Stevens of Hartwick College and Professor Barry Balof of Whitman College. My past and present colleagues in the Mathematics Department at Rose-Hulman Institute of Technology have been very supportive during the duration of the project. In particular, I thank Diane Evans, Al Holder, Leanne Holder, Tanya Jajcay, John J. Kinney, Thomas Langley, Jeffery J. Leader, David Rader, and John Rickert. I must also thank Dean Art Western for approving the sabbatical which provided the time for me to start writing this book.

I thank Larry Alldredge for his help in dealing with the computer science material, Professor Rebecca DeVasher of the Rose-Hulman Institute of Technology for
her guidance on the applications in chemistry, and Professor Jerome Wagner of the Rose-Hulman Institute of Technology for his enlightening remarks on the physics applications.

A book of this nature requires the use of many references. The members of the library staff of the Rose-Hulman Institute of Technology were always so helpful when books and articles were needed. Consequently, it is only fitting to recognize the behind-the-scenes efforts of Jan Jerrell and, especially, Amy Harshbarger.

The last note of thanks belongs to Mrs. Mary Lou McCullough, the now-retired one-time secretary of the Rose-Hulman mathematics department. Although not directly involved in this project, her friendship and encouragement in working with me on several editions of another book and numerous research articles had a tremendous effect on my work in writing this new book. I shall remain ever grateful for everything she has done for me.

Unfortunately, any remaining errors, ambiguities, or misleading comments are the sole responsibility of the author.

Ralph P. Grimaldi
Terre Haute, Indiana
PART ONE

THE FIBONACCI NUMBERS
CHAPTER 1

Historical Background

Born around 1170 into the Bonacci family of Pisa, Leonardo of Pisa was the son of the prosperous merchant Guglielmo, who sought to have his son follow in his footsteps. Therefore, when Guglielmo was appointed the customs collector for the Algerian city of Bugia (now Bejaia), around 1190, he brought Leonardo with him. It was here that the young man studied with a Muslim schoolmaster who introduced him to the Hindu-Arabic system of enumeration along with Hindu-Arabic methods of computation. Then, as he continued his life in the mercantile business, Leonardo found himself traveling to Constantinople, Egypt, France, Greece, Rome, and Syria, where he continued to investigate the various arithmetic systems then being used. Consequently, upon returning home to Pisa around 1200, Leonardo found himself an advocate of the elegant simplicity and practical advantage of the Hindu-Arabic number system—especially, when compared with the Roman numeral system then being used in Italy. As a result, by the time of his death in about 1240, Italian merchants started to recognize the value of the Hindu-Arabic number system and gradually began to use it for business transactions. By the end of the sixteenth century, most of Europe had adjusted to the system.

In 1202, Leonardo published his pioneering masterpiece, the Liber Abaci (The Book of Calculation or The Book of the Abacus). Therein he introduced the Hindu-Arabic number system and arithmetic algorithms to the continent of Europe. Leonardo started his work with the introduction of the Hindu-Arabic numerals: the nine Hindu figures 1, 2, 3, 4, 5, 6, 7, 8, 9, along with the figure 0, which the Arabs called “zephirum” (cipher). Then he addressed the issue of a place value system for the integers. As the text progresses, various types of problems are addressed, including one type on determinate and indeterminate linear systems of equations in more than two unknowns, and another on perfect numbers (that is, a positive integer whose value equals the sum of the values of all of its divisors less than itself—for example, \(6 = 1 + 2 + 3\) and \(28 = 1 + 2 + 4 + 7 + 14\)). Inconspicuously tucked away between these two types of problems lies the one problem that so many students and teachers of mathematics seem to know about—the notorious “Problem of the Rabbits.”

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Before continuing at this point, let us mention that although Leonardo is best known for the Liber Abaci, he also published three other prominent works. The Practica Geometriae (Practice of Geometry) was written in 1220. The Flos (Flower or Blossom) was published in 1225, as was the Liber Quadratorum (The Book of Square Numbers). The latter work established Leonardo as a renowned number theorist.
In the now famous “Problem of the Rabbits,” Leonardo introduces us to a person who has a pair of newborn rabbits—one of each gender. We are interested in determining the number of pairs of rabbits that can be bred from (and include) this initial pair in a year if

1. each newborn pair, a female and a male, matures in one month and then starts to breed;
2. two months after their birth, and every month thereafter, a now mature pair breed at the beginning of each month. This breeding then results in the birth of one (newborn) pair, a female and a male, at the end of that month; and,
3. no rabbits die during the course of the year.

If we start to examine this situation on the first day of a calendar year, we find the results in Table 2.1 on p. 6.

We need to remember that at the end of each month, a newborn pair (born at the end of the month) grows to maturity, regardless of the number of days—be it 28, 30, or 31—in the next month. This makes the new maturity entry equal to the sum of the prior maturity entry plus the prior newborn entry. Also, since each mature pair produces a newborn pair at the end of that month, the newborn entry for any given month equals the maturity entry for the prior month. From the third column in Table 2.1, we see that at the end of the year, the person who started with this one pair of newborn rabbits now has a total of 233 pairs of rabbits, including the initial pair.

This sequence of numbers—namely, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... —is often called the Fibonacci sequence. The name Fibonacci is a contraction of Filius Bonaccii, the Latin form for “son of Bonaccio,” and the name was given to the sequence in May of 1876 by the renowned French number theorist François Édouard Anatole Lucas (pronounced Lucah) (1842–1891). In reality, Leonardo was not the first to describe the sequence, but he did publish it in the Liber Abaci, which introduced it to the West.
The Fibonacci sequence has proved to be one of the most intriguing and ubiquitous number sequences in all of mathematics. Unfortunately, when these numbers arise, far too many students, and even teachers of mathematics, are only aware of the connection between these numbers and the "Problem of the Rabbits." However, as the reader will soon learn, these numbers possess a great number of fascinating properties and arise in so many different areas.

### Table 2.1

<table>
<thead>
<tr>
<th></th>
<th>Number of Pairs of Newborn Rabbits</th>
<th>Number of Pairs of Mature Rabbits</th>
<th>Total Number of Pairs of Rabbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Start</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>January 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>One Month Later</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>February 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Two Months Later</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>March 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Three Months Later</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>April 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Four Months Later</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>May 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Five Months Later</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>June 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Six Months Later</td>
<td>5</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>July 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seven Months Later</td>
<td>8</td>
<td>13</td>
<td>21</td>
</tr>
<tr>
<td>August 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eight Months Later</td>
<td>13</td>
<td>21</td>
<td>34</td>
</tr>
<tr>
<td>September 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nine Months Later</td>
<td>21</td>
<td>34</td>
<td>55</td>
</tr>
<tr>
<td>October 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ten Months Later</td>
<td>34</td>
<td>55</td>
<td>89</td>
</tr>
<tr>
<td>November 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eleven Months Later</td>
<td>55</td>
<td>89</td>
<td>144</td>
</tr>
<tr>
<td>December 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>One Year Later</td>
<td>89</td>
<td>144</td>
<td>233</td>
</tr>
<tr>
<td>January 1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Upon examining the sequence in the middle column of Table 2.1, we see that after the first two entries, each entry is the sum of the two preceding entries. For example, 
\[ 1 = 1 + 0, \quad 2 = 1 + 1, \quad 3 = 2 + 1, \quad 5 = 3 + 2, \quad 8 = 5 + 3, \quad \ldots, \quad 55 = 34 + 21. \]
So we are able to determine later numbers in the sequence when we know the values of earlier numbers in the sequence. This property now allows us to define what we shall henceforth consider to be the Fibonacci numbers. Consequently, the sequence of Fibonacci numbers is defined, recursively, as follows:

For \( n \geq 0 \), if we let \( F_n \) denote the \( n \)th Fibonacci number, we have

1. \( F_0 = 0, \quad F_1 = 1 \) (The Initial Conditions)
2. \( F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \) (The Recurrence Relation)

Therefore, the sequence \( F_0, F_1, F_2, F_3, \ldots \), which appears in the middle column of Table 2.1, now has a different starting point, namely, \( F_0 \), from the sequence \( F_1, F_2, F_3, \ldots \), which appears in the third column of Table 2.1. This sequence—\( F_0, F_1, F_2, F_3, \ldots \)—is now accepted as the standard definition for the sequence of Fibonacci numbers. It is one of the earliest examples of a recursive sequence in mathematics. Many feel that Fibonacci was undoubtedly aware of the recursive nature of these numbers. However, it was not until 1634, when mathematical notation had sufficiently progressed, that the Dutch mathematician Albert Girard (1595–1632) wrote the formula in his posthumously published work L’Arithmetique de Simon Stevin de Bruges.

Using the recursive definition above, we find the first 25 Fibonacci numbers in Table 3.1.

<table>
<thead>
<tr>
<th>( F_0 )</th>
<th>( F_5 )</th>
<th>( F_{10} )</th>
<th>( F_{15} )</th>
<th>( F_{20} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>55</td>
<td>610</td>
<td>6765</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>89</td>
<td>987</td>
<td>10,946</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>144</td>
<td>1597</td>
<td>17,711</td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>233</td>
<td>2584</td>
<td>28,657</td>
</tr>
<tr>
<td>4</td>
<td>34</td>
<td>377</td>
<td>4181</td>
<td>46,368</td>
</tr>
</tbody>
</table>

TABLE 3.1

---

Properties of the Fibonacci Numbers

As we examine the entries in Table 3.1, we find that the greatest common divisor of $F_5 = 5$ and $F_6 = 8$ is 1. This is due to the fact that the only positive integers that divide $F_5 = 5$ are 1 and 5, and the only positive integers that divide $F_6 = 8$ are 1, 2, 4, and 8. We shall denote this by writing $\gcd(F_5, F_6) = 1$. Likewise, $\gcd(F_9, F_{10}) = 1$, since 1, 2, 17, and 34 are the only positive integers that divide $F_9 = 34$, while the only positive integers that divide $F_{10} = 55$ are 1, 5, 11, and 55. Hopefully we see a pattern developing here, and this leads us to our first general property for the Fibonacci numbers.

**Property 4.1:** For $n \geq 0$, $\gcd(F_n, F_{n+1}) = 1$.

**Proof:** We note that $\gcd(F_0, F_1) = \gcd(0, 1) = 1$. Consequently, if the result is false, then there is a first case, say $n = r > 0$, where $\gcd(F_r, F_{r+1}) > 1$. However, $\gcd(F_{r-1}, F_r) = 1$. So there is a positive integer $d$ such that $d > 1$ and $d$ divides $F_r$ and $F_{r+1}$. But we know that

$$F_{r+1} = F_r + F_{r-1}.$$ 

So if $d$ divides $F_r$ and $F_{r+1}$, it follows that $d$ divides $F_{r-1}$. This then contradicts $\gcd(F_{r-1}, F_r) = 1$. Consequently, $\gcd(F_n, F_{n+1}) = 1$ for $n \geq 0$. 

Using a similar argument and Property 4.1, the reader can establish our next result.

**Property 4.2:** For $n \geq 0$, $\gcd(F_n, F_{n+2}) = 1$.

To provide some motivation for the next property, we observe that

$$F_0 + F_1 + F_2 + F_3 + F_4 + F_5 = 0 + 1 + 1 + 2 + 3 + 5 = 12 = 4 \cdot 3$$
$$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 = 1 + 1 + 2 + 3 + 5 + 8 = 20 = 4 \cdot 5$$
$$F_2 + F_3 + F_4 + F_5 + F_6 + F_7 = 1 + 2 + 3 + 5 + 8 + 13 = 32 = 4 \cdot 8.$$
These results suggest the following:

**Property 4.3:** The sum of any six consecutive Fibonacci numbers is divisible by 4. Even further, for \( n \geq 0 \) (with \( n \) fixed),

\[
\sum_{r=0}^{5} F_{n+r} = F_n + F_{n+1} + F_{n+2} + F_{n+3} + F_{n+4} + F_{n+5} = 4F_{n+4}.
\]

**Proof:** For \( n \geq 0 \),

\[
\sum_{r=0}^{5} F_{n+r} = F_n + F_{n+1} + F_{n+2} + F_{n+3} + F_{n+4} + F_{n+5}
\]

\[
= (F_n + F_{n+1}) + F_{n+2} + F_{n+3} + F_{n+4} + (F_{n+3} + F_{n+4})
\]

\[
= 2F_{n+2} + 2F_{n+3} + 2F_{n+4} = 2(F_{n+2} + F_{n+3}) + 2F_{n+4}
\]

\[
= 4F_{n+4}.
\]

\( \square \)

In a similar manner, one can likewise verify the following:

**Property 4.4:** The sum of any ten consecutive Fibonacci numbers is divisible by 11. In fact, for \( n \geq 0 \) (with \( n \) fixed),

\[
\sum_{r=0}^{9} F_{n+r} = 11F_{n+6}.
\]

Our next property was discovered by Edouard Lucas in 1876. A few observations help suggest the general result:

\[
F_0 + F_1 + F_2 = 2 = 3 - 1 = F_4 - 1
\]

\[
F_0 + F_1 + F_2 + F_3 = 4 = 5 - 1 = F_5 - 1
\]

\[
F_0 + F_1 + F_2 + F_3 + F_4 = 7 = 8 - 1 = F_6 - 1.
\]

**Property 4.5:** For \( n \geq 0 \), \( \sum_{r=0}^{n} F_r = F_{n+2} - 1 \).

**Proof:** Although this summation formula can be established using the Principle of Mathematical Induction, here we choose to use the recursive definition of the Fibonacci numbers and consider the following:

\[
F_0 = F_2 - F_1
\]

\[
F_1 = F_3 - F_2
\]

\[
F_2 = F_4 - F_3
\]

\[
\vdots \quad \vdots \quad \vdots
\]

\[
F_{n-1} = F_{n+1} - F_n
\]

\[
F_n = F_{n+2} - F_{n+1}.
\]
When we add these \(n + 1\) equations, the left-hand side gives us \(\sum_{r=0}^{n} F_r\), while the right-hand side provides \((F_2 - F_1) + (F_3 - F_2) + \cdots + (F_{n+1} - F_n) + (F_{n+2} - F_{n+1}) = -F_1 + (F_2 - F_2) + (F_3 - F_3) + \cdots + (F_n - F_n) + (F_{n+1} - F_{n+1}) + F_{n+2} = F_{n+2} - F_1 = F_{n+2} - 1\). \(\square\)

Passing from first powers to squares, we find that

\[
\begin{align*}
F_0^2 = 0^2 &= 0 = 0 \times 1 \\
F_1^2 + F_0^2 &= 0^2 + 1^2 = 1 = 1 \times 1 \\
F_2^2 + F_1^2 + F_0^2 &= 0^2 + 1^2 + 1^2 = 2 = 2 \times 1 \\
F_3^2 + F_2^2 + F_1^2 + F_0^2 &= 0^2 + 1^2 + 1^2 + 2^2 = 6 = 2 \times 3 \\
F_4^2 + F_3^2 + F_2^2 + F_1^2 + F_0^2 &= 0^2 + 1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \times 5.
\end{align*}
\]

From what is suggested in these five results, we conjecture the following:

**Property 4.6:** For \(n \geq 0\), \(\sum_{r=0}^{n} F_r^2 = F_n \times F_{n+1}\).

**Proof:** Here we shall use the Principle of Mathematical Induction. For \(n = 0\), we have \(\sum_{r=0}^{0} F_r^2 = F_0^2 = 0^2 = 0 = 0 \times 1 = F_0 \times F_1 = F_0 \times F_{0+1}\). This demonstrates that the conjecture is true for this first case and provides the basis step for our inductive proof. So now we assume the conjecture true for some fixed (but arbitrary) \(n = k \geq 0\). This gives us \(\sum_{r=0}^{k} F_r^2 = F_k \times F_{k+1}\). Turning to the case where \(n = k + 1 \geq 1\), we have

\[
\sum_{r=0}^{k+1} F_r^2 = \left(\sum_{r=0}^{k} F_r^2\right) + F_{k+1}^2 = (F_k \times F_{k+1}) + F_{k+1}^2 = F_{k+1} \times (F_k + F_{k+1}) = F_{k+1} \times F_{k+2}.
\]

Consequently, the truth of the case for \(n = k + 1\) follows from the case for \(n = k\). So our conjecture is true for all \(n \geq 0\), by the Principle of Mathematical Induction. \(\square\)

At this point, let us mention three more properties exhibited by the Fibonacci numbers. There are so many! The reader should find a wealth of such results in References [38, 50]. The first two of these properties are also due to Edouard Lucas from 1876. The third was discovered in 1680 by the Italian-born French astronomer and mathematician Giovanni Domenico (Jean Dominique) Cassini (1625–1712). This result was also discovered independently in 1753 by the Scottish mathematician and landscape artist Robert Simson (1687–1768). We shall leave the proofs of all three results for the reader. However, we shall obtain the result due to Cassini in another way, when we introduce a \(2 \times 2\) matrix whose components are Fibonacci numbers in Chapter 14.

**Property 4.7** For \(n \geq 1\), \(\sum_{r=1}^{n} F_{2r-1} = F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}\).

**Property 4.8** For \(n \geq 1\), \(\sum_{r=1}^{n} F_{2r} = F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1\).
**Property 4.9:** For \( n \geq 1 \), \( F_{n-1}F_{n+1} - F_n^2 = (-1)^n \).

At this point, we realize that the Fibonacci numbers do possess some interesting properties. But surely there must be places where these numbers arise—other than the “Problem of the Rabbits.” In Chapter 5, we shall encounter some examples where these numbers arise, and start to learn why these numbers are often referred to as ubiquitous.

**EXERCISES FOR CHAPTER 4**

1. Prove Property 4.2—that is, for \( n \geq 0 \), \( \gcd(F_n, F_{n+2}) = 1 \).
2. Provide an example to show that \( \gcd(F_n, F_{n+3}) \neq 1 \) for some \( n \geq 0 \).
3. For \( n \geq 1 \), prove that \( F_{2(n+1)} = 2F_{2n} + F_{2n-1} \).
4. For \( n \geq 2 \), prove that \( F_{n+2} + F_{n-2} = 3F_n \).
5. For \( n \geq 2 \), prove that \( F_{n+2} + F_n + F_{n-2} = 4F_n \).
6. For \( n \geq 1 \), prove that \( F_{n+1}^3 = F_n^3 + F_{n-1}^3 + 3F_{n-1}F_nF_{n+1} \).
7. For \( n \geq 2 \), prove that \( F_{3n} = 4F_{3n-3} + F_{3n-6} \).
8. Prove that \( \sum_{r=0}^{9} F_{n+r} = 11F_{n+6} \).
9. Use the Principle of Mathematical Induction to prove that for \( n \geq 0 \), \( \sum_{r=0}^{n} F_r = F_{n+2} - 1 \).
10. Fix \( n \geq 0 \). Prove that \( \sum_{r=1}^{m} F_{n+r} = F_{m+n+2} - F_{n+2} \).
11. Prove Property 4.7—that is, for \( n \geq 1 \), \( \sum_{r=1}^{n} F_{2r-1} = F_{2n} \).
12. Prove Property 4.8—that is, for \( n \geq 1 \), \( \sum_{r=1}^{n} F_{2r} = F_{2n+1} - 1 \).
13. Verify the result due to Giovanni Cassini in Property 4.9 for \( n = 3, 4, 5, \) and 6.
14. Use the Principle of Mathematical Induction to prove Property 4.9—that is, for \( n \geq 1 \), \( F_{n+1}F_{n+2} = F_n^2 + (-1)^n \).
15. For \( n \geq 1 \), prove that \( F_nF_{n+1}F_{n+2} = F_{n+1}^3 + F_n^2 + (-1)^{n+1} \).
16. Jodi starts to write the Fibonacci numbers on the board in her office, using the recursive definition. She writes the correct values for the numbers \( F_0, F_1, F_2, \ldots, F_{n-1} \), but then Professor Brooks distracts her and she writes \( F_{n+1} \) instead of the actual value \( F_n \). (a) If she does not make any further mistakes in using the recursive definition for the Fibonacci numbers, what value does she write next? (b) What does she write instead of the actual value of \( F_{n+2} \)? (c) In general, for \( r > 0 \), what value does she write instead of the actual value of \( F_{n+r} \)?
17. Use the Principle of Mathematical Induction to prove that for \( n \geq 1 \),
\[
\sum_{r=1}^{n} rF_r = F_1 + 2F_2 + 3F_3 + \cdots + nF_n = nF_{n+2} - F_{n+3} + 2.
\]
(This formula is an example of a weighted sum involving the Fibonacci numbers.)
18. For \( n \geq 0 \), prove that \( F_{n}^2 + F_{n+3}^2 = 2(F_{n+1}^2 + F_{n+2}^2) \). (H. W. Gould, 1963) [24].

19. For \( n \geq 1 \), prove that \( \sum_{i=1}^{n}(-1)^{i+1}F_{i+1} = (-1)^{n-1}F_{n} \).

20. For \( n \geq 1 \), prove that \( F_{n+1}^2 - F_{n}^2 = F_{n-1}F_{n+2} \).

21. For \( n \geq 0 \), prove that \( F_{n+1}^2 + F_{n+4}^2 = F_{n+1}^2 + F_{n+3}^2 + 4F_{n+2}^2 \). (M. N. S. Swamy, 1966) [52].

22. For \( n \geq 1 \), prove that \( \sum_{i=1}^{2n}F_{i}F_{i+1} = F_{2n+1}^2 - 1 \). (K. S. Rao, 1953) [45].

23. For \( n \geq 1 \), prove that \( \sum_{i=1}^{n}F_{i}F_{i+1} = F_{2n+1}^2 - 1 \) \(2 \quad \] (T. Koshy, 1998) [37].

24. (a) For any real numbers \( a \) and \( b \), prove that \( \left[ a^2 + b^2 + (a + b)^2 \right]^2 = 2 \left[ a^4 + b^4 + (a + b)^4 \right] \).

   [This result is known as Candido’s identity, in honor of the Italian mathematician Giacomo Candido (1871–1941).]

   (b) For \( n \geq 0 \), prove that \( \left( F_{n}^2 + F_{n+1}^2 + F_{n+2}^2 \right)^2 = 2(F_{n}^4 + F_{n+1}^4 + F_{n+2}^4) \).

25. For \( n \geq 0 \), prove that \( F_{n+5} - 3F_{n} \) is divisible by 5. [Alternatively, this can be stated as \( F_{n+5} \equiv 3F_{n} \pmod{5} \).]

26. For \( n \geq m \geq 1 \), prove that \( \sum_{r=m}^{n}F_{r} = F_{n+2} - F_{m+1} \).

27. (a) Verify that \( (F_{3} + F_{4} + F_{5} + F_{6}) + F_{4} = F_{8} \).

   (b) For what value of \( n \) is it true that \( (F_{4} + F_{5} + F_{6} + F_{7} + F_{8}) + F_{5} = F_{n} \)?

   (c) Fix \( n \geq 1 \) and \( m \geq 1 \). What is \( (F_{n} + F_{n+1} + F_{n+2} + \cdots + F_{n+m}) + F_{n+1} \)? (This fascinating tidbit was originally recognized by W. H. Huff.)

28. For \( n \geq 3 \), prove that

\[
F_{n} + F_{n-1} + F_{n-2} + 2F_{n-3} + 2^2F_{n-4} + 2^3F_{n-5} + \cdots + 2^{n-4}F_{2} + 2^{n-3}F_{1} = 2^{n-1}.
\]
As the title indicates, this chapter will provide some examples where the Fibonacci numbers arise. In particular, one such example will show us how to write a Fibonacci number as a sum of binomial coefficients. In addition, even more examples will arise from some of the exercises for the chapter.

Example 5.1: [Irving Kaplansky (1917-2006)]: For \( n \geq 1 \), let \( S_n = \{1, 2, 3, \ldots, n\} \), and let \( S_0 = \emptyset \), the null, or empty, set. Then the number of subsets of \( S_n \) is \( 2^n \). But now let us count the number of subsets of \( S_n \) with no consecutive integers. So, for \( n \geq 0 \), we shall let \( a_n \) count the number of subsets of \( S_n \) that contain no consecutive integers. We consider the situation for \( n = 3, 4, \) and \( 5 \). In each case, we find the empty set \( \emptyset \); otherwise, there would have to exist two integers in \( \emptyset \) of the form \( x \) and \( x + 1 \). Either such integer contradicts the definition of the null set. So the subsets with no consecutive integers for these three cases are as follows:

\[
\begin{align*}
  n = 3: & \quad S_3 = \{1, 2, 3\} \\
  \text{Subsets:} & \quad \emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\} \\
  n = 4: & \quad S_4 = \{1, 2, 3, 4\} \\
  \text{Subsets:} & \quad \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\} \\
  n = 5: & \quad S_5 = \{1, 2, 3, 4, 5\} \\
  \text{Subsets:} & \quad \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{5\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{1, 3, 5\}
\end{align*}
\]

Note that when we consider the case for \( n = 5 \), only two situations can occur, and they cannot occur simultaneously:

(i) 5 is not in the subset: Here we can use any of the eight subsets for \( S_4 \)—as we see from the first line of subsets for \( S_5 \).

(ii) 5 is in the subset: Then we cannot have 4 in the subset. So we place the integer 5 in each of the five subsets for \( S_3 \) and arrive at the five subsets in the second line of subsets for \( S_5 \). Consequently, we have \( a_5 = a_4 + a_3 \).
The above argument generalizes to give us
\[ a_n = a_{n-1} + a_{n-2}, \quad n \geq 2, \quad a_0 = 1, \quad a_1 = 2. \]
The recurrence relation in this case is the same as the one for the Fibonacci numbers, but the initial conditions are different. Here we have \( a_0 = 1 = F_2 \) and \( a_1 = 2 = F_3 \). Therefore,
\[ a_n = F_{n+2}, \quad n \geq 0. \]

**Example 5.2:** As in Example 5.1, we shall let \( \text{Sn} = \{1, 2, 3, \ldots, n\} \) for \( n \geq 1 \). Then for any nonempty subset \( A \) of \( \text{Sn} \), we define \( A + 1 = \{a + 1 \mid a \in A\} \). So if \( n = 4 \) and \( A = \{1, 2, 4\} \), then \( A + 1 = \{2, 3, 5\} \), and we see that \( A \cup (A + 1) = S_5 \). Consequently, for \( n \geq 1 \), we shall now let \( g_n \) count the number of subsets \( A \) of \( \text{Sn} \) such that \( A \cup (A + 1) = S_{n+1} \). Such subsets \( A \) of \( \text{Sn} \) are called generating sets for \( \text{S}_{n+1} \). We realize that for any such subset \( A \), it follows that \( 1 \in A \) and, for \( n \geq 2, n \in A \). For \( n = 3, \ 4, \ 5 \), we find the following examples of generating sets:
\[
\begin{align*}
  n = 3: & \quad \{1, 3\}, \{1, 2, 3\} \\
  n = 4: & \quad \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\} \\
  n = 5: & \quad \{1, 2, 3, 5\}, \{1, 2, 3, 4\} \\
        & \quad \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 4, 5\}.
\end{align*}
\]

Here we see that the \( g_5 \) generating sets for \( S_6 \) (where \( n = 5 \)) are obtained from those of \( S_5 \) (where \( n = 4 \)) and from those of \( S_4 \) (where \( n = 3 \)), by placing 5 in each of the \( g_4 \) generating subsets for \( S_5 \) and in each of the \( g_3 \) generating subsets for \( S_4 \). Consequently, \( g_5 = g_4 + g_3 \) and this particular case generalizes to
\[
g_n = g_{n-1} + g_{n-2}, \quad n \geq 3, \quad g_1 = 1 \text{ (for } \{1\}), \quad g_2 = 1 \text{ (for } \{1, 2\}).
\]
(Note that we could define \( g_0 = 0 \), by extending the given recurrence relation to \( n \geq 2 \) and solving the equation \( g_0 = g_2 - g_1 \) to obtain \( g_0 = 1 - 1 = 0 \).) Here we find that
\[
g_n = F_n, \quad n \geq 1.
\]
(M ore on generating sets can be found in Reference [26]. A generalization of this idea is found in Reference [54].)

**Example 5.3:** Next we examine binary strings made up of 0’s and 1’s. For \( n \geq 1 \), there are \( 2^n \) binary strings of length \( n \)—that is, the strings are made up of \( n \) symbols, each a 0 or a 1. We wish to count those strings of length \( n \), where there are no consecutive 1’s. So we shall let \( b_n \) count the number of such strings of length \( n \) and learn, for example, that (i) \( b_3 = 5 \), for the strings 000, 100, 010, 001, 101; and (ii) \( b_4 = 8 \), for the strings 0000, 1000, 0100, 0010, 0001, 1010, 1001, 0101. In