Frontiers of Modern Asset Allocation
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Frontiers of Modern Asset Allocation

PAUL D. KAPLAN

John Wiley & Sons, Inc.
To my children, Ruth, Rachel, and Benjamin.
And to all LGBT people everywhere who serve investors.
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Foreword

The breadth and depth of the articles in this book suggest that Paul Kaplan has been thinking about markets for about as long as markets have existed. That’s not true, of course; he’s only 50. And, in fact, having preceded him into this world and having been the first employee of Ibbotson Associates in 1979, I experienced the pleasure of having Paul work for me in the early years of that organization. We were a small firm, we were young, and we were usually broke—the ideal conditions for pushing the frontiers of knowledge to make new discoveries and bring them to market.

The discovery for which Ibbotson Associates (now a part of Morningstar, Inc.) first became known was the long-run return on equities and the tremendous superiority of that return to the returns of fixed-principal assets, at least in the United States and for the period over which the return was measured (1926 to the present). (The fixed-principal assets with which Ibbotson was concerned were corporate and government bonds, Treasury bills, and a hypothetical asset returning the rate of consumer price inflation: “Stocks, Bonds, Bills, and Inflation.”) In other words, Ibbotson Associates, drawing on work by Roger Ibbotson and Rex Sinquefield, estimated the equity risk premium—the extra return that investors had received for taking the risks involved in holding stocks instead of one of the riskless-asset proxies enumerated above. Note the tense of the language: “that investors had received.” Ibbotson Associates estimated the $ex post$ or realized equity risk premium.

The stability of the equity risk premium, as thus measured, was remarkable. Looking at Figure F.1, which covers the period from 1871 through 2010, the cumulative index line for stocks looks almost straight. The illusion of no risk for long-term equity investors is so powerful that only a Scrooge would point out that the downward wiggle between 1929 and 1932 represented an 89 percent loss of value. And no matter how far backward in time one extended the study, the line still appeared straight. Adding Harold Cowles’ data for 1871 to 1925, which are already incorporated in Figure F.1, to Ibbotson’s data for 1926 to the present only made the line straighter. Adding in G. William Schwert’s data for 1802 to 1870 (not shown) has the same effect. What does change as you vary the time window is the return on $bonds$, so the equity risk premium over bonds does change, a little.
FOREWORD

Hadn’t anyone before Ibbotson Associates, or Roger Ibbotson and Rex Sinquefield, estimated the equity risk premium? Of course the thought had occurred to many, but the preexisting methodology—to use a kind of Dividend Discount Model (DDM) for the aggregate of all stocks in the market—gave forecasts, or estimates of the \textit{ex ante} or expected risk premium, not backward looks at history. Hindsight showed that DDM-based forecasts had been much too low. A typical DDM estimate of the forward-looking, or expected, equity risk premium over bonds was in the range of 2 to 3 percent.\(^3\) In contrast, Ibbotson Associates showed that stocks had out-returned intermediate-term Treasury bonds by much more, 5.4 percent, using 1926 to 1979 as the measurement period (1979 being very early in the history of the firm).

Drawing on a variety of logical arguments, Ibbotson Associates supported the use of the historically achieved risk premium as the best forecast of the future, and subsequent events supported this method. By 1999, the historical (1926 to 1999) equity risk premium over bonds had swelled slightly, to 5.7 percent, as the bull market in equities outpaced the contemporaneous bull market in bonds. (These numbers are all compound annual returns, or geometric means. The Ibbotson forecasts used arithmetic means; in other words, they said that the future arithmetic mean risk premium would equal the past arithmetic mean. But this is a minor technical point.)
Foreword

The Ibbotson forecasts became the linchpin of financial planning, asset allocation for institutions, cost-of-capital estimation, and a host of other practices. The equity risk premium is arguably the most important single variable in finance, because it helps you plan ahead and decide how much to save and invest, and Ibbotson found what appeared to be a reliable estimator of it.

Why were the original DDM forecasts too low? One reason is that they assumed that future repricing—change in the valuation (say, price/earnings ratio) of the market—would be zero. But in an environment of falling interest rates, repricing is not zero. Stocks were very cheap in 1979, partly because interest rates were historically high. Equities have a longer duration than intermediate Treasury bonds, so falling interest rates helped stocks more than they helped bonds. (Whether falling interest rates were part of investors’ expectations in, say, 1979 is a question that is unlikely to be resolved, but it is plausible to suppose that they were.) Additional light is shed on the low prices of equities in the late 1970s in a classic article by Modigliani and Cohn (1979). Another factor is that corporate profit growth surprised on the upside in the 1980s and 1990s, as takeovers, the threat of takeovers, and other factors caused companies to focus more carefully on maximizing shareholder value.

But these trends clearly couldn’t continue forever. As the economist Herbert Stein said, any trend that cannot continue forever won’t. The bear markets of 2000 to 2002 and 2007 to 2009 shaved away a good portion of the historical realized risk premium, even when the measured period begins in 1926. The fact that bonds rallied strongly during most of the bear-market period further eroded the historical equity risk premium.

The events of the 2000s have had a tremendous impact on the prevailing thinking (including Ibbotson’s) about the equity risk premium. Most notably, the original Ibbotson method has been shown to be procyclical, when what is needed is a forecast method that is either countercyclical or not cyclical at all.

A procyclical forecast is one that extrapolates past trends forward. An example of a procyclical forecast is as follows: The more baseball games that the Chicago Cubs win, the more games that the Cubs are expected to win. Maybe the Cubs have gotten better at playing ball or maybe some other factor is at play. At any rate, such a forecast is not ridiculous. Some teams improve over a given time period while others deteriorate, and an increased percentage of wins is evidence that the team has in fact improved its skills and will continue its winning ways. (Realists, of course, understand that the Cubs will never really be any good.)

If the stock market has a winning streak, however, it is just becoming more expensive, suggesting that further returns will be lower, not higher!
FOREWORD

Although fundamentals, such as corporate profits, can change, I am speaking of stock price changes not explained by changes in fundamentals. Bondholders know this better than equity holders. If the price of a 5 percent Treasury bond rises because yields have fallen to 4 percent, do bond investors expect the past high returns to continue, or do they expect to now earn 4 percent? Even fairly naive bond investors expect the latter.

As the stock market soared in the late 1990s causing the historical average equity risk premium to increase, rate-of-return forecasts using the Ibbotson method also became larger (because the forecasts embodied the future-equals-past assumption). Thus, forecasts that were reasonable in 1974 or 1979, and that were vindicated by later results, seemed extravagantly optimistic at the price levels that prevailed in 1999. Because the market’s price/earnings ratio (P/E) had risen from less than 10 in 1979 to well over 20 in 1999, the future-equals-past method implied a further doubling of the P/E every 20 years into the indefinite future. Such a forecast is obviously not reasonable.

So what, at bottom, was wrong with the future-equals-past method? It not only assumed that the future would resemble the past but that the market is fairly priced. In a certain circle, the idea that the overall market might be mispriced was too politically incorrect in the 1970s and 1980s to be seriously considered and to make its way into forecasts. But by the late 1990s, the strong form of the efficient market hypothesis was no longer in vogue; the market could be over- or underpriced. And if the market is substantially mispriced, you have to use a different forecasting method, one that includes the current price as an input. The DDM fits this criterion, and the past 15 years or so have seen a return to the DDM for forecasting the equity risk premium. We will return to the modern use of the DDM shortly.

To Morningstar’s (and Ibbotson’s) credit, the firm now uses multiple methods, including the future-equals-past method, a version of the DDM, and a method (Ibbotson and Chen 2003) that combines aspects of both.

ARE STOCKS RISKY IN THE LONG RUN?

Let’s examine my earlier observation a little more closely. Figure F.1 gives a powerful illusion of no risk in the stock market.

Now that we’ve seen where the beginning and end points in Figure F.1 are, we can draw a straight line through them (or a best-fit regression line through the full data set) and see that whenever there is a deviation from the straight path, the market eventually snaps back to it and crosses it. Thus, there’s no risk to the truly long-term investor. Returns are self-evidently
mean reverting; if you wait long enough, you'll earn the long-run average return!

Well, maybe not. A little logic shows that returns must be in some sense mean reverting. If extraordinarily good returns cause stocks to become overpriced, they are more likely to be followed by poor ones, and vice versa. But wait a minute. Although there's risk in the deviations around the line, as 1929 to 1932 and more recent episodes demonstrate, the biggest risk comes from the fact that we didn't know in advance what the slope of the line would be. In other words, you don't know what the mean you're reverting to is. And you never will.

Ibbotson Associates, or Roger Ibbotson and Rex Sinquefield, never themselves said that stocks were riskless or almost riskless in the long run. To the contrary, their method emphasized the risk of stocks by drawing wide confidence bands around the forecast means. That stocks are riskless (or have low risk) if you wait long enough is a misunderstanding of Ibbotson and Sinquefield, promoted by others. Those who adhere to that misunderstanding sometimes use Ibbotson and Sinquefield's data to support their cause, but they shouldn't.

SURVIVAL BIAS: DID YOU KNOW IN ADVANCE THAT THE UNITED STATES AND UNITED KINGDOM WOULD SUCCEED?

A number of investigators, including Roger Ibbotson himself in his collaboration with Gary Brinson, pointed out that historically based forecasts of long-run rates of return may be biased because one is observing only markets that were lucky enough to have survived. This principle is best illustrated relative to a hypothetical portfolio of country index funds, purchased at the beginning of the last century, when there were no developed markets, and Europe, North and South America, and other parts of the world were bursting with emerging markets. (See Figure F.2.) As it turned out, an investor who held funds in the United States, the United Kingdom, and a few other small countries would have enjoyed uninterrupted equity markets up to the present, but investors who bought in Germany, Japan, Russia, Austria-Hungary, China, and so forth would not have. All of these countries now have stock markets, but, at some point, the investor would have lost everything and, in order to remain an investor, would have had to inject new capital earned in the labor market. I developed this theme and William Goetzmann of Yale has applied it to everything under the sun.

This observation implies that the return achieved after the fact by investors in the U.S. or U.K. equity index is an overestimate of the return they
expected before the fact. In other words, the U.S. or U.K. result is one of the better outcomes an investor in 1900 might have hoped for. Most investors fared much worse. That such a wide range of outcomes was not only possible but likely must have been known by investors in 1900, a date when the U.S. Civil War and various upheavals in Europe were within living memory. Thus, the expected return on equities is almost surely lower than the historical number. Fama and French (2000) appear to have confirmed this conjecture by using the Dividend Discount Model to estimate the returns that U.S. investors expected or required at each point in historical time; they come up with an equity risk premium of about 3 percent.

**SAMPLE PERIOD BIAS: WERE THE PAST 200 YEARS REALLY TYPICAL?**

Another likely source of upward bias in using historical returns as forecasts comes from the time period that was studied. Although Roger Ibbotson and Rex Sinquefield originally studied only the time period from 1926 to 1974, they and others extended the period both backward (to 1802, in the case of Schwert) and forward (to the present), and found that the extended results confirmed the 1926–1974 finding of a high equity risk premium. It was comforting to those expecting a high equity premium that, no matter what period you looked at, you got roughly the same result. But is a 210-year look at history really long enough? Angus Maddison, who estimated global GDP data from the year one (that’s right, the beginning of the Christian era) through the present, would say no. His celebrated work shows that human economic progress was painfully slow—with annual per capita real GDP growth rates of 0.0 percent to 0.1 percent—until about 1820, when the rate zoomed, approaching 2 percent on a global basis (Maddison 2007).
Note that the period during which humanity made almost all of its economic progress—1820 to the present—is almost identical with the period from which the very long estimate of the historical equity premium is taken. No wonder the number is high.

Now, we could get into interesting debates about whether Shakespeare’s London or Mozart’s Vienna had really experienced no economic progress relative to London and Vienna a millennium earlier. Although Professor Maddison (1926–2010) is no longer around to answer the question, I think he would tell us that these civilizations really had experienced growth, but that they are outliers and that his estimates of almost-zero growth in almost all countries for almost all of human history are correct.

Hans Rosling, a Swedish statistician who has become popular for his lively computer animations of economic history, would probably agree. Figure F.3 shows Rosling’s mapping of countries in 1800 according to their real GDP per capita and their life expectancy. All of the countries had real

**FIGURE F.3** The World in 1800

*Source: gapminder.org. Used by permission.*
Life Expectancy at Birth

Income per Person (GDP/capita, inflation-adjusted 2005 US$)

GDPs per capita of less than $3,000 (in today’s dollars)—roughly that of India now. The United Kingdom had the highest GDP, followed by the Netherlands and then the fledgling United States. All of the countries had life expectancies of 40 years or less. (The bubble representing each country is drawn with its area proportional to that country’s population.)

Figure F.4 is the same map, but showing data as of 2009. The sickest country today (Zimbabwe) has a longer life expectancy than the healthiest country in 1800. Regrettably, the poorest countries today are still poorer than the richest country in 1800, but most countries, including many considered developing or in the emerging markets, are much richer than the richest were in 1800.

This progress did not take place evenly. The Western countries, Japan, and a few others first pulled ahead in what I would like to call the Great Decompression (roughly 1820 to 1945), wherein the rich and healthy left the poor and sick far behind. Then, starting about 1945, a Great
Recompression began, with the poor and sick moving dramatically toward the rich and healthy upper-right portion of the graph. Sub-Saharan Africa represents most of the laggards, but Africa will develop, too; we are seeing particularly fast movement in that direction right now. The world is becoming, on average, moderately rich. Amazingly, the world average per capita GDP today, about $10,500, is equal to the U.S. per capita GDP around 1940, when the United States (despite the then-recent Great Depression) was the richest country in the world and a First World society by any reasonable standard.

For those who become depressed reading the news, these observations may seem like no more than an enjoyable digression. But they bear directly on the issue we’ve been discussing all along: This two-century spasm of material and bodily improvement is exactly the period over which we’ve measured the equity risk premium. Would that it could happen over and over in the future, but it can’t.

As noted earlier, the Dividend Discount Model, or DDM, had long existed at the time Roger Ibbotson and Rex Sinquefield did their seminal work and also gave forecasts of the return on equities. The DDM, reduced to its simplest form, says that investing in a stock is like owning a savings account. That portion of earnings that is not spent (on capital improvements, paid out as dividends, and so forth) is added to the stock’s fundamental value. As the fundamental value of a stock rises, so does its market value. Ibbotson, in a personal communication, even once told me that he thought that the DDM gave the theoretically best forecasts, because it is forward-looking and, if the dividend growth rate estimate is correctly formed, avoids the survival and procyclical biases, and other measurement errors, of the future-equals-past method.

The DDM, however, wasn’t widely used in forecasting of long-run rates of return on market benchmarks because forecasts of the dividend growth rate were (and are) notoriously inaccurate. It seemed as difficult to forecast the dividend growth rate—needed for a DDM estimate of the expected return on the stock market—as to forecast the stock market total return itself; so why bother?

In 1984, however, Jeffrey Diermeier (then of First Chicago Investment Advisors) pointed out, in the course of cowriting an article with Ibbotson and me, that corporate profits cannot reasonably be expected to grow indefinitely as a share of gross domestic product; otherwise, corporate profits will soon be larger than the entire economy, which cannot happen (Diermeier, Ibbotson, and Siegel 1984). Thus, the long-run growth rate of real (inflation-adjusted) GDP—which, in the United States, has been about 3 percent—is also likely to be a good estimate of the real growth rate of aggregate corporate earnings and, by extension, of dividends. One can use this information
to rough-in a forecast of the stock market:

\[ E(R_s) = \text{dividend yield} + \text{real GDP growth} + \text{inflation} \]
\[ = 2\% + 3\% + 2.5\% = 7.5\% \]

The numerical estimate represents conditions in early 2011. We’d add an extra one-half point to reflect the likelihood that the dividend yield, long depressed by tax policy, will rise over time. This is the expected return on all corporations, including those privately held, and is not calculated on a per-share basis. The estimate then needs to be diluted (reduced) somewhat to reflect the fact that companies need to issue new shares over time in order to raise enough capital to grow at the rates they expect; and to recognize any difference in expected growth rates between corporations in general and those in a particular capitalization-weighted benchmark (say, the S&P 500). Any expected change in valuation (the price/earnings or price/dividend ratio) also needs to be included; we’ve assumed no such change. While these adjustments can be complex, 7 percent seems like a good current estimate of the expected geometric mean equity return. This number is lower than that produced by the future-equals-past method.

The use of a DDM with a link to GDP to estimate equity returns did not become widely accepted after my 1984 article with Diermeier and Ibbotson. One reason is that the low forecasts it gave were not vindicated by subsequent results; the higher future-equals-past forecasts were more on the money for quite a number of years. But today, DDM-based equity forecasts with a link to GDP are commonplace, and among investment professionals, they have largely supplanted the future-equals-past method. (Some financial planners and other participants in the retail investment market still place strong emphasis on future-equals-past.) Robert Arnott, Clifford Asness, Peter Bernstein, Richard Grinold, and Kenneth Kroner, among others, have refined the DDM (and argued for its logical superiority) to the point where one can refer without irony to a DDM counter-revolution. The state of the art for estimating expected equity returns has come full circle.

Here, I’ve done the easy work. I’ve looked at two methods of forecasting aggregate equity returns, in excess of bonds, and have suggested that a method with a link to expected aggregate economic growth is more likely to give good forecasts than a method with no such link (but with a link to events of the past, whether those events were expected or surprising). In the ensuing volume, Paul Kaplan does the hard work of investigating issues
Foreword

with much more subtlety than those I've touched on here. The reader will find great benefit and profit in reading it.

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February 2011

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NOTES

1. Ibbotson and Sinquefield (1976).
3. In addition to the well-known published literature on the equity risk premium, I recommend Grinold and Kroner (2002), and Leibowitz, Hammond, and Siegel (2011).
5. For example, see Brown, Goetzmann, and Ross (1995).
6. Professor Rosling’s web site is www.gapminder.org. The software used to generate Figures F.3 and F.4 is available as a free download from the web site.

REFERENCES

FOREWORD


Introduction

In our analyses the [portfolio weights] might represent individual securities or they might represent aggregates such as, say, bonds, stocks and real estate.

—Harry Markowitz (Markowitz 1952)

I think the most important thing that happened between 1959 [i.e., when Markowitz (1959) was published] and the present is the notion of doing your analysis on asset classes in the first instance. This has become part of the infrastructure that we now rely on. In 1959, I had a theory. I had a rationale, and so on. Now, we have an industry.

—Harry Markowitz (Markowitz, Savage, and Kaplan 2010)

The asset-allocation paradigm—in which a portfolio is first divided up among a set of asset classes and then separately managed within each asset class—is the nearly universal approach to asset management. Putting this simple idea into practice, however, raises many issues, among them:

- How should the asset classes be defined? In particular, should equities be divided up based on investment style, geography, or other factors?
- Should asset classes be represented by market-value-weighted indexes or should other principles be used to create representative portfolios?
- How do actively managed funds fit into asset-class mixes?
- Should an optimization model be used to help determine the asset-class mix? If so, is the mean-variance model the right model or should another approach be used?
- How can inputs to the mean-variance model (expected returns, standard deviations, and correlations) be estimated for illiquid asset classes or asset classes that have small amounts of market data?
- How should market crashes and fat tails in return distributions be taken into account in asset-allocation models?
- What is the contribution of asset-allocation policy to overall portfolio performance, as opposed to market-timing and security selection?
- When building a retirement portfolio, how does an asset-allocation model account for regular withdrawals to fund a retiree’s expenditures?
- How can Monte Carlo simulation be used effectively as an asset-allocation-modeling tool?
- How can complicated asset-allocation models be made more interactive and graphical, and thus more accessible and powerful to investors?
INTRODUCTION

I have spent much of my 20-year career tackling these questions—first, as a researcher with Ibbotson Associates then for Morningstar. In this book, I gather what I believe to be my best answers—articles I have written or co-written that seek to understand the theories and practicalities of building asset-allocation models to create suitable investment portfolios.

However, I did not arrive at these answers all by myself. One of the best things about my job is that I get to discuss these issues with the leading thinkers in economics and finance. And although we don’t always agree, these discussions always lead me to a greater understanding of the issues at hand. Therefore, I also include several of my interviews with intellectuals such as the late Benoît Mandelbrot, who challenged the conventional approach to asset-class modeling back when it all started, and Harry Markowitz, the father of portfolio theory.

One of the reasons that many of the issues in the asset-allocation discussion are so controversial is that asset allocation is an inherently active approach to asset management. The only truly passive approach to asset allocation is holding a market-value-weighted portfolio of asset classes at all times. Although it is common practice to hold market-valued portfolios within asset classes (especially equities), rarely is such an approach recommended for doing asset allocation, at least in practice. Setting asset-class weightings not aligned with the market is betting against the market.

I organize the articles into four general topics:

I. Equities
II. Fixed Income, Real Estate, and Alternatives
III. Crashed and Fat Tails
IV. Doing Asset Allocation

At the head of each group of articles, I provide an overview. The overviews bring together the common themes among the articles within each section and throughout the book. Except for slight edits, each article appears as it did when it was originally published. Although I wrote many of these articles years ago, the reader should still find their main points applicable to today’s investing environment.

Unless the reader has a clear understanding of the distinction between expected return and geometric mean, I highly recommend reviewing the following note before reading the main body of the articles.

REFERENCES

Markowitz, Harry M., Sam Savage, and Paul D. Kaplan. 2010. “What Does Harry Markowitz Think?” Morningstar Advisor (June/July) [Chapter 27].
A Note on Expected Return and Geometric Mean

Paul D. Kaplan

The focus of mean-variance analysis, as formulated by Harry Markowitz (1952), is the tradeoff between expected return and standard deviation. Many investors are unfamiliar with the concept of expected return as used by Markowitz and confuse it with geometric mean. This is because in Markowitz’s investment model, the expected return is the relevant measure of reward. For long-term investors, however, what matters is the long-term rate of portfolio growth, or the geometric mean.

This is not to say that the two concepts are not related. In fact, in his 1959 book, Markowitz himself discussed how to approximate geometric mean with a function of expected return and standard deviation and proposed that the mean-variance model be used to find the portfolio that has the highest geometric mean (see Markowitz 1959, Chapter 6.).

The difference between expected return and geometric mean increases with return volatility. For asset classes with low volatility, such as short-term bonds, expected return and geometric mean are close in value. For highly volatile asset classes, such as emerging- and frontier-markets equity and venture capital, the two are quite different. Some of the articles in this book present methods for estimating expected returns of these highly volatile assets with results that, at first glance, may appear to be highly exaggerated.

In valuation practice, the expected return is known as the cost of capital. Sometimes, the two terms are used interchangeably, as Andrew Clare and I did in Chapter 9 (Clare and Kaplan 1999). Unfortunately, the confusion between expected return and geometric mean also occurs among valuation practitioners. Some practitioners use the geometric mean as if it were the cost of capital. This can result in severe overvaluations, especially for high-risk assets.
A NOTE ON EXPECTED RETURN AND GEOMETRIC MEAN

The purpose of this note is to clarify the difference between expected return and geometric mean and explain how they can be related in mean-variance analysis. To do so, I first present a mathematical formulation of the concepts and then illustrate them using numerical examples based on results in some of the articles in this book.

MATHEMATICAL FORMULATION

Suppose that there is an asset that at some time in the future, T, will make a single payment, \( \tilde{X}_T \), that will not be known until time T. However, from now, time 0, until T, the value of the asset has some fair value, which would be the market value if the asset were traded in an informational-efficient market. At each time \( t \), this value, \( \tilde{X}_t \), will become known. The rate of change in the asset’s value from time \( t - 1 \) to \( t \) is its time \( t \) return, \( \tilde{R}_t \). Hence,

\[
\tilde{R}_t = \frac{\tilde{X}_t}{\tilde{X}_{t-1}} - 1
\]

Thus, we can write:

\[
\tilde{X}_T = X_0 (1 + \tilde{R}_1) (1 + \tilde{R}_2) \cdots (1 + \tilde{R}_T)
\]

where \( X_0 \) is the known value of the asset at time 0.

We assume that each \( \tilde{R}_t \) is independent and identically distributed, so that they all have the same mathematical expectation, \( ER \). Hence,

\[
E_0 [\tilde{X}_T] = (1 + ER)^T X_0
\]

where \( E_0 [\tilde{X}_T] \) denotes the mathematical expectation of \( \tilde{X}_T \) based on all information available at time 0. It is important to remember that \( E_0 [\tilde{X}_T] \) is not necessarily the most likely value of \( \tilde{X}_T \) nor the median of \( \tilde{X}_T \). Rather, it is the probability-weighted average of all possible values of \( \tilde{X}_T \), based on everything that is known about \( \tilde{X}_T \) at time 0.

The problem that a valuation analyst faces is to estimate \( X_0 \) from an estimate of \( E_0 [\tilde{X}_T] \) at an appropriately risk-adjusted discount rate, the cost of capital. Rewriting equation N.3 as

\[
X_0 = \frac{E_0 [\tilde{X}_T]}{(1 + ER)^T}
\]

it is evident that the cost of capital is the expected return.
A Note on Expected Return and Geometric Mean

Now suppose that the asset in question is an investor’s portfolio. The investor may be interested in the growth rate of the portfolio from time 0 to time $T$, which I denote $\tilde{g}_T$. We have:

$$\tilde{g}_T = \sqrt[\frac{T}{\sqrt{X_T}}]{X_0} - 1 = \sqrt[\frac{T}{\sqrt{(1 + \tilde{R}_1)(1 + \tilde{R}_2)\cdots(1 + \tilde{R}_T)}}]{(1 + \tilde{R}_1)(1 + \tilde{R}_2)\cdots(1 + \tilde{R}_T)} - 1 \quad (N.5)$$

The geometric mean from time 0 to time $T$, $GM_T$ is the mathematical expectation of $\tilde{g}_T$:

$$GM_T = E_0[\tilde{g}_T] = E_0\left[\sqrt[\frac{T}{\sqrt{(1 + \tilde{R}_1)(1 + \tilde{R}_2)\cdots(1 + \tilde{R}_T)}}]{(1 + \tilde{R}_1)(1 + \tilde{R}_2)\cdots(1 + \tilde{R}_T)} - 1\right] \quad (N.6)$$

It can be shown that if each $\tilde{R}_t$ is not only independent and identically distributed but is lognormally distributed with standard deviation $SD$, that as $T$ grows large, $GM_T$ tends toward:

$$GM = \frac{(1 + ER)^2}{\sqrt{(1 + ER)^2 + SD^2}} - 1 \quad (N.7)$$

Equation N.7 is the link between expected return and geometric mean. It shows that when there is no return volatility ($SD = 0$), the two are the same, but as volatility increases, they grow apart.

**NUMERICAL EXAMPLES**

For the purpose of illustration, from the chapters in this book on emerging- and frontier-markets equity, real estate, and venture capital, I compiled expected returns and standard deviations for U.S. stocks, U.S. bonds, U.S. real estate, Zimbabwe stocks, and U.S. venture capital and used equation N.7 to derive estimates of the geometric means. Table N.1 presents the results.

Note that although the expected returns of Zimbabwe stocks and U.S. venture capital are extremely high, so are their standard deviations. The result is that these high-risk asset classes have geometric means that are in line with those of risky U.S. stocks.

In Chapter 12 (Kaplan 1995), I introduce a graphical device to illustrate the relationship between geometric mean, expected return, and standard deviation: isogeometric mean curves. An isogeometric mean curve traces
A NOTE ON EXPECTED RETURN AND GEOMETRIC MEAN

TABLE N.1  Capital Market Assumptions for Five Asset Classes

<table>
<thead>
<tr>
<th>Article</th>
<th>Asset Class</th>
<th>Geometric Mean %</th>
<th>Expected Return %</th>
<th>Standard Deviation %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kaplan (1995) Chapter 12</td>
<td>U.S. Bonds</td>
<td>5.15</td>
<td>5.30</td>
<td>5.57</td>
</tr>
<tr>
<td>Clare and Kaplan (1999) Chapter 9</td>
<td>Zimbabwe Stocks</td>
<td>12.21</td>
<td>26.00</td>
<td>64.35</td>
</tr>
</tbody>
</table>

out the relationship between expected return and standard deviation implied by equation N.7 for a given value of geometric mean in the standard Markowitz space. Figure N.1 shows the isogeometric mean curves for the five asset classes in Table N.1. For example, the topmost curve consists of all combinations of expected return and standard deviation that are consistent with an expected long-term geometric mean of 13.38 percent. These include the combination of an expected return of 45.00 percent and a standard deviation of 115.60 percent, which are our estimates of these parameters for

FIGURE N.1  Isogeometric Mean Curves