The authors should be applauded for providing a unique and very readable account of probability metrics and the application of this specialized field to financial problems.

Professor Carol Alexander, Henley Business School at Reading

This self-contained book covering the important field of probability metrics is a wonderful addition to the literature in financial economics. What makes it unique is that it presents this area at a level accessible to those without extensive prior experience – academic and practitioner alike.

Professor Peter N. Kolm, New York University

Is the behavior of the stocks in our portfolio close to their behavior during the most recent crisis? How close is the strategy of hedge fund A to the strategy of hedge fund B? In which proportions do we invest in a given universe of stocks so that the resulting portfolio matches as much as possible the strategy of fund C?

All of these questions are essential to finance and they have one feature in common: measuring distances between random quantities. Problems of this kind have been explored for many years in areas other than finance. In A Probability Metrics Approach to Financial Risk Measures, the field of probability metrics and risk measures are related to one another and applied to finance for the first time, revealing groundbreaking new classes of risk measures, finding new relations between existing classes of risk measures, and providing answers to the question of which risk measure is best for a given problem. Applications include optimal portfolio choice, risk theory, and numerical methods in finance.

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A Probability Metrics Approach to Financial Risk Measures
A Probability Metrics Approach to Financial Risk Measures

Svetlozar T. Rachev
Stoyan V. Stoyanov
Frank J. Fabozzi

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STR
To my grandchildren Iliana, Zoya, and Zari

SVS
To my parents Veselin and Evgeniya Kolevi and
my brother Pavel Stoyanov

FJJ
To my wife Donna and
my children Francesco, Patricia, and Karly
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The theory of probability metrics is a branch of probability theory. It finds application in different theoretical and applied fields such as probability theory, queuing theory, insurance risk theory, and finance. The theory of probability metrics looks for answers to the following basic question: How can one measure the difference between random quantities? In finance, for example, we assume a stochastic model for asset return distributions and, in order to estimate the risk of a portfolio of assets, we sample from the fitted distribution. Then, we use the generated simulations to calculate portfolio risk. In this context, there are two issues arising on two different levels. First, the assumed stochastic model should be “close” to the empirical data. In this sense, we say that we need a realistic model in the first place. Second, since the risk estimate is essentially computed from random scenarios, we have to be aware of the variability of the estimator and how it depends on the assumed asset return distributions.

Although based on universal principles and ideas, the field of probability metrics is very specialized. Most of the literature is highly technical and is accessible mostly to specialists in probability theory. As far as applications are concerned, apart from our book Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: Ideal Risk, Uncertainty, and Performance Measures (John Wiley & Sons, 2008), we are unaware of other literature describing applications in finance.
PREFACE

This book has two goals. The first goal is to describe applications in finance and extend them where possible. The second goal is to present the theory of probability metrics in a more accessible form which would be appropriate for non-specialists in the field. Topics requiring more mathematical rigor and detail are included in technical appendices to chapters.

The book is organized in the following way. Chapter 1 provides a conceptual description of the method of probability metrics and reviews direct and indirect applications in the field of finance. Chapter 2 provides an introduction to the theory of probability metrics. The classical theory describing investor choice under uncertainty is provided in Chapter 3. Chapter 4 discusses the classification of probability distances to primary, simple, and compound types. The information in Chapter 2 is a prerequisite. Chapters 5, 6, and 7 are devoted to risk and uncertainty measures and discuss in detail AVaR and the Monte Carlo method for AVaR estimation. Chapter 6 is a prerequisite to Chapter 7. Finally, Chapter 8 considers the problem of quantifying stochastic dominance relations and takes advantage of the terms introduced in Chapter 3.

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Svetlozar (Zari) T. Rachev completed his Ph.D. degree in 1979 from Moscow State (Lomonosov) University, and his Doctor of Science Degree in 1986 from Steklov Mathematical Institute in Moscow. Currently he is Chair-Professor in Statistics, Econometrics and Mathematical Finance at the University of Karlsruhe in the School of Economics and Business Engineering. He is also Professor Emeritus at the University of California, Santa Barbara in the Department of Statistics and Applied Probability. He has published seven monographs, eight handbooks and special-edited volumes, and over 300 research articles. His recently coauthored books published by Wiley in mathematical finance and financial econometrics include Fat-Tailed and Skewed Asset Return Distributions: Implications for Risk Management, Portfolio selection, and Option Pricing (2005), Operational Risk: A Guide to Basel II Capital Requirements, Models, and Analysis (2007), Financial Econometrics: From Basics to Advanced Modeling Techniques (2007), and Bayesian Methods in Finance (2008). Professor Rachev is cofounder of Bravo Risk Management Group, specializing in financial risk-management software. Bravo Group was recently acquired by FinAnalytica, for which he currently serves as Chief-Scientist.

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Chapter 1

Introduction

In this chapter, we provide a conceptual description of the method of probability metrics and discuss direct and indirect applications in the field of finance, which are described in more detail throughout the book.

1.1 Probability Metrics

The development of the theory of probability metrics started with the investigation of problems related to limit theorems in probability theory. Limit theorems occupy a very important place in probability theory, statistics, and all their applications. A well-known example is the celebrated central limit theorem (CLT) but there are many other limit theorems, such as the generalized CLT, the max-stable CLT, functional limit theorems, etc. In general, the applicability of the limit theorems stems from the fact that the limit law can be regarded as an approximation to the stochastic model under consideration and, therefore, can be accepted as an approximate substitute. The central question arising is how large an error we make by adopting the approximate model and this question can be investigated by
CHAPTER 1 INTRODUCTION

studying the distance between the limit law and the stochastic model. It turns out that this distance is not influenced by the particular problem. Rather, it can be studied by a theory based on some universal principles.

Generally, the theory of probability metrics studies the problem of measuring distances between random quantities. On one hand, it provides the fundamental principles for building probability metrics – the means of measuring such distances. On the other, it studies the relationships between various classes of probability metrics. Another realm of study concerns problems which require a particular metric while the basic results can be obtained in terms of other metrics. In such cases, the metrics relationship is of primary importance.

Certainly, the problem of measuring distances is not limited to random quantities only. In its basic form, it originated in different fields of mathematics. Nevertheless, the theory of probability metrics was developed due to the need of metrics with specific properties. Their choice is very often dictated by the stochastic model under consideration and to a large extent determines the success of the investigation. Rachev (1991) provides more details on the methods of the theory of probability metrics and its numerous applications in both theoretical and more practical problems.

1.2 Applications in Finance

There are no limitations in the theory of probability metrics concerning the nature of the random quantities. This makes its methods fundamental and appealing. Actually, in the general case, it is more appropriate to refer to the random quantities as random elements. They can be random variables, random vectors, random functions or random elements in general spaces. For instance, in the context of financial applications, we can study the distance between two random stocks prices, or between vectors of financial variables that are used to construct portfolios, or between yield curves which are much more complicated objects. The methods of the theory remain
the same, irrespective of the nature of the random elements. This represents the most direct application of the theory of probability metrics in finance: that is, it provides a method for measuring how different two random elements are. We explain the axiomatic construction of probability metrics and provide financial interpretations in Chapter 2.

Financial economics, like any other science relying on statistical methods, considers statistical information about the objects it studies on several levels. In some theories in the area of finance, conclusions are drawn only on the basis of certain characteristics of the corresponding distributions. For example, an investor would oftentimes use a risk-reward ratio to rank investment opportunities. Essentially, this reduces to computing the measure of reward (e.g., the expected return) and the measure of risk (e.g., value-at-risk, conditional value-at-risk, standard deviation). Both the measure of reward and the measure of risk represent two characteristics of the corresponding distributions. In effect, the final decision is made on the basis of these two characteristics which, from the investor’s perspective, aggregate the information available in the distribution functions.

The theory describing investor choice under uncertainty, the fundamentals of which we discuss in Chapter 3, uses a different approach. Various criteria were developed for first-, second-, and higher-order stochastic dominance based on the distributions themselves. As a consequence, investment opportunities are compared directly through their distribution functions, which is a superior approach from the standpoint of the utilized information.

As another example, consider the problem of building a diversified portfolio. The investor would be interested not only in the marginal distribution characteristics (i.e., the characteristics of the assets on a stand-alone basis), but also in how the assets depend on each another. This requires an additional piece of information which cannot be recovered from the distribution functions of the asset returns. The notion of stochastic dependence can be described by considering the joint behavior of assets returns.

The theory of probability metrics offers a systematic approach towards such a hierarchy of ways to utilize statistical information.
CHAPTER 1 INTRODUCTION

It distinguishes between primary, simple, and compound types of distances which are defined on the space of characteristics, the space of distribution functions, and the space of joint distributions, respectively. Therefore, depending on the particular problem, one can choose the appropriate distance type and this represents another direct application of the theory of probability metrics in the field of finance. This classification of probability distances is explained in Chapter 4.

Besides direct applications, there are also a number of indirect ones. For instance, one of the most important problems in risk estimation is formulating a realistic hypothesis for the asset return distributions. This is largely an empirical question because no arguments exist that can be used to derive a model from some general principles. Therefore, we have to hypothesize a model that best describes a number of empirically confirmed phenomena about asset returns: (1) volatility clustering, (2) autoregressive behavior, (3) short- and long-range dependence, and (4) fat-tailed behavior of the building blocks of the time-series model which varies depending on the frequency (e.g., intra-day, daily, monthly). The theory of probability metrics can be used to suggest a solution to (4). The fact that the degree of heavy-tailedness varies with the frequency may be related to the process of aggregation of higher-frequency returns to obtain lower frequency returns. Generally, the residuals from higher-frequency return models tend to have heavier tails and this observation together with a result known as a pre-limit theorem can be used to derive a suggestion for the overall shape of the return distribution. Furthermore, the probability distance used in the pre-limit theorem indicates that the derived shape is most relevant for the body of the distribution. As a result, through the theory of probability metrics we can obtain an approach to construct reasonable models for asset return distributions. We discuss in more detail limit and pre-limit theorems in Chapter 7.

Another central topic in finance is quantification of risk and uncertainty. The two notions are related but are not synonymous. Functionals quantifying risk are called risk measures and functionals quantifying uncertainty are called deviation measures or dispersion
1.2 APPLICATIONS IN FINANCE

measures. Axiomatic constructions are suggested in the literature for all of them. It turns out that the axioms defining measures of uncertainty can be linked to the axioms defining probability distances, however, with one important modification. The axiom of symmetry, which every distance function should satisfy, appears unnecessarily restrictive. Therefore, we can derive the class of deviation measures from the axiomatic construction of asymmetric probability distances which are also called probability quasi-distances. The topic is discussed in detail in Chapter 5.

As far as risk measures are concerned, we consider in detail advantages and disadvantages of value-at-risk, average value-at-risk (AVaR), and spectral risk measures in Chapter 5 and Chapter 6. Since Monte Carlo-based techniques are quite common among practitioners, we discuss in Chapter 7 Monte Carlo-based estimation of AVaR and the problem of stochastic stability in particular. The discussion is practical, based on simulation studies, and is inspired by the classical application of the theory of probability metrics in estimating the stochastic stability of probabilistic models. We apply the CLT and the Generalized CLT to derive the asymptotic distribution of the AVaR estimator under different distributional hypotheses and we discuss approaches to improve its stochastic stability.

We mentioned that adopting stochastic dominance rules for prospect selection rather than rules based on certain characteristics leads to a more efficient use of the information contained in the corresponding distribution functions. Stochastic dominance rules, however, are of the type “X dominates Y” or “X does not dominate Y”: that is, the conclusion is qualitative. As a consequence, computational problems are hard to solve in this setting. A way to overcome this difficulty is to transform the nature of the relationship from qualitative to quantitative. We describe how this can be achieved in Chapter 8, which is the last chapter in the book. Our approach is fundamental and is based on asymmetric probability semidistances, which are also called probability quasi-semidistances.

The link with probability metrics theory allows a classification of stochastic dominance relations in general. They can be primary, simple, or compound but also, depending on the underlying structure,
they may or may not be generated by classes of investors, which is a typical characterization in the classical theory of choice under uncertainty. This is also a topic discussed in Chapter 8.

References

The goals of this chapter are the following:

- To provide examples of metrics in probability theory and interpretations from a financial economics perspective.
- To introduce formally the notions of a probability metric and a probability distance.
- To consider the general setting of random variables defined on a given probability space \((\Omega, \mathcal{A}, \text{Pr})\) taking values in a separable metric space \(U\), allowing a unified treatment of problems involving one-dimensional random variables, random vectors or stochastic processes, for example.
- To consider the alternative setting of probability distances on the space of probability measures \(\mathcal{P}_2\) defined on the \(\sigma\)-algebras of Borel subsets of \(U^2 = U \times U\) where \(U\) is a separable metric space.
- To examine the equivalence of the notion of a probability distance on the space of probability measures \(\mathcal{P}_2\) and on the space of joint distributions \(\mathcal{L}X_2\) generated by pairs of random variables \((X, Y)\) taking values in a separable metric space \(U\).
## Chapter 2  Probability Distances and Metrics

Notation introduced in this chapter:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>EN</td>
<td>The engineer’s metric</td>
</tr>
<tr>
<td>$\mathcal{X}^p$</td>
<td>The space of real-valued r.v. with $E</td>
</tr>
<tr>
<td>$\rho$</td>
<td>The uniform (Kolmogorov) metric</td>
</tr>
<tr>
<td>$\mathcal{X} = \mathcal{X}(\mathbb{R})$</td>
<td>The space of real-valued r.v.s</td>
</tr>
<tr>
<td>$L$</td>
<td>The Lévy metric</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>The Kantorovich metric</td>
</tr>
<tr>
<td>$\theta_p$</td>
<td>The $L_p$-metric between distribution functions</td>
</tr>
<tr>
<td>$K, K^*$</td>
<td>The Ky Fan metrics</td>
</tr>
<tr>
<td>$L_p$</td>
<td>The $L_p$-metric between r.v.s</td>
</tr>
<tr>
<td>MOM$_p$</td>
<td>The metric between $p$-th moments</td>
</tr>
<tr>
<td>$(S, \rho)$</td>
<td>Metric space with a metric $\rho$</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>The $n$-dimensional vector space</td>
</tr>
<tr>
<td>$r(C_1, C_2)$</td>
<td>The Hausdorff metric (semimetric between sets)</td>
</tr>
<tr>
<td>$s(F, G)$</td>
<td>The Skorokhod metric</td>
</tr>
<tr>
<td>$K = K_\rho$</td>
<td>Parameter of a distance space</td>
</tr>
<tr>
<td>$\mathcal{H}$</td>
<td>The class of Orlicz’s functions</td>
</tr>
<tr>
<td>$\rho_H$</td>
<td>The Birnbaum–Orlicz distance</td>
</tr>
<tr>
<td>$\text{Kr}$</td>
<td>The Kruglov distance</td>
</tr>
<tr>
<td>$(U, d)$</td>
<td>Separable metric space with metric $d$</td>
</tr>
<tr>
<td>s.m.s.</td>
<td>Separable metric space</td>
</tr>
<tr>
<td>$U^k$</td>
<td>The $k$-fold Cartesian product of $U$</td>
</tr>
<tr>
<td>$B_k = B_k(U)$</td>
<td>The Borel $\sigma$-algebra on $U^k$</td>
</tr>
<tr>
<td>$\mathcal{P}_k = \mathcal{P}_k(U)$</td>
<td>The space of probability laws on $B_k$</td>
</tr>
<tr>
<td>$T_{\alpha, \beta, \ldots, \gamma} P$</td>
<td>The marginal of $P \in \mathcal{P}_k$ on the coordinates $\alpha, \beta, \ldots, \gamma$</td>
</tr>
<tr>
<td>$\mathcal{P}_{X}$</td>
<td>The distribution of $X$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>A probability semidistance</td>
</tr>
<tr>
<td>$\mathcal{X} := \mathcal{X}(U)$</td>
<td>The set of $U$-valued random variables</td>
</tr>
<tr>
<td>$\mathcal{L}\mathcal{X}_2 := \mathcal{L}\mathcal{X}_2(U)$</td>
<td>The space of $\mathcal{P}_{X,Y}, X, Y \in \mathcal{X}(U)$</td>
</tr>
<tr>
<td>u.m.</td>
<td>Universally measurable</td>
</tr>
<tr>
<td>u.m.s.m.s.</td>
<td>Universally measurable separable metric space</td>
</tr>
</tbody>
</table>
2.2 SOME EXAMPLES OF PROBABILITY METRICS

Important terms introduced in this chapter:

<table>
<thead>
<tr>
<th>Term</th>
<th>Concise explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(semi)metric function</td>
<td>A special function satisfying properties making it uniquely positioned for computing distances</td>
</tr>
<tr>
<td>(semi)metric space</td>
<td>A space equipped with a (semi)metric function for measuring distances between space elements</td>
</tr>
<tr>
<td>probability (semi)metric</td>
<td>A (semi)metric function designed to measure distances between random elements</td>
</tr>
</tbody>
</table>

2.1 Introduction

Generally speaking, a functional which measures the distance between random quantities is called a probability metric. These random quantities can be of a very general nature. For instance, they can be random variables, such as the daily returns of equities, the daily change of an exchange rate, etc., or stochastic processes, such as a price evolution in a given period, or much more complex objects, such as the daily movement of the shape of the yield curve.

In this chapter, we provide examples of probability metrics and interpretations from the perspective of financial economics, limiting the discussion to one-dimensional random variables. Then we proceed with the axiomatic definition of probability metrics. In the appendix, we provide a more technical discussion of the axiomatic construction in a much more general context.

2.2 Some Examples of Probability Metrics

Below is a list of various metrics commonly found in probability and statistics. In this section, we limit the discussion to one-dimensional variables only.
CHAPTER 2  PROBABILITY DISTANCES AND METRICS

2.2.1  Engineer’s metric

The engineer’s metric is

\[ \text{EN}(X, Y) := |E(X) - E(Y)| \quad X, Y \in \mathfrak{X}^1 \]  \hspace{1cm} (2.2.1)

where \( \mathfrak{X}^p \) is the space of all real-valued random variables (r.v.s) with \( E|X|^p < \infty \). In the case of the engineer’s metric, we measure the distance between the random variables \( X \) and \( Y \) only in terms of the deviation of their means. For example, if \( X \) and \( Y \) describe the return on two common stocks, then the engineer’s metric computes the distance between their expected returns.

2.2.2  Uniform (or Kolmogorov) metric

The uniform (or Kolmogorov) metric is

\[ \rho(X, Y) := \sup \{|F_X(x) - F_Y(x)| : x \in \mathbb{R}\} \quad X, Y \in \mathfrak{X} = \mathfrak{X}(\mathbb{R}) \]  \hspace{1cm} (2.2.2)

where \( F_X \) is the distribution function (d.f.) of \( X, \mathbb{R} = (-\infty, +\infty) \), and \( \mathfrak{X} \) is the space of all real-valued r.v.s.

Figure 2.1 illustrates the Kolmogorov metric. The c.d.f.s of two random variables are plotted on the top plot and the bottom plot shows the absolute difference between them, \( |F_X(x) - F_Y(x)| \), as a function of \( x \). The Kolmogorov metric is equal to the largest absolute difference between the two c.d.f.s. A arrow shows where it is attained.

If the random variables \( X \) and \( Y \) describe the return distribution of the common stocks of two corporations, then the Kolmogorov metric has the following interpretation. The distribution function \( F_X(x) \) is by definition the probability that \( X \) loses more than a level \( x \), \( F_X(x) = P(X \leq x) \). Similarly, \( F_Y(x) \) is the probability that \( Y \) loses more than \( x \). Therefore, the Kolmogorov distance \( \rho(X, Y) \) is the maximum deviation between the two probabilities that can be attained.
2.2 SOME EXAMPLES OF PROBABILITY METRICS

by varying the loss level $x$. If $\rho(X, Y) = 0$, then the probabilities that $X$ and $Y$ lose more than a loss level $x$ coincide for all loss levels.

Usually, the loss level $x$, for which the maximum deviation is attained, is close to the mean of the return distribution, i.e. the mean return. Thus, the Kolmogorov metric is completely insensitive to the tails of the distribution which describe the probabilities of extreme events – extreme returns or extreme losses.

2.2.3 Lévy metric

The Lévy metric is

$$L(X, Y) := \inf \{ \epsilon > 0 : F_X(x - \epsilon) - \epsilon \leq F_Y(x) \leq F_X(x + \epsilon) + \epsilon \quad \forall x \in \mathbb{R} \}. \quad (2.2.3)$$

The Lévy metric is difficult to calculate in practice. Figure 2.2 contains an illustration. The Lévy metric has important theoretic
CHAPTER 2 PROBABILITY DISTANCES AND METRICS

Figure 2.2: Illustration of the Lévy metric. $L(X, Y)\sqrt{2}$ is the maximum distance between the graphs of $F_X$ and $F_Y$ along a 45 degrees direction. The arrow indicates where the maximum is attained.

application in probability theory as it metrizes the weak convergence. It can be viewed as measuring the closeness between the graphs of the distribution functions while the Kolmogorov metric is a uniform metric between the distribution functions. The general relationship between the two is

$$L(X, Y) \leq \rho(X, Y)$$

(2.2.4)

For example, suppose that $X$ is a random variable describing the return distribution of a portfolio of stocks and $Y$ is a deterministic benchmark with a return of 2.5% ($Y = 2.5\%$). (The deterministic benchmark in this case could be either the cost of funding over a specified time period or a target return requirement to satisfy a liability such as a guaranteed investment contract.) Assume also that the portfolio return has a normal distribution with mean equal to 2.5% and a volatility $\sigma$, $X \in N(2.5\%, \sigma^2)$. Since the expected portfolio return is exactly equal to the deterministic benchmark, the Kolmogorov distance between them is always equal to 1/2 irrespective of how small