Output Coupling in Optical Cavities and Lasers

A Quantum Theoretical Approach
Edited by
Kikuo Ujihara

Output Coupling in
Optical Cavities and Lasers
Related Titles

W. Vogel, D.-G. Welsch

Quantum Optics
2006
ISBN: 978-3-527-40507-7

S.M. Dutra

Cavity Quantum Electrodynamics
The Strange Theory of Light in a Box
2004
ISBN: 978-0-471-44338-4

H.-A. Bachor, T.C. Ralph

A Guide to Experiments in Quantum Optics
2004
ISBN: 978-3-527-40393-6
Kikuo Ujihara

Output Coupling in Optical Cavities and Lasers

A Quantum Theoretical Approach
All books published by Wiley-VCH are carefully produced. Nevertheless, authors, editors, and publisher do not warrant the information contained in these books, including this book, to be free of errors. Readers are advised to keep in mind that statements, data, illustrations, procedural details or other items may inadvertently be inaccurate.

Library of Congress Card No.: applied for

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Bibliographic information published by
the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.d-nb.de

© 2010 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim

All rights reserved (including those of translation into other languages). No part of this book may be reproduced in any form – by photoprinting, microfilm, or any other means – nor transmitted or translated into a machine language without written permission from the publishers.

Registered names, trademarks, etc. used in this book, even when not specifically marked as such, are not to be considered unprotected by law.

Printed in the Federal Republic of Germany
Printed on acid-free paper

Cover Design Adam-Design, Weinheim
Typesetting Macmillan Publishing Solutions, Bangalore, India
Printing and Binding Strauss GmbH, Mörlenbach

ISBN: 978-3-527-40763-7
For Mieko
This page intentionally left blank
## Contents

Preface  XIII  
Acknowledgments  XVII

### 1 A One-Dimensional Optical Cavity with Output Coupling:  
#### Classical Analysis  1  
1.1 Boundary Conditions at Perfect Conductor and Dielectric Surfaces  1  
1.2 Classical Cavity Analysis  2  
1.2.1 One-Sided Cavity  2  
1.2.2 Symmetric Two-Sided Cavity  5  
1.3 Normal Mode Analysis: Orthogonal Modes  7  
1.3.1 One-Sided Cavity  7  
1.3.2 Symmetric Two-Sided Cavity  12  
1.4 Discrete versus Continuous Mode Distribution  15  
1.5 Expansions of the Normalization Factor  16  
1.6 Completeness of the Modes of the “Universe”  17  

### 2 A One-Dimensional Optical Cavity with Output Coupling:  
#### Quantum Analysis  23  
2.1 Quantization  23  
2.2 Energy Eigenstates  24  
2.3 Field Commutation Relation  26  
2.4 Thermal Radiation and the Fluctuation–Dissipation Theorem  28  
2.4.1 The Density Operator of the Thermal Radiation Field  28  
2.4.2 The Correlation Function and the Power Spectrum  29  
2.4.3 The Response Function and the Fluctuation–Dissipation Theorem  31  
2.4.4 Derivation of the Langevin Noise for a Single Cavity Resonant Mode  33
2.4.5 Excitation of the Cavity Resonant Mode by a Current Impulse 37
2.5 Extension to an Arbitrarily Stratified Cavity 38
2.5.1 Description of the Cavity Structure 38
2.5.2 The Modes of the “Universe” 40

3 A One-Dimensional Quasimode Laser: General Formulation 47
3.1 Cavity Resonant Modes 47
3.2 The Atoms 49
3.3 The Atom–Field Interaction 49
3.4 Equations Governing the Atom–Field Interaction 51
3.5 Laser Equation of Motion: Introducing the Langevin Forces 53
3.5.1 The Field Decay 53
3.5.2 Relaxation in Atomic Dipole and Atomic Inversion 55

4 A One-Dimensional Quasimode Laser: Semiclassical and Quantum Analysis 61
4.1 Semiclassical Linear Gain Analysis 61
4.2 Semiclassical Nonlinear Gain Analysis 64
4.3 Quantum Linear Gain Analysis 67
4.4 Quantum Nonlinear Gain Analysis 74

5 A One-Dimensional Laser with Output Coupling: Derivation of the Laser Equation of Motion 81
5.1 The Field 81
5.2 The Atoms 83
5.3 The Atom–Field Interaction 84
5.4 Langevin Forces for the Atoms 85
5.5 Laser Equation of Motion for a Laser with Output Coupling 86

6 A One-Dimensional Laser with Output Coupling: Contour Integral Method 91
6.1 Contour Integral Method: Semiclassical Linear Gain Analysis 91
6.2 Contour Integral Method: Semiclassical Nonlinear Gain Analysis 94
6.3 Contour Integral Method: Quantum Linear Gain Analysis 95
6.4 Contour Integral Method: Quantum Nonlinear Gain Analysis 100
## Contents

### 7 A One-Dimensional Laser with Output Coupling: Semiclassical Linear Gain Analysis 103

7.1 The Field Equation Inside the Cavity 104  
7.2 Homogeneously Broadened Atoms and Uniform Atomic Inversion 106  
7.3 Solution of the Laser Equation of Motion 108  
7.3.1 The Field Equation for Inside the Cavity 108  
7.3.2 Laplace-Transformed Equations 109  
7.3.3 The Field Inside the Cavity 113  
7.3.4 The Field Outside the Cavity 114

### 8 A One-Dimensional Laser with Output Coupling: Semiclassical Nonlinear Gain Analysis 119

8.1 The Field Equation Inside the Cavity 119  
8.2 Homogeneously Broadened Atoms and Uniform Pumping 121  
8.3 The Steady State 122  
8.4 Solution of the Coupled Nonlinear Equations 125  
8.5 The Field Outside the Cavity 129

### 9 A One-Dimensional Laser with Output Coupling: Quantum Linear Gain Analysis 133

9.1 The Equation for the Quantum Linear Gain Analysis 134  
9.2 Homogeneously Broadened Atoms and Uniform Atomic Inversion 137  
9.3 Laplace-Transformed Equations 138  
9.4 Laplace-Transformed Noise Forces 140  
9.5 The Field Inside the Cavity 144  
9.5.1 Thermal Noise 146  
9.5.2 Quantum Noise 148  
9.5.3 The Total Field 151  
9.6 The Field Outside the Cavity 154  
9.7 The Field Correlation Function 156  
9.8 The Laser Linewidth and the Correction Factor 162

### 10 A One-Dimensional Laser with Output Coupling: Quantum Nonlinear Gain Analysis 167

10.1 The Equation for the Quantum Nonlinear Gain Analysis 167  
10.2 Homogeneously Broadened Atoms and Uniform Pumping 170
10.3 The Steady-State and Laplace-Transformed Equations 171
10.4 The Lowest-Order Solution 176
10.5 The First-Order Solution: Temporal Evolution 178
10.5.1 The Formal Temporal Differential Equation 178
10.5.2 Thermal Noise 182
10.5.3 Quantum Noise 182
10.5.4 The Temporal Differential Equation 186
10.5.5 Penetration of Thermal Noise into the Cavity 187
10.6 Phase Diffusion and the Laser Linewidth 188
10.7 Phase Diffusion in the Nonlinear Gain Regime 190
10.7.1 Phase Diffusion 190
10.7.2 Evaluation of the Sum \( \sum_m \left( |A_m|^2 + |B_m|^2 \right) \) 196
10.7.3 The Linewidth and the Correction Factors 199
10.8 The Field Outside the Cavity 202

11 Analysis of a One-Dimensional Laser with Two-Side Output Coupling: The Propagation Method 211
11.1 Model of the Laser and the Noise Sources 211
11.2 The Steady State and the Threshold Condition 214
11.3 The Time Rate of the Amplitude Variation 218
11.4 The Phase Diffusion of the Output Field 221
11.5 The Linewidth for the Nonlinear Gain Regime 223
11.6 The Linewidth for the Linear Gain Regime 228

12 A One-Dimensional Laser with Output Coupling: Summary and Interpretation of the Results 235
12.1 Models of the Quasimode Laser and Continuous Mode Laser 235
12.2 Noise Sources 236
12.2.1 Thermal Noise and Vacuum Fluctuation as Input Noise 236
12.2.2 Quantum Noise 237
12.3 Operator Orderings 238
12.4 Longitudinal Excess Noise Factor 239
12.4.1 Longitudinal Excess Noise Factor Below Threshold 239
12.4.2 Longitudinal Excess Noise Factor Above Threshold 240
12.5 Mathematical Relation between Below-Threshold and Above-Threshold Linewidths 241
12.6 Detuning Effects 243
12.7 Bad Cavity Effect 245
12.8 Incomplete Inversion and Level Schemes 246
12.9 The Constants of Output Coupling 247
12.10 Threshold Atomic Inversion and Steady-State Atomic Inversion 249
15 Quantum Theory of the Output Coupling of an Optical Cavity 335
15.1 Quantum Field Theory 336
15.1.1 Normal Mode Expansion 336
15.1.2 Natural Mode Quantization 344
15.1.3 Projection Operator Method 348
15.2 Quantum Noise Theory 349
15.2.1 The Input–Output Theory by Time Reversal 349
15.2.2 The Input–Output Theory by the Boundary Condition 351
15.2.3 Another Quantum Noise Theory 354
15.3 Green’s Function Theory 355
15.4 Quasimode Theory 355
15.5 Summary 355
15.6 Equations for the Output Coupling and Input–Output Relation 356

Appendices 359

Index 365
Preface

After a half-century from the birth of the laser, we now see lasers in a variety of locations in academic institutions and industrial settings, as well as in everyday life. The species of laser are diverse. The core of quantum-mechanical laser theory was established in the 1960s by the Haken school and Scully. Semiclassical gas laser theory was also established in the 1960s by Lamb. Subsequently, many theoretical works on lasers have appeared for specific types of lasers or for specific operation modes. So, laser science is now mature and seems to leave little to be elucidated. Laser science has evolved into many branches of quantum-optical science, including coherent interaction, nonlinear optics, optical communications, quantum-optical information, quantum computation, laser-cooled atoms, and Bose–Einstein condensation, as well as gravitational wave detection by laser interferometer. Laser light is typical classical light, in that it closely simulates the coherent state of light, while in recent years light with non-classical quality has claimed more and more attention.

The role of laser theory is to clarify the character and quality of laser light and to show how it arises. The Haken school considered the laser linewidth and the amplitude distribution, while Scully considered the number distribution of laser photons. Laser linewidth and photon number distribution are complementary aspects of the same laser phenomenon viewed from wave phase or corpuscular viewpoints. Analysis of a laser from these viewpoints is involved because of the interaction of many atoms and the optical field as well as the pumping and damping processes. Thus, a common recipe for treating the laser field is to assume a single-mode field and reduce the number of degrees of freedom of the field to one. Then one has a single time-dependent variable for the field or a photon distribution for a single mode. The cost of reducing the number of degrees of freedom for the field to one is to lose information regarding the spatial field distribution, especially the relation between the fields inside and outside the laser cavity.

The theme of this book is to discuss how to deal with this defect of standard laser theories. To fully incorporate the field degrees of freedom in a laser is to treat the output coupling of the laser cavity rigorously. When the output coupling loss of the cavity is incorporated, cavity mode quantization becomes a difficult task because of the associated losses. Usual field quantization, relying on the field expansion in terms of orthogonal modes, becomes impossible because decaying
cavity modes are non-orthogonal. A direct approach to this problem is to set the laser cavity in a much larger cavity that simulates the “universe.” Quantization is accomplished using the normal modes of the larger cavity that includes the laser cavity. The cost of this procedure is to have an infinite number of field modes instead of the single mode in conventional theories.

The burden of the infinite number of field modes can be relaxed if we go to a collective field variable expressing the total electric field. Then, the laser equation of motion can be solved for the total electric field. Thermal or vacuum fluctuation affecting the laser field is incorporated automatically in this procedure. Quantum noise is introduced as the fluctuating force associated with the decay of the atomic dipole. The resulting expressions for the laser linewidth both below and above threshold have a common correction factor compared with the formula resulting from the theory assuming a single mode. This factor, called the excess noise factor, attracted the attention of many scientists, who discussed the origin of the factor. Various approaches to derive the factor have been published. In particular, Siegman proposed that the excess noise factor is the result of non-orthogonality or bi-orthogonality of cavity modes. The non-orthogonality is, in turn, a consequence of the open character of the laser cavity as compared to the closed structure of a fictitious “single”-mode cavity.

Using the orthogonal modes of the “universe,” it can be shown that the relation between the field inside and outside the cavity is not determined simply by the transmission coefficient of the cavity mirror, because the thermal or vacuum field exists everywhere. Outside the cavity, the total field is the sum of the transmitted field and the ambient thermal or vacuum field.

In this book, we present a laser theory that takes into account the output coupling of the cavity and uses the orthogonal modes of the “universe.” We analyze the wave aspect of the laser field in both a semiclassical and quantum-mechanical manner. In the quantum-mechanical analysis, we obtain the excess noise factor. We also present a simplified method to avoid the use of the modes of the “universe” where again the excess noise factor is derived. We analyze the spontaneous emission process in a cavity with output coupling to show that the respective spontaneous emission process in a cavity is not enhanced by the excess noise factor. In order to consider the physical origin of the excess noise factor, the theories of the excess noise factor are surveyed. Also, to compare the method taken in this book with other methods to treat output coupling, quantum theories on cavity output coupling or the input–output relation are surveyed.

We begin in Chapter 1 with a classical analysis of one-dimensional optical cavities with output coupling. Chapter 2 gives a quantum-mechanical analysis of the same cavities embedded in a larger cavity. Chapter 3 describes the necessary preliminaries for a quantum-mechanical laser analysis. This includes the Langevin force for the field in the case of the single-mode approximation, and those for atomic polarization and atomic inversion. As a reference for a full laser analysis that incorporates cavity output coupling, a laser theory assuming a single mode, which we call a quasimode, is presented in Chapter 4. Standard, conventional results on laser operation, especially on laser linewidth, are derived. Chapter 5
displays, for a laser with output coupling, the complete equations of motion for the field modes, atomic polarizations, and atomic inversions. The atomic variables are eliminated to obtain an equation for the total field. In Chapter 6 is shown the contour integral method of solution for the field equation utilizing the poles in the normalization constant in the field mode functions.

Chapter 7 gives semiclassical, linear gain analysis. Ignoring the Langevin forces and the gain saturation effect, it solves the field equation of motion using the Fourier expansion of the normalization constant of the field mode functions. The space-time structure of the linear build-up of the field is clarified. Chapter 8, giving a semiclassical, nonlinear gain analysis, improves Chapter 7 by incorporating the saturation effect in atomic inversion, but still ignoring the Langevin forces. The steady-state field distribution is determined. Chapter 9 improves Chapter 7 by incorporating the Langevin forces, but ignoring the saturation effect. This amounts to a quantum, linear gain analysis. The laser linewidth below threshold is determined. Chapter 10 gives a quantum, nonlinear gain analysis. This includes both the Langevin forces and the saturation effect, summarizing the results of the previous three chapters. The expression for the linewidth is shown to have two corrections compared to that in Chapter 4, one of which is the excess noise factor. Chapter 11 presents a simplified method of laser analysis that combines the effects of the Langevin forces and optical boundary conditions for traveling waves. Chapter 12 summarizes the results obtained in Chapters 7–11 and discusses various physical aspects of laser oscillation. The spontaneous emission process in a cavity with output coupling is analyzed in Chapter 13. Chapter 14 surveys theories of excess noise factor and finally Chapter 15 surveys quantum theories of output coupling.

The book is structured so that the reader can begin with basic quantum-mechanical knowledge and step up to rather complicated laser wave analyses. Knowledge of preliminary quantum mechanics, some preliminary operator algebra, simple contour integrals, Fourier transforms, and differential equations is assumed. Knowledge of elementary laser theory is also assumed. Knowledge of basic semiclassical laser analysis is preferable. Leaps in transforming one equation into the next are avoided as often as possible. Wherever the description of a topic is short and poor, the relevant literature is cited for the reader’s reference. Problems are provided in Chapters 1–5.

Fully quantum-mechanical theories of excess noise or output coupling exist, but most of them are in a sophisticated form. Unfortunately, to treat a realistic cavity is involved, as we will see in this book. But theories to be compared easily with experiments will be of particular importance in view of the developing field of non-classical light. We hope some of the published papers cited in this book meet this demand. There is no doubt that future papers will appear to improve the situation.

Kikuo Ujihara  
Tokyo, August 2009
This page intentionally left blank
Acknowledgments

Thanks are due to Ichiro Takahashi for reading the manuscript.

The author is grateful to the authors of various papers who readily granted permission to cite portions of their work or to reproduce their figures. He also acknowledges the publishers of the journals or books, The American Physical Society (Ref. [30] of Chapter 14, Refs [6, 7, 14] of Chapter 15, Figure 13.1–13.4, and Figures 14.2–14.4), The Optical Society of America (Ref. [15] of Chapter 15, and Figure 15.1), The IEEE (Refs [6, 8] of Chapter 14), and Springer-Verlag (Ref. [25] of Chapter 14), and Ref. [2] of Appendices), for giving permission to cite or reproduce figures from their journals or books. The origins of the citations and figures are individually given in the book.
This page intentionally left blank
1
A One-Dimensional Optical Cavity with Output Coupling: Classical Analysis

In this chapter, a one-dimensional optical cavity with output coupling is considered. The optical cavity has transmission loss at one or both of the end surfaces. The classical, natural cavity mode is defined, and decaying or growing mode functions are derived using the cavity boundary conditions. A series of resonant modes appears. But these modes are not orthogonal to each other and are not suitable for quantum-mechanical analysis of the optical field inside or outside of the cavity. Hypothetical boundaries are added at infinity in order to obtain orthogonal wave mode functions that satisfy the cavity and infinity boundary conditions. These new mode functions are suitable for field quantization, where each mode is quantized separately and the electric field of an optical wave is made up of contributions from each mode. Some results of quantization are described in the next chapter. Chapter 3 deals with the usual quasimode model: a perfect cavity with distributed internal loss or with a fictitious loss reservoir.

1.1
Boundary Conditions at Perfect Conductor and Dielectric Surfaces

In a source-free space, the electric field \( E \) and the magnetic field \( H \) described using a vector potential \( A \) satisfy the following equations:

\[
\nabla^2 A(r) - \frac{1}{c^2} \left( \frac{\partial}{\partial t} \right)^2 A(r) = 0
\]

(1.1)

\[
E(r) = -\frac{\partial}{\partial t} A(r)
\]

(1.2)

\[
H(r) = \frac{1}{\mu} \nabla \times A(r)
\]

(1.3)
where \( c \) is the velocity of light and \( \mu \) is the magnetic permeability of the medium.

We work in a Coulomb gauge where

\[
\text{div} \mathbf{A}(\mathbf{r}) = 0
\]  

(1.4)

In this chapter we consider one-dimensional, plane vector waves that are polarized in the \( x \)-direction and propagated to the \( z \)-direction. Therefore we write

\[
\mathbf{A}(\mathbf{r}) = A(z, t) \mathbf{x}
\]  

(1.5)

where \( \mathbf{x} \) is the unit vector in the \( x \)-direction. At the surface of a perfect conductor that is vertical to the \( z \)-axis, the tangential component of the electric vector vanishes. The tangential component of the magnetic field should be proportional to the surface current. In the absence of a forced current, this condition is automatically satisfied: the magnetic field that is consistent with the electric field induces the necessary surface current. At the interface between two dielectric media, or at the interface between a dielectric medium and vacuum, the tangential components of both the electric and magnetic fields must be continuous. Thus, at the surface \( z_c \) of a perfect conductor,

\[
\frac{\partial}{\partial t} A(z_c, t) = 0
\]  

(1.6)

and at the interface \( z_i \) of dielectrics 1 and 2,

\[
\frac{\partial}{\partial t} A_1(z_i, t) = \frac{\partial}{\partial t} A_2(z_i, t)
\]  

(1.7)

\[
\left. \frac{\partial}{\partial z} A_1(z, t) \right|_{z=z_i} = \left. \frac{\partial}{\partial z} A_2(z, t) \right|_{z=z_i}
\]  

(1.8)

In Equation 1.8 we have dropped the magnetic permeability \( \mu_1 \) and \( \mu_2 \), assuming that both of them are equal to that in vacuum, \( \mu_0 \), which is usually valid in the optical region of the frequency spectrum.

1.2 Classical Cavity Analysis

1.2.1 One-Sided Cavity

Consider a one-sided cavity depicted in Figure 1.1. This cavity consists of a lossless non-dispersive dielectric of dielectric constant \( \varepsilon_1 \), which is bounded by a perfect conductor at \( z = -d \) and vacuum at \( z = 0 \). The outer space \( 0 < z \) is a vacuum of dielectric constant \( \varepsilon_0 \). Subscripts 1 and 0 will be used for the regions \(-d < z < 0\) and \( 0 < z \), respectively. The velocity of light in the regions 1 and 0 are \( c_1 \) and \( c_0 \), respectively.
The natural oscillating field mode of the cavity, the cavity resonant mode, is defined as the mode that has only an outgoing wave in the outer space $0 < z$. For reasons that will be described in Chapter 14, we also derive a mode that has only an incoming wave outside. For simplicity, let us call these the outgoing mode and incoming modes, respectively. Let the mode functions be

$$A(z, t) = u(z)e^{-ikt}, \quad -d < z < 0$$

$$= ve^{-i(\omega t + k_0 z)}, \quad 0 < z$$

where $v$ is a constant. We define the wavenumber $k$ by

$$k_i = \omega / c_i, \quad i = 0, 1$$

The upper and lower signs in the second line in Equation 1.9 are for the outgoing mode and the incoming mode, respectively. Substituting Equation 1.9 into Equation 1.1 via Equation 1.5 we obtain

$$-\frac{\omega^2}{c_i^2} u = \left( \frac{d}{dz} \right)^2 u, \quad -d < z < 0$$

$$k_0 = \frac{\omega}{c_0}, \quad 0 < z$$

Thus we can set

$$u(z) = Ae^{ik_1 z} + Be^{-ik_1 z}$$

$$v = C$$

where $k_1 = \omega c_1 / c_0$. Putting this into Equation 1.6 for $z = -d$ and into Equations 1.7 and 1.8 for $z = 0$, we obtain

$$Ae^{-ik_1 d} + Be^{ik_1 d} = 0$$

$$A + B = C$$

$$ik_1 (A - B) = \pm ik_0 C$$

We then have

$$e^{2ik_1 d} = \frac{\mp k_0 - k_1}{k_1 \mp k_0} = \frac{\mp c_1 - c_0}{c_0 \mp c_1}$$
For the outgoing mode (upper sign) we have

\[ e^{2ik_1d} = \frac{-c_1 - c_0}{c_0 - c_1} = -\frac{c_0 + c_1}{c_0 - c_1} \]  

(1.15)

Because we are assuming that both \( c_1 \) and \( c_0 \) are real and that the velocity of light in the dielectric is smaller than that in vacuum \( (c_1 < c_0) \), \( k_1 \) is a complex number \( K_{1\text{out}} \). We reserve \( k_1 \) for the real part of \( K_{1\text{out}} \). Then we obtain

\[ K_{1\text{out},m} = k_{1m} - i\gamma \]

\[ k_{1m} = \frac{1}{2d}(2m + 1)\pi, \quad m = 0, 1, 2, 3, \ldots \]  

(1.16)

\[ \gamma = \frac{1}{2d} \ln\left(\frac{c_0 + c_1}{c_0 - c_1}\right) = \frac{1}{2d} \ln\left(\frac{1}{r}\right) \]

There is an eigenmode every \( \pi/d \) in the wavenumber. Note that the imaginary part is independent of the mode number. The coefficient

\[ r = \frac{c_0 - c_1}{c_0 + c_1} \]  

(1.17)

is the amplitude reflectivity of the coupling surface, \( z = 0 \), for the wave incident from the left, that is, from inside the cavity. The corresponding eigenfrequency of the mode is

\[ \Omega_m \equiv \Omega_{1\text{out},m} = \omega_{cm} - i\gamma_c \]

\[ \omega_{cm} = \frac{c_1}{2d}(2m + 1)\pi, \quad m = 0, 1, 2, 3, \ldots \]  

(1.18a)

\[ \gamma_c = \frac{c_1}{2d} \ln\left(\frac{c_0 + c_1}{c_0 - c_1}\right) = \frac{c_1}{2d} \ln\left(\frac{1}{r}\right) \]

where we have defined the complex angular frequency \( \Omega_m \). In subsequent chapters, a typical cavity eigenfrequency, with a certain large number \( m \), will be denoted as

\[ \Omega_c = \omega_c - i\gamma_c \]  

(1.18b)

The separation of the mode frequencies is \( \Delta \omega_c = c_1 \pi/d \).

Likewise, for the incoming mode (lower sign) we have

\[ e^{2ik_1d} = \frac{c_1 - c_0}{c_0 + c_1} = -\frac{c_0 - c_1}{c_0 + c_1} \]  

(1.19)

from which we obtain

\[ K_{1\text{in},m} = K^*_{1\text{out},m} = k_{1m} + i\gamma \]  

(1.20a)

and

\[ \Omega_{1\text{in},m} = \Omega^*_{1\text{out},m} = \omega_{cm} + i\gamma_c \equiv \Omega^*_m \]  

(1.20b)
Going back to Equation 1.13 we now get the ratios of $A$, $B$, and $C$. Thus, except for an undetermined constant factor, for the outgoing mode we have

$$A(z,t) = u_m(z)e^{-i\Omega_m t}$$  \hspace{1cm} (1.21a)

$$u_m(z) = \begin{cases} \sin(\Omega_m(z + d)/c_1), & -d < z < 0 \\ \sin(\Omega_m d/c_1)e^{i\Omega_m(z)/\omega_0}, & 0 < z \end{cases}$$  \hspace{1cm} (1.21b)

and for the incoming mode we have

$$A(z,t) = \tilde{u}_m(z)e^{-i\Omega_m^* t}$$  \hspace{1cm} (1.22a)

$$\tilde{u}_m(z) = \begin{cases} \sin(\Omega_m^*(z + d)/c_1), & -d < z < 0 \\ \sin(\Omega_m^* d/c_1)e^{-i\Omega_m(z)/\omega_0}, & 0 < z \end{cases}$$  \hspace{1cm} (1.22b)

where the suffix $m$ signifies the cavity mode. We note that the outgoing mode is temporally decaying whereas the incoming mode is growing. Inside the cavity, the field is a superposition of a pair of right-going and left-going waves with decaying or growing amplitudes. We note that $\tilde{u}_m(z) = u^*_m(z)$, meaning that the complex conjugate of the incoming mode function is the time-reversed outgoing mode function.

We also note that different members of the outgoing mode are non-orthogonal in the sense that

$$\int_{-d}^{0} u^*_m(z) u_{m'}(z)dz \neq 0, \hspace{1cm} m \neq m'$$  \hspace{1cm} (1.23)

Similarly, members of the incoming mode are mutually non-orthogonal. However, a member of the outgoing mode and a member of the incoming mode are approximately orthogonal. That is, if normalized properly, it can be shown that

$$\int_{-d}^{0} \tilde{u}^*_{out,m}(z) u_{in,m'}(z)dz \simeq \delta_{m,m'}$$  \hspace{1cm} (1.24)

The approximation here neglects the integrals of spatially rapidly oscillating terms. This is justified when the cavity length $d$ is much larger than the optical wavelength $\lambda_k = 2\pi c_1/\omega_k$ or when $m \gg 1$ in Equation 1.16. These relationships among the outgoing and incoming mode functions will be discussed in Chapter 14 in relation to the quantum excess noise or the excess noise factor of a laser.

### 1.2.2 Symmetric Two-Sided Cavity

Consider a symmetrical, two-sided cavity depicted in Figure 1.2. This cavity consists of a lossless non-dispersive dielectric of dielectric constant $\epsilon_1$, which is
bounded by external vacuum at both \( z = -d \) and \( z = d \). Subscripts 1 and 0 will be used for the internal region \(-d < z < d\) and external region \( d < z \) and \( z < -d\), respectively. The velocity of light in the regions 1 and 0 are \( c_1 \) and \( c_0 \), respectively.

Let the mode functions be

\[
A(z, t) = u(z)e^{-i\omega t}, \quad -d < z < d \\
= ve^{-i(\omega t + k_0 z)}, \quad d < z \\
= we^{-i(\omega t - k_0 z)}, \quad z < -d
\]

where again the upper signs are for the outgoing mode and the lower ones are for the incoming mode, and both \( v \) and \( w \) are constants. Following a similar procedure as above, this time we get symmetric and antisymmetric mode functions for both outgoing and incoming modes.

The symmetric outgoing mode function is (problem 1-1)

\[
A(z, t) = \begin{cases} 
\cos(\Omega z/c_1)e^{-i\Omega t}, & -d < z < d \\
\cos(\Omega d/c_1)e^{-i\Omega (1-z-d)/c_0}, & d < z \\
\cos(\Omega d/c_1)e^{-i\Omega (1+z+d)/c_0}, & z < -d
\end{cases}
\]

where

\[
\Omega = \Omega_m = \omega_m - i\gamma_c \\
\omega_m = \frac{c_1}{d} m\pi, \quad m = 0, 1, 2, 3, \ldots \\
\gamma_c = \frac{c_1}{2d} \ln \left( \frac{c_0 + c_1}{c_0 - c_1} \right) = \frac{c_1}{2d} \ln \left( \frac{1}{r} \right)
\]

The antisymmetric outgoing mode function is

\[
A(z, t) = \begin{cases} 
\sin(\Omega z/c_1)e^{-i\Omega t}, & -d < z < d \\
\sin(\Omega d/c_1)e^{-i\Omega (1-z-d)/c_0}, & d < z \\
-\sin(\Omega d/c_1)e^{-i\Omega (1+z+d)/c_0}, & z < -d
\end{cases}
\]

Figure 1.2 The symmetrical two-sided cavity model.
where

\begin{align*}
\Omega &= \Omega_m = \omega_m - \mathrm{i} \gamma_c \\
\omega_m &= \frac{c_1}{2d} (2m + 1) \pi, \quad m = 0, 1, 2, 3, \ldots \\
\gamma_c &= \frac{c_1}{2d} \ln \left( \frac{c_0 + c_1}{c_0 - c_1} \right) = \frac{c_1}{2d} \ln \left( \frac{1}{r} \right) 
\end{align*}

The symmetric and antisymmetric incoming mode functions are given by Equations 1.26 and 1.28, respectively, with \( \Omega_m \) replaced by \( \Omega_m^* \). Note that the antisymmetric mode functions for \( 0 < z \), if shifted to the left by \( d (z \to z + d) \), coincide with the mode functions for the one-sided cavity in Equations 1.21a and 1.22a, as is expected from the mirror symmetry of the two-sided cavity. The relations 1.23 and 1.24 also hold in this cavity model.

1.3 Normal Mode Analysis: Orthogonal Modes

As we have seen in the previous section, the natural resonant modes (outgoing mode) of the cavity, as well as the associated incoming modes, are non-orthogonal and associated with time-decaying or growing factors. This feature is not suitable for straightforward quantization. For straightforward quantization, we need orthogonal, stationary modes describing the cavity. For this purpose, we introduce artificial boundaries at large distances so as to get such field modes.

1.3.1 One-Sided Cavity

1.3.1.1 Mode Functions of the “Universe”

For the one-sided cavity, we add a perfectly reflective boundary of a perfect conductor at \( z = L \) as in Figure 1.3. Then we have three boundaries: at \( z = -d \) and \( z = L \) the boundary condition 1.6 applies, whereas at \( z = 0 \) the conditions 1.7 and 1.8 apply. The region \( -d < z < L \) is our “universe,” within which the region \( -d < z < 0 \) is the cavity and the region \( 0 < z < L \) is the outside space.

\begin{center}
\textbf{Figure 1.3} The one-sided cavity embedded in a large cavity.
\end{center}
Here, again, subscripts 1 and 0 will be used for the regions \(-d < z < 0\) and \(0 < z < L\), respectively. Assuming, again, the form of Equation 1.5 for the field, we assume the following form of the field:

\[
A(z, t) = Q(t) U(z)
\]  

(1.30)

We try solutions of the form:

\[
A_1(z, t) = Q(t) U_1(z, t), \quad -d < z < 0
\]

(1.31a)

\[
A_0(z, t) = Q(t) U_0(z, t), \quad 0 < z < L
\]

(1.31b)

Then Equation 1.1 gives

\[
\left( \frac{d}{dt} \right)^2 U(t) + \omega^2 Q(t) = 0
\]  

(1.32a)

and

\[
\left( \frac{d}{dz} \right)^2 U_1(z) + (k_1)^2 U_1(z) = 0
\]  

(1.32b)

\[
\left( \frac{d}{dz} \right)^2 U_0(z) + (k_0)^2 U_0(z) = 0
\]

where

\[
k_i = \omega/c_i = \omega(c_i \mu_0)^{1/2}, \quad i = 0, 1
\]  

(1.33)

Thus we assume the following spatial form:

\[
U(z) = \begin{cases} 
U_1(z) = a_1 e^{ik_1z} + b_1 e^{-ik_1z}, & -d < z < 0 \\
U_0(z) = a_0 e^{ik_0z} + b_0 e^{-ik_0z}, & 0 < z < L
\end{cases}
\]  

(1.34)

Applying the boundary conditions yields

\[
a_1 e^{-ik_0L} + b_1 e^{ik_0L} = 0
\]  

(1.35a)

\[
a_1 + b_1 = a_0 + b_0
\]  

(1.35b)

\[
a_1 k_1 - b_1 k_1 = a_0 k_0 - b_0 k_0
\]  

(1.35c)

\[
a_0 e^{ik_0L} + b_0 e^{-ik_0L} = 0
\]  

(1.35d)

For non-vanishing coefficients, we need the determinantal equation (problem 1-2)

\[
tan(k_0L) = -(k_0/k_1) tan(k_1d)
\]  

(1.36)
or
\[
c_1 \tan \frac{\omega d}{c_1} + c_0 \tan \frac{\omega L}{c_0} = 0 \tag{1.37}
\]

Under this condition, the function \(A\) can be determined except for a constant factor as
\[
A_1(z,t) = \int \sin k_1(z + d) \cos(\omega t + \phi), \\
A_0(z,t) = \int \frac{k_1 \cos k_1 d}{k_0 \cos k_0 L} \sin k_0(z - L) \cos(\omega t + \phi)
\]
\[
= \int \left( \frac{k_1}{k_0} \cos k_1 d \sin k_0 z + \sin k_1 d \cos k_0 z \right) \cos(\omega t + \phi), \\
0 < z < L
\]

where \(\phi\) is an arbitrary phase and \(f\) is an arbitrary constant. Equation 1.37 has been used in the last line.

1.3.1.2 Orthogonal Spatial Modes of the “Universe”
Now the allowed values of \(k_0, 1\) or \(\omega\) are determined by Equation 1.37. If we choose a large \(L, L \gg d\), it can be seen that the solution is distributed rather uniformly with approximate frequency, in \(k_0\), of \(\pi / L\), and that there is no degeneracy in \(k_0\) and thus in \(\omega\). It can be shown that the space part of the \(j\)th mode functions in Equation 1.38, that is,
\[
U_j(z) = \begin{cases} 
\sin k_j(z + d), & -d < z < 0 \\
\left( \frac{k_j}{k_0} \cos k_1 d \sin k_0 z + \sin k_1 d \cos k_0 z \right), & 0 < z < L
\end{cases}
\]

form an orthogonal set in the sense that
\[
\int_{-d}^{L} \varepsilon(z) U_i(z) U_j(z) dz = 0, \quad i \neq j \tag{1.40a}
\]

To show this relation, let us consider the integral
\[
I = \int_{-d}^{L} \frac{1}{\mu_0} \frac{\partial}{\partial z} U_i(z) \frac{\partial}{\partial z} U_j(z) dz
\]
\[
= \frac{1}{\mu_0} U_i(z) \frac{\partial}{\partial z} U_j(z) \bigg|_{-d}^{0} + \frac{1}{\mu_0} U_i(z) \frac{\partial}{\partial z} U_j(z) \bigg|_{L}^{0} - \frac{1}{\mu_0} \int_{-d}^{0} U_i(z) \left( \frac{\partial}{\partial z} \right)^2 U_j(z) dz - \frac{1}{\mu_0} \int_{0}^{L} U_i(z) \left( \frac{\partial}{\partial z} \right)^2 U_j(z) dz
\]
\[
\frac{k_{ij}^2}{\mu_0} \int_{-d}^{0} U_i(z) U_j(z) dz + \frac{k_{0j}^2}{\mu_0} \int_{0}^{L} U_i(z) U_j(z) dz = \omega_j^2 \left( \int_{-d}^{0} \varepsilon_1 U_i(z) U_j(z) dz + \int_{0}^{L} \varepsilon_0 U_i(z) U_j(z) dz \right) 
\]

\[
= \omega_j^2 \int_{-d}^{L} \varepsilon(z) U_i(z) U_j(z) dz 
\]

(1.40b)

In the second line, the values at \( z = -d \) and \( z = L \) vanish because of the condition on the perfect boundary, while the values at \( z = 0 \) cancel because of the continuity of both the function and its derivative. The Helmholtz equation 1.32a and 1.32b was used on going from the third to the fourth line. Finally, Equation 1.33 was used to go to the fifth line. Because we can interchange \( U_i(z) \) and \( U_j(z) \) in the first line, we also have

\[
I = \omega_i^2 \int_{-d}^{L} \varepsilon(z) U_i(z) U_j(z) dz 
\]

(1.40c)

Thus we have

\[
0 = \left( \omega_j^2 - \omega_i^2 \right) \int_{-d}^{L} \varepsilon(z) U_i(z) U_j(z) dz 
\]

(1.40d)

Since the modes are non-degenerate, the integral must vanish, which proves Equation 1.40a.

1.3.1.3 Normalization of the Mode Functions of the “Universe”

For later convenience, we normalize the mode function 1.39 as

\[
\begin{align*}
U_j(z) &= N_j u_j(z) \\
U_j(z) &= \begin{cases} 
\sin k_{ij}(z + d), & -d < z < 0 \\
\left( \frac{k_{ij}}{k_{0j}} \cos k_{0j}d \sin k_{0j}z + \sin k_{ij}d \cos k_{0j}z \right), & 0 < z < L 
\end{cases} 
\end{align*}
\]

(1.41b)

with the orthonormality property

\[
\int_{-d}^{L} \varepsilon(z) U_i(z) U_j(z) dz = \delta_{ij} 
\]

(1.42a)

where the Kronecker delta symbol

\[
\delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j 
\end{cases} 
\]

(1.42b)