HEAT CONDUCTION
To Allison
-DWH

To Gül
-MNO
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The decision to take on the third edition of Professor Özişik’s book was not one that I considered lightly. Having taught from the second edition for more than a dozen years and to nearly 500 graduate students, I was intimately familiar with the text. For the last few years I had considered approaching Professor Özişik with the idea for a co-authored third edition. However, with his passing in October 2008 at the age of 85, before any contact between us, I was faced with the decision of moving forward with a revision on my own. Given my deep familiarity and appreciation for Professor Özişik’s book, it was ultimately an easy decision to proceed with the third edition.

My guiding philosophy to the third edition was twofold: first, to preserve the spirit of the second edition as the standard contemporary analytic treatment of conduction heat transfer, and, second, to write a truly major revision with goals to improve and advance the presentation of the covered material. The feedback from literally hundreds of my students over the years combined with my own pedagogical ideas served to shape my overall approach to the revision. At the end of this effort, I sincerely feel that the result is a genuine collaboration between me and Professor Özişik, equally combining our approaches to conduction heat transfer. As noted in the second edition, this book is meant to serve as a graduate-level textbook on conduction, as well as a comprehensive reference for practicing engineers and scientists. The third edition remains true to these goals.

I will now attempt to summarize the changes and additions to the third edition, providing some commentary and motivating thoughts. Chapter 1 is very much in the spirit of a collaborative effort, combining the framework from the second edition with significant revision. Chapter 1 presents the concepts of conduction starting with the work of Fourier and providing a detailed derivation of the
heat equation. We now present both differential and integral derivations, which together present additional insight into the governing heat equation, notably with regard to conservation of energy. Extension of the heat equation to other coordinate systems, partial lumping of the heat equation, and the detailed treatment of relevant boundary conditions complete the chapter.

Chapter 2 is completely new in the third edition, presenting the characteristic value problem (the Sturm–Liouville problem) and the concept of orthogonal functions in considerable detail. We then develop the trigonometric orthogonal functions, Bessel functions, and the Legendre polynomials, followed by a rigorous presentation of the Fourier series, and finally the Fourier integrals. This material was dispersed in Chapters 2–4 in the second edition. The current unified treatment is envisioned to provide a more comprehensive foundation for the following chapters, while avoiding discontinuity through dispersion of the material over many chapters.

Chapters 3, 4, and 5 present the separation of variables method for Cartesian, cylindrical, and spherical coordinate systems, respectively. The organization differs from the second edition in that we first emphasize the steady-state solutions and then the transient solutions, with the concept of superposition presented in Chapter 3. In particular, the superposition schemes are presented using a more systematic approach and added illustrative figures. The many tables of characteristic value problems and resulting eigenfunctions and eigenvalues are retained in this edition in Chapter 2, although the approach from the second edition of developing more generic solutions in conjunction with these tables was deemphasized. Finally, each of these chapters now ends with a capstone example problem, attempting to illustrate the numerical implementation of representative analytic solutions, with goals of discussing the numerical convergence of realization of our analytic solutions, as well as emphasizing the concepts of energy conservation. A significant change in the third edition was the removal of all semi-infinite and infinite domain material from these three chapters and consolidating this material into Chapter 6, a new chapter dedicated to the semi-infinite and infinite domain problems in the context of the Fourier integral. Pedagogically, it was felt that this topic was better suited to a dedicated chapter, rather than treated along with separation of variables. Overall, the revised treatment in Chapters 2–6 will hopefully assist students in learning the material, while also greatly improving the utility of the third edition as a reference book. Additional homework problems have been added throughout.

Chapters 7, 8, and 9 focus on the treatment of nonhomogeneities in heat conduction problems, notably time-dependent nonhomogeneities, using Duhamel’s theorem, the Green’s function approach, and the Laplace transform method, respectively. A new derivation of Duhamel’s theorem is presented along with a new presentation of Duhamel’s various solutions that is intended to clarify the use and limitations of this method. Also included in Chapter 7 is a new table giving closed-form solutions for various surface temperature functions. The Green’s function chapter contains new sample problems with a greater variety of boundary condition types. The Laplace transform chapter contains additional
problems focusing on the general solution method (i.e., not limited to the small-time approximations), and the Laplace transform tables are greatly expanded to facilitate such solutions. Together, Chapters 1–9 are considered the backbone of a graduate course on conduction heat transfer, with the remainder of the text providing additional topics to pursue depending on the scope of the class and the interests of the instructor and students.

Chapters 10, 11, and 12 cover the one-dimensional composite medium, the moving heat source problem, and phase-change heat transfer, respectively, much along the lines of the original treatment by Professor Özisik, although efforts were made to homogenize the style with the overall revision. Chapter 13 is on approximate analytic methods and takes a departure from the exact analytic treatment of conduction presented in the first dozen chapters. The emphasis on the integral method and method of residuals has applications to broader techniques (e.g., finite-element methods), although efforts are made in the current text to emphasize conservation of energy and formulation in the context of these methods. Chapter 14 presents the integral transform technique as a means for solution of the heat equation under a variety of conditions, setting the foundation for use of the integral transform technique in Chapter 15, which focuses on heat conduction in anisotropic solids.

Chapter 16 is a totally new contribution to the text and presents an introduction to microscale heat conduction. There were several motivating factors for the inclusion of Chapter 16. First, engineering and science fields are increasingly concerned with the micro- and nanoscales, including with regard to energy transfer. Second, given that the solution of the heat equation in conjunction with Fourier’s law remains at the core of this book, it is useful to provide a succinct treatment of the limitations of these equations.

The above material considerably lengthened the manuscript with regard to the corresponding treatment in the second edition, and therefore, some material was omitted from the revised edition. It is here that I greatly missed the opportunity to converse with Professor Özisik, but ultimately, such decisions were mine alone. The lengthy treatment of finite-difference methods and the chapter on inverse heat conduction from the second edition were omitted from the third edition. With regard to the former, the logic was that the strength of this text is the analytic treatment of heat conduction, noting that many texts on numerical methods are available, including treatment of the heat equation. The latter topic has been the subject of entire monographs, and, therefore, the brief treatment of inverse conduction was essentially replaced with the introduction to microscale heat transfer.

I would like to express my sincere gratitude to the many heat transfer students that have provided me with their insight into a student’s view of heat conduction and shared their thoughts, both good and bad, on the second edition. You remain my primary motivation for taking on this project. A few of my former students have reviewed chapters of the third edition, and I would like to express my thanks to Fotouh Al-Ragom, Roni Plachta, Lawrence Stratton and Nadim Zgheib for their feedback. I would also like to thank my graduate student Philip Jackson for his
invaluable help with the numerical implementation of the Legendre polynomials for the capstone problem in Chapter 5. I would like to acknowledge several of my colleagues at the University of Florida, including Andreas Haselbacher for useful discussions and feedback regarding Chapter 1, Renwei Mei for his reviews of several chapters and his very generous contribution of Table 1 in Chapter 7, Greg Sawyer for his suggestion of the capstone problem in Chapter 5, as well as for many motivating discussions, and Simon Phillpot for his very insightful review of Chapter 16. I would like to thank my editors at John Wiley & Sons, Daniel Magers and Bob Argentieri, for their support and enthusiasm from the very beginning, as well as for their generous patience as this project neared completion.

On a more personal note, I want to thank my wonderful children, Katherine, William, and Mary-Margaret, for their support throughout this process. Many of the hours that I dedicated to this manuscript came at their expense, and I will never forget their patience with me to the very end. I also thank Mary-Margaret for her admirable proof-reading skills when needed. The three of you never let me down. Finally, I thank the person that gave me the utmost support throughout this project, my wife and dearest partner in life, Allison. This effort would never have been completed without your unwavering support, and it is to you that I dedicate this book.

David W. Hahn

Gainesville, Florida
In preparing the second edition of this book, the changes have been motivated by the desire to make this edition a more application-oriented book than the first one in order to better address the needs of the readers seeking solutions to heat conduction problems without going through the details of various mathematical proofs. Therefore, emphasis is placed on the understanding and use of various mathematical techniques needed to develop exact, approximate, and numerical solutions for a broad class of heat conduction problems. Every effort has been made to present the material in a clear, systematic, and readily understandable fashion. The book is intended as a graduate-level textbook for use in engineering schools and a reference book for practicing engineers, scientists and researchers. To achieve such objectives, lengthy mathematical proofs and developments have been omitted, instead examples are used to illustrate the applications of various solution methodologies.

During the twelve years since the publication of the first edition of this book, changes have occurred in the relative importance of some of the application areas and the solution methodologies of heat conduction problems. For example, in recent years, the area of inverse heat conduction problems (IHCP) associated with the estimation of unknown thermophysical properties of solids, surface heat transfer rates, or energy sources within the medium has gained significant importance in many engineering applications. To answer the needs in such emerging application areas, two new chapters are added, one on the theory and application of IHCP and the other on the formulation and solution of moving heat source problems. In addition, the use of enthalpy method in the solution of phase-change problems has been expanded by broadening its scope of applications. Also, the chapters on the use of Duhamel’s method, Green’s function, and finite-difference
methods have been revised in order to make them application-oriented. Green’s function formalism provides an efficient, straightforward approach for developing exact analytic solutions to a broad class of heat conduction problems in the rectangular, cylindrical, and spherical coordinate systems, provided that appropriate Green’s functions are available. Green’s functions needed for use in such formal solutions are constructed by utilizing the tabulated eigenfunctions, eigenvalues and the normalization integrals presented in the tables in Chapters 2 and 3.

Chapter 1 reviews the pertinent background material related to the heat conduction equation, boundary conditions, and important system parameters. Chapters 2, 3, and 4 are devoted to the solution of time-dependent homogeneous heat conduction problems in the rectangular, cylindrical, and spherical coordinates, respectively, by the application of the classical method of separation of variables and orthogonal expansion technique. The resulting eigenfunctions, eigenconditions, and the normalization integrals are systematically tabulated for various combinations of the boundary conditions in Tables 2-2, 2-3, 3-1, 3-2, and 3-3. The results from such tables are used to construct the Green functions needed in solutions utilizing Green’s function formalism.

Chapters 5 and 6 are devoted to the use of Duhamel’s method and Green’s function, respectively. Chapter 7 presents the use of Laplace transform technique in the solution of one-dimensional transient heat conduction problems.

Chapter 8 is devoted to the solution of one-dimensional, time-dependent heat conduction problems in parallel layers of slabs and concentric cylinders and spheres. A generalized orthogonal expansion technique is used to solve the homogeneous problems, and Green’s function approach is used to generalize the analysis to the solution of problems involving energy generation.

Chapter 9 presents approximate analytical methods of solving heat conduction problems by the integral and Galerkin methods. The accuracy of approximate results are illustrated by comparing with the exact solutions. Chapter 10 is devoted to the formulation and the solution of moving heat source problems, while Chapter 11 is concerned with the exact, approximate, and numerical methods of solution of phase-change problems.

Chapter 12 presents the use of finite difference methods for solving the steady-state and time-dependent heat conduction problems. Chapter 13 introduces the use of integral transform technique in the solution of general time-dependent heat conduction equations. The application of this technique for the solution of heat conduction problems in rectangular, cylindrical, and spherical coordinates requires no additional background, since all basic relationships needed for constructing the integral transform pairs have already been developed and systematically tabulated in Chapters 2 to 4. Chapter 14 presents the formulation and methods of solution of inverse heat conduction problems and some background information on statistical material needed in the inverse analysis. Finally, Chapter 15 presents the analysis of heat conduction in anisotropic solids. A host of useful information, such as the roots of transcendental equations, some properties of Bessel functions, and the numerical values of Bessel functions and Legendre polynomials are included in Appendixes IV and V for ready reference.
I would like to express my thanks to Professors J. P. Bardon and Y. Jarny of University of Nantes, France, J. V. Beck of Michigan State University, and Woo Seung Kim of Hanyang University, Korea, for valuable discussions and suggestions in the preparation of the second edition.

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HEAT CONDUCTION
1

HEAT CONDUCTION FUNDAMENTALS

No subject has more extensive relations with the progress of industry and the natural sciences; for the action of heat is always present, it penetrates all bodies and spaces, it influences the processes of the arts, and occurs in all the phenomena of the universe.


All matter when considered at the macroscopic level has a definite and precise energy. Such a state of energy may be quantified in terms of a thermodynamic energy function, which partitions energy at the atomic level among, for example, electronic, vibrational, and rotational states. Under local equilibrium, the energy function may be characterized by a measurable scalar quantity called temperature. The energy exchanged by the constituent particles (e.g., atoms, molecules, or free electrons) from a region with a greater local temperature (i.e., greater thermodynamic energy function) to a region with a lower local temperature is called heat. The transfer of heat is classically considered to take place by conduction, convection, and radiation, and although it cannot be measured directly, the concept has physical meaning because of the direct relationship to temperature. Conduction is a specific mode of heat transfer in which this energy exchange takes place in solids or quiescent fluids (i.e., no convective motion resulting from the macroscopic displacement of the medium) from the region of high temperature to the region of low temperature due to the presence of a temperature gradient within the system. Once the temperature distribution $T(\hat{r}, t)$ is known within the medium as a function of space (defined by the position vector $\hat{r}$) and time (defined by scalar $t$), the flow of heat is then prescribed from the governing heat transfer laws. The study of heat conduction provides an enriching
HEAT CONDUCTION FUNDAMENTALS

combination of fundamental science and mathematics. As the prominent ther-
moodynamicist H. Callen wrote: “The history of the concept of heat as a form
of energy transfer is unsurpassed as a case study in the tortuous development
of scientific theory, as an illustration of the almost insuperable inertia presented
by accepted physical doctrine, and as a superb tale of human ingenuity applied
to a subtle and abstract problem” [2]. The science of heat conduction is princi-
pally concerned with the determination of the temperature distribution and flow
of energy within solids. In this chapter, we present the basic laws relating the
heat flux to the temperature gradient in the medium, the governing differential
equation of heat conduction, the boundary conditions appropriate for the analysis
of heat conduction problems, the rules of coordinate transformation needed for
working in different orthogonal coordinate systems, and a general discussion of
the various solution methods applicable to the heat conduction equation.

1-1 THE HEAT FLUX

Laws of nature provide accepted descriptions of natural phenomena based on
observed behavior. Such laws are generally formulated based on a large body
of empirical evidence accepted within the scientific community, although they
usually can be neither proven nor disproven. To quote Joseph Fourier from the
opening sentence of his Analytical Theory of Heat: “Primary causes are unknown
to us; but are subject to simple and constant laws, which may be discovered by
observation” [1]. These laws are considered general laws, as their application
is independent of the medium. Well-known examples include Newton’s laws of
motion and the laws of thermodynamics. Problems that can be solved using only
general laws of nature are referred to as deterministic and include, for example,
simple projectile motion.

Other problems may require supplemental relationships in addition to the
general laws. Such problems may be referred to as nondeterministic, and their
solution requires laws that apply to the specific medium in question. These addi-
tional laws are referred to as particular laws or constitutive relations. Well-known
examples include the ideal gas law, the relationship between shear stress and the
velocity gradient for a Newtonian fluid, and the relationship between stress and
strain for a linear-elastic material (Hooke’s law).

The particular law that governs the relationship between the flow of heat and
the temperature field is named after Joseph Fourier. For a homogeneous, isotropic
solid (i.e., material in which thermal conductivity is independent of direction),
Fourier’s law may be given in the form

\[ q''(\hat{r}, t) = -k\nabla T(\hat{r}, t) \quad \text{W/m}^2 \quad (1-1) \]

where the temperature gradient \( \nabla T(\hat{r}, t) \) is a vector normal to the isothermal
surface, the heat flux vector \( q''(\hat{r}, t) \) represents the heat flow per unit time, per unit
area of the isothermal surface in the direction of decreasing temperature gradient,
and \( k \) is the thermal conductivity of the material. The thermal conductivity is a positive, scalar quantity for a homogeneous, isotropic material. The minus sign is introduced in equation (1-1) to make the heat flow a positive quantity in the positive coordinate direction (i.e., opposite of the temperature gradient), as described below. This text will consider the heat flux in the SI units W/m\(^2\) and the temperature gradient in K/m (equivalent to the unit °C/m), giving the thermal conductivity the units of W/(m·K). In the Cartesian coordinate system (i.e., rectangular system), equation (1-1) is written as

\[
q''(x, y, z, t) = -\hat{i}k \frac{\partial T}{\partial x} - \hat{j}k \frac{\partial T}{\partial y} - \hat{k}k \frac{\partial T}{\partial z}
\]

(1-2)

where \( \hat{i}, \hat{j}, \) and \( \hat{k} \) are the unit direction vectors along the \( x, y, \) and \( z \) directions, respectively. One may consider the three components of the heat flux vector in the \( x, y, \) and \( z \) directions, respectively, as given by

\[
q''_x = -k \frac{\partial T}{\partial x} \quad q''_y = -k \frac{\partial T}{\partial y} \quad \text{and} \quad q''_z = -k \frac{\partial T}{\partial z}
\]

(1-3a,b,c)

Clearly, the flow of heat for a given temperature gradient is directly proportional to the thermal conductivity of the material. Equation (1-3a) is generally used for one-dimensional (1-D) heat transfer in a rectangular coordinate system. Figure 1-1 illustrates the sign convention of Fourier’s law for the 1-D Cartesian coordinate system. Both plots depict the heat flux (W/m\(^2\)) through the plane at \( x = x_0 \) based on the local temperature gradient. In Figure 1-1(a), the gradient \( dT/dx \) is negative with regard to the Cartesian coordinate system; hence the resulting flux is mathematically positive, and by convention is in the positive \( x \) direction, as shown in the figure. In contrast, Figure 1-1(b) depicts a positive gradient \( dT/dx \). This yields a mathematically negative heat flux, which by convention

Figure 1-1  Fourier’s law illustrated for a (a) positive heat flux and (b) a negative heat flux.
is in the negative \( x \) direction, as indicated in the figure. As defined, Fourier’s law is directly tied to the coordinate system, with positive heat flux always flowing in the positive coordinate direction. While determining the actual direction of heat flow is often trivial for 1-D problems, multidimensional problems, and notably transient problems, can present considerable difficulty in determining the direction of the local heat flux terms. Adherence to the sign convention of Fourier’s law will avoid any such difficulties of flux determination, which is useful in the context of overall energy conservation for a given heat transfer problem.

In addition to the heat flux, which is the flow of heat \textit{per unit area} normal to the direction of flow (e.g., a plane perpendicular to the page in Fig. 1-1), one may define the total heat flow, often called the \textit{heat rate}, in the unit of watts (W). The heat rate is calculated by multiplying the heat flux by the total cross-sectional area through which the heat flows for a 1-D problem or by integrating over the area of flow for a multidimensional problem. The heat rate in the \( x \) direction for one-, two-, and three-dimensional (1-D, 2-D, and 3-D) Cartesian problems is given by

\[
q_x = -k A_x \frac{dT}{dx} \text{ W} \quad (1-4)
\]

\[
q_x = -k H \int_{y=0}^{L} \frac{\partial T(x, y)}{\partial x} dy \text{ W} \quad (1-5)
\]

\[
q_x = -k \int_{y=0}^{L} \int_{z=0}^{H} \frac{\partial T(x, y, z)}{\partial x} dz dy \text{ W} \quad (1-6)
\]

where \( A_x \) is the total cross-sectional area for the 1-D problem in equation (1-4). The total cross-sectional area for the 2-D problem in equation (1-5) is defined by the surface from \( y = 0 \) to \( L \) in the second spatial dimension and by the length \( H \) in the \( z \) direction, for which there is no temperature dependence [i.e., \( T \neq f(z) \)]. The total cross-sectional area for the 3-D problem in equation (1-6) is defined by the surface from \( y = 0 \) to \( L \) and \( z = 0 \) to \( H \) in the second and third spatial dimensions, noting that \( T = f(x, y, z) \).

\section*{1-2 THERMAL CONDUCTIVITY}

Given the direct dependency of the heat flux on the thermal conductivity via Fourier’s law, the thermal conductivity is an important parameter in the analysis of heat conduction. There is a wide range in the thermal conductivities of various engineering materials. Generally, the highest values are observed for pure metals and the lowest value by gases and vapors, with the amorphous insulating materials and inorganic liquids having thermal conductivities that lie in between. There are important exceptions. For example, natural type IIa diamond (nitrogen free) has the highest thermal conductivity of any bulk material \((\sim 2300 \text{ W/m} \cdot \text{K})\) at ambient...
temperature), due to the ability of the well-ordered crystal lattice to transmit thermal energy via vibrational quanta called phonons. In Chapter 16, we will explore in depth the physics of energy carriers to gain further insight into this important material property.

To give some idea of the order of magnitude of thermal conductivity for various materials, Figure 1-2 illustrates the typical range for various material classes. Thermal conductivity also varies with temperature and may change with orientation for nonisotropic materials. For most pure metals the thermal conductivity decreases with increasing temperature, whereas for gases it increases with increasing temperature. For most insulating materials it increases with increasing temperature. Figure 1-3 provides the effect of temperature on the thermal conductivity for a range of materials. At very low temperatures, thermal conductivity increases rapidly and then exhibits a sharp decrease as temperatures approach absolute zero, as shown in Figure 1-4, due to the dominance of energy carrier scattering from defects at extreme low temperatures. A comprehensive compilation of thermal conductivities of materials may be found in references 3–6. We present in Appendix I the thermal conductivity of typical engineering materials together with the specific heat $c_p$, density $\rho$, and the thermal diffusivity $\alpha$. These latter properties are discussed in more detail in the following section.

Figure 1-2  Typical range of thermal conductivity of various material classes.
1.3 DIFFERENTIAL EQUATION OF HEAT CONDUCTION

We now derive the differential equation of heat conduction, often called the heat equation, for a stationary, homogeneous, isotropic solid with heat generation within the body. Internal heat generation may be due to nuclear or chemical reactivity, electrical current (i.e., Joule heating), absorption of laser light, or other sources that may in general be a function of time and/or position. The heat equation may be derived using either a differential control volume approach or an integral approach. The former is perhaps more intuitive and will be presented first, while the latter approach is more general and readily extends the derivation to moving solids.