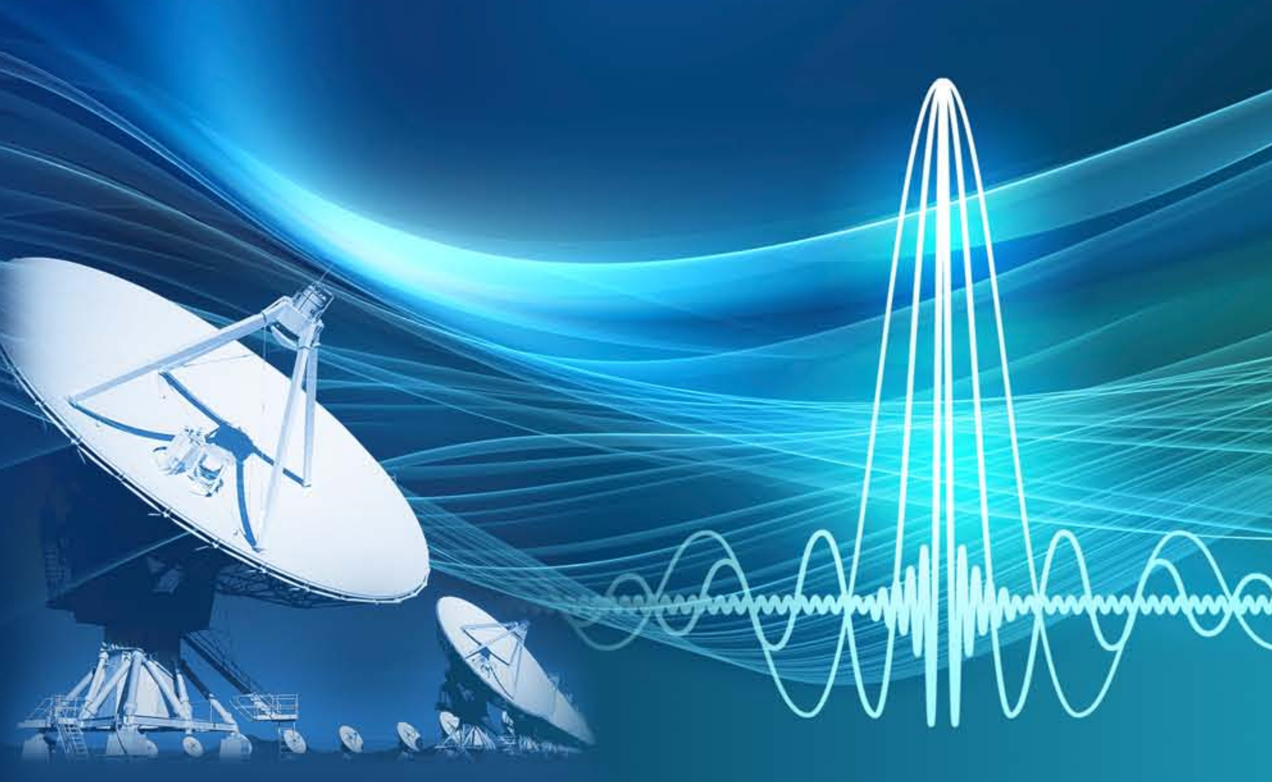
Antonio Napolitano

Generalizations of Cyclostationary Signal Processing

Spectral Analysis and Applications



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SPECTRAL ANALYSIS AND APPLICATIONS

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To Ubalda and Asha

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About the Author

Antonio Napolitano was born in Naples, Italy, in 1964. He received his Ph.D. in Electronic Engineering and Computer Science in 1994 from the University of Napoli Federico II, Italy. Since 2005 he has been Professor of Telecommunications at the University of Napoli "Parthenope", Italy. He is an IEEE Senior Member. From 2006 to 2009, and since 2011 he has been an Associate Editor of the IEEE Transactions on Signal Processing; and since 2008 he has been on the Editorial Board of Signal Processing (Elsevier). He is a Member of the Signal Processing Theory and Method Technical Committee (SPTM-TC) of the IEEE Signal Processing Society. In 1995 and 2006, he received the Best Paper Award from the European Association for Signal Processing (EURASIP) for articles on higher order cyclostationarity and the functional approach for signal analysis, respectively. In 2008, he received from Elsevier the Most Cited Paper Award for a review article on cyclostationarity.

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AN

Preface

Many processes in nature arise from the interaction of periodic phenomena with random phenomena. The results are processes which are not periodic, but whose statistical functions are periodic functions of time. These processes are called cyclostationary and are an appropriate mathematical model for signals encountered in telecommunications, radar, sonar, telemetry, astronomy, mechanics, econometric, biology. In contrast, the classical model of stationary processes considers statistical functions which do not depend on time. More generally, if different periodicities are present in the generation mechanism of the process, the process is called almost cyclostationary (ACS). Almost all modulated signals adopted in communications, radar, and sonar can be modeled as ACS. Thus, in the past twenty years the exploitation of almost-cyclostationarity properties in communications and radar has allowed the design of signal processing algorithms for detection, estimation, and classification that significantly outperform classical algorithms based on a stationary description of signals. The gain in performance is due to a proper description of the nonstationarity of the signals, that is, the time variability of their statistical functions.

In this book, mathematical models for two general classes of nonstationary processes are presented: generalized almost-cyclostationary (GACS) processes and spectrally correlated (SC) processes. Both classes of processes include cyclostationary and ACS processes as special cases. SC and GACS processes are appropriate models for the received signal in mobile communications or radar scenarios when the transmitted signal is ACS and the propagation channel is a (possibly multipath) Doppler channel due to the relative motion between transmitter, receiver, and/or surrounding scatters or targets. SC processes are shown to be useful in the description of processes encountered in multirate systems and spectral analysis with nonuniform frequency spacing. GACS processes find application in the description of communications signals with slowly varying parameters such as carrier frequency, baud rate, etc.

The problem of statistical function estimation is addressed for both GACS and SC processes. This problem is challenging and of great interest at the applications level. In fact, once the nonstationary behavior of the observed signal has been characterized, statistical functions need to be estimated to be exploited in applications. The existence of reliable statistical function estimators for ACS processes is one of the main motivations for the success of signal processing algorithms based on this model. The results presented in this book extend most of the techniques used for ACS signals to the more general classes of GACS and SC signals. Mean-square consistency and asymptotic Normality properties are proved for the considered statistical function estimators. Both continuous- and discrete-time cases are considered and the problem of sampling and aliasing is addressed. Extensive simulation results are presented to corroborate the theoretical results.

How to use this book

The book is organized so that it can be used by readers with different requirements. Chapter 1 contains background material for easy reference in the subsequent chapters. Chapters 2 and 4 contain the main results presented in the form of theorems with sketches of proofs and illustrative examples. Thus, these chapters can be used by the non-specialist who is only interested in recipes or results and wants to grasp the main ideas. Each of these two chapters is followed by a chapter containing complements and proofs (Chapters 3 and 5). Each proof is divided into two parts. The first part consists of the formal manipulations to find the result. This part is aimed at advanced readers with a background of graduate students in engineering. The second part of the proof consists of the justification of the formal manipulations and is therefore aimed at specialists (e.g., mathematicians).

Book outline

In **Chapter 1**, the statistical characterization of persistent (finite-power) nonstationary stochastic processes is presented. Both strict-sense and wide-sense characterizations are considered. Harmonizability and time-frequency representations are treated. Definition and properties of almost periodic functions are provided. A brief review of ACS processes is also presented. The chapter ends with some properties of cumulants.

In **Chapter 2**, GACS processes are presented and characterized. GACS processes have multivariate statistical functions that are almost-periodic function of time. The (generalized) Fourier series of these functions have both coefficients and frequencies, named lag-dependent cycle frequencies, which depend on the lag shifts of the processes. ACS processes are obtained as special case when the frequencies do not depend on the lag parameters. The problems of linear filtering and sampling of GACS processes are addressed. The cyclic correlogram is shown to be, under mild conditions, a mean-square consistent and asymptotically Normal estimator of the cyclic autocorrelation function. Such a function allows a complete second-order characterization in the wide-sense of GACS processes. Numerical examples of communications through Doppler channels due to relative motion between transmitter and receiver with constant relative radial acceleration are considered. Simulation results on statistical function estimation are carried out to illustrate the theoretical results. Proofs of the results in Chapter 2 are reported in Chapter 3.

In **Chapter 3**, complements and proofs for the results presented in Chapter 2 are reported. Each proof consists of two parts. The first part contains formal manipulations that lead to the result. The second part contains the justifications of the mathematical manipulations of the first part. Thus, proofs can be followed with two different levels of rigor, depending on the background and interest of the reader.

In **Chapter 4**, SC processes are presented and characterized. SC processes have the Loève bifrequency spectrum with spectral masses concentrated on a countable set of support curves in the bifrequency plane. ACS processes are obtained as a special case when the curves are lines with a unit slope. The problems of linear filtering and sampling of SC processes are addressed. The time-smoothed and the frequency-smoothed cross-periodograms are considered as estimators of the spectral correlation density. Consistency and asymptotic Normality properties are analyzed. Illustrative examples and simulation results are presented. Proofs of the results in Chapter 4 are reported in Chapter 5.

In **Chapter 5**, complements and proofs for the results presented in Chapter 4 are reported. The system used is the same as in Chapter 3.

In **Chapter 6**, the problem of signal modeling and statistical function estimation is addressed in the functional or fraction-of-time (FOT) approach. Such an approach is an alternative to the classical one where signals are modeled as sample paths or realizations of a stochastic process. In the FOT approach, a signal is modeled as a single function of time and a probabilistic model is constructed by this sole function of time. Nonstationary models that can be treated in this approach are discussed.

In **Chapter 7**, applications in mobile communications and radar/sonar systems are presented. A model for the wireless channel is developed. It is shown how, in the case of relative motion between transmitter and receiver or between radar and target, the ACS transmitted signal is modified into a received signal with a different kind of nonstationarity. Conditions under which the GACS or SC model are appropriate for the received signal are derived.

In **Chapter 8**, citations are classified into categories and listed in chronological order.

List of Abbreviations

a.e. = almost everywhere
ACS = almost cyclostationary
AP = almost periodic
CDMA = code-division multiple access
DFS = discrete Fourier series
DFT = discrete Fourier transform
DSSS = direct-sequence spread-spectrum
DT-CCC = discrete-time cyclic cross-correlogram
fBm = fractional Brownian motion
FOT = fraction-of-time
FRESH = frequency shift
GACS = generalized almost-cyclostationary
H-CCC = hybrid cyclic cross-correlogram
LAPTV = linear almost-periodically time-variant
lhs = left-hand side
LTI = linear time-invariant
LTV = linear time-variant
MIMO = multi-input multi-output
MMSE = minimum mean-square error
PAM = pulse amplitude modulated
pdf = probability density function
QTI = quadratic time-invariant
RCS = radar cross-section
rhs = right-hand side
RX = receiver
SC = spectrally correlated
SCD = spectral correlation density
STFT = short-time Fourier transform
t.m.s.s. = temporal mean-square sense
TX = transmitter
w.a.p. = weakly almost periodic
w.p.1 = with probability 1
w.r.t. = with respect to
WSS = wide-sense stationary

1

Background

In this chapter, background material that will be referred to in the subsequent chapters is reviewed. In Section 1.1, the statistical characterization of persistent (finite-power) nonstationary stochastic processes is presented. Second-order statistics in both time, frequency, and time-frequency domains are considered. In Section 1.2, definitions of almost-periodic functions and their generalizations (Besicovitch 1932) and related results are reviewed. Almost-cyclostationary (ACS) processes (Gardner 1985, 1987d) are treated in Section 1.3. Finally, in Section 1.4, some results on cumulants are reviewed.

1.1 Second-Order Characterization of Stochastic Processes

1.1.1 Time-Domain Characterization

In the classical stochastic-process framework, statistical functions are defined in terms of ensemble averages of functions of the process and its time-shifted versions. Nonstationary processes have these statistical functions that depend on time.

Let us consider a continuous-time real-valued process $\{x(t, \omega),\ t \in \mathbb{R},\ \omega \in \Omega\}$, with abbreviate notation $x(t)$ when it does not create ambiguity, where Ω is a sample space equipped with a σ-field $\mathcal{F}$ and a probability measure P defined on the elements of $\mathcal{F}$. The cumulative distribution function of $x(t)$ is defined as (Doob 1953)

$$F_x(\xi; t) \triangleq P[x(t, \omega) \leqslant \xi] = \int_\Omega \mathbf{1}_{\{\omega\,:\,x(t,\omega)\leqslant\xi\}}\,\mathrm{d}P(\omega) \triangleq \mathrm{E}\left\{\mathbf{1}_{\{\omega\,:\,x(t,\omega)\leqslant\xi\}}\right\} \tag{1.1}$$

where

$$\mathbf{1}_{\{\omega\,:\,x(t,\omega)\leqslant\xi\}} \triangleq \begin{cases} 1, & \omega\ :\ x(t,\omega) \leqslant \xi, \\ 0, & \omega\ :\ x(t,\omega) > \xi \end{cases} \tag{1.2}$$

Generalizations of Cyclostationary Signal Processing: Spectral Analysis and Applications, First Edition.

is the indicator of the set $\{\omega \in \Omega : x(t,\omega) \leqslant \xi\}$ and $\mathrm{E}\{\cdot\}$ denotes statistical expectation (ensemble average). The expected value corresponding to the distribution $F_x(\xi;t)$ is the statistical mean

$$\int_{\mathbb{R}} \xi \,\mathrm{d}F_x(\xi;t) = \int_{\Omega} x(t,\omega)\,\mathrm{d}P(\omega) = \mathrm{E}\left\{x(t,\omega)\right\}. \tag{1.3}$$

Analogously, at second-order, the process is characterized by the second-order joint distribution function (Doob 1953)

$$\begin{aligned} F_x(\xi_1,\xi_2;t,\tau) &\triangleq P[x(t+\tau,\omega) \leqslant \xi_1, x(t,\omega) \leqslant \xi_2] \\ &= \mathrm{E}\left\{\mathbf{1}_{\{\omega\,:\,x(t+\tau,\omega)\leqslant\xi_1\}}\,\mathbf{1}_{\{\omega\,:\,x(t,\omega)\leqslant\xi_2\}}\right\} \end{aligned} \tag{1.4}$$

and the autocorrelation function

$$\mathrm{E}\left\{x(t+\tau,\omega)\,x(t,\omega)\right\} = \int_{\mathbb{R}^2} \xi_1\xi_2 \,\mathrm{d}F_x(\xi_1,\xi_2;t,\tau). \tag{1.5}$$

If $F_x(\xi;t)$ and $F_x(\xi_1,\xi_2;t,\tau)$ depend on t, the process is said to be nonstationary in the strict sense. If $F_x(\xi;t)$ $[F_x(\xi_1,\xi_2;t,\tau)]$ does not depend on t, the process $x(t)$ is said to be 1st-order [2nd-order] stationary in the strict sense. If both mean and autocorrelation function do not depend on t, the process is said to be wide-sense stationary (WSS) (Doob 1953).

In the following, we will focus on the second-order statistics of complex-valued nonstationarity processes.

The complex-valued stochastic process $x(t)$ is said to be a *second-order process* if the second-order moments

$$\mathcal{R}_{\boldsymbol{x}}(t,\tau) \triangleq \mathrm{E}\left\{x(t+\tau)\,x^{(*)}(t)\right\} \tag{1.6}$$

exist $\forall t$ and $\forall \tau$. In Equation (1.6), superscript $(*)$ denotes optional complex conjugation, and subscript $\boldsymbol{x} \triangleq [x, x^{(*)}]$. That is, $\mathcal{R}_{\boldsymbol{x}}(t,\tau)$ denotes one of two different functions depending if the complex conjugation is considered or not in subscript $\boldsymbol{x}$. If conjugation is present, then (1.6) is the *autocorrelation function*. If the conjugation is absent, then (1.6) is the *conjugate autocorrelation function* also referred to as *relation function* (Picinbono and Bondon 1997) or *complementary correlation* (Schreier and Scharf 2003a). Note that, in the complex case the order of the distribution functions turns out to be doubled with respect to the real case. For example, the joint distribution function of $x(t)$ and $x(t+\tau)$ is a fourth-order joint distribution of the real and imaginary parts of $x(t)$ and $x(t+\tau)$.

The *(conjugate) autocovariance* is the (conjugate) autocorrelation of the process reduced to be zero mean by subtracting its mean value

$$\mathcal{C}_{\boldsymbol{x}}(t,\tau) \triangleq \mathrm{E}\Big\{\big[x(t+\tau) - \mathrm{E}\{x(t+\tau)\}\big]\,\big[x(t) - \mathrm{E}\{x(t)\}\big]^{(*)}\Big\}. \tag{1.7}$$

Even if $\mathcal{C}_{\boldsymbol{x}}(t,\tau) = \mathcal{R}_{\boldsymbol{x}}(t,\tau)$ only for zero-mean processes, in some cases the terms autocorrelation, autocovariance, and covariance are used interchangeably. When the terms autocovariance or covariance are adopted, from the context it is understood if the mean value is subtracted or not. In statistics, the definition of autocorrelation includes in (1.6) also a normalization by the standard deviations of $x(t)$ and $x(t+\tau)$.

1.1.2 Spectral-Domain Characterization

The characterization of stochastic processes in the spectral domain can be made by resorting to the concept of harmonizability (Loève 1963). A second-order stochastic process $x(t)$ is said to be *harmonizable* if its (conjugate) autocorrelation function can be expressed by the Fourier-Stieltjes integral

$$\mathrm{E}\left\{x(t_1)\, x^{(*)}(t_2)\right\} = \int_{\mathbb{R}^2} e^{j2\pi[f_1 t_1 + (-) f_2 t_2]}\, \mathrm{d}\gamma_{\boldsymbol{x}}(f_1, f_2) \tag{1.8}$$

where $\gamma_{\boldsymbol{x}}(f_1, f_2)$ is a spectral correlation function of bounded variation (Loève 1963):

$$\int_{\mathbb{R}^2} |\, \mathrm{d}\gamma_{\boldsymbol{x}}(f_1, f_2)| < \infty. \tag{1.9}$$

In (1.8), $(-)$ is an optional minus sign that is linked to $(*)$. $\gamma_{\boldsymbol{x}}(f_1, f_2)$ denotes one of two different functions depending if the complex conjugation is considered or not in subscript $\boldsymbol{x}$.

Under the harmonizability condition, $x(t)$ is said to be *(strongly) harmonizable* and can be expressed by the Cramér representation (Cramér 1940)

$$x(t) = \int_{\mathbb{R}} e^{j2\pi f t}\, \mathrm{d}\chi(f) \tag{1.10}$$

where $\chi(f)$ is the *integrated spectrum* of $x(t)$.

In (Loève 1963), it is shown that a necessary condition for a stochastic process to be harmonizable is that it is second-order continuous (or mean-square continuous) (Definition 2.2.11, Theorem 2.2.12). Moreover, it is shown that a stochastic process is harmonizable if and only if its covariance function is harmonizable. In fact, convergence of integrals in (1.8) and (1.10) is in the mean-square sense. In (Hurd 1973), the harmonizability of processes obtained by some processing of other harmonizable processes is studied.

If the absolutely continuous and the discrete component of $\chi(f)$ are (possibly) nonzero and the singular component of $\chi(f)$ is zero with probability 1 (w.p.1) (Cramér 1940), we can formally write $\mathrm{d}\chi(f) = X(f)\,\mathrm{d}f$ (w.p.1) (Gardner 1985, Chapter 10.1.2), where

$$X(f) = \int_{\mathbb{R}} x(t)\, e^{-j2\pi f t}\, \mathrm{d}t \tag{1.11}$$

is the Fourier transform of $x(t)$ which possibly contains Dirac deltas in correspondence of the jumps of the discrete component of $\chi(f)$. For *finite-power processes,* that is such that the time-averaged power

$$P_x \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{E}\left\{|x(t)|^2\right\}\, \mathrm{d}t \tag{1.12}$$

exists and is finite, relation (1.11) is intended in the sense of distributions (Gelfand and Vilenkin 1964, Chapter 3), (Henniger 1970).

Let $x(t)$ be an harmonizable stochastic process. Its *bifrequency spectral correlation function* or *Loève bifrequency spectrum* (Loève 1963; Thomson 1982), also called *generalized spectrum* in (Gerr and Allen 1994), *cointensity spectrum* in (Middleton 1967), or *dual frequency spectral correlation* in (Hanssen and Scharf 2003), is defined as

$$\mathcal{S}_{\boldsymbol{x}}(f_1, f_2) \triangleq \mathrm{E}\left\{X(f_1)\, X^{(*)}(f_2)\right\} \tag{1.13}$$

and if $\chi(f)$ and $\gamma_{\boldsymbol{x}}(f_1, f_2)$ do not contain singular components w.p.1, in the sense of distributions the result is that

$$\mathrm{d}\gamma_{\boldsymbol{x}}(f_1, f_2) = \mathrm{E}\left\{\mathrm{d}\chi(f_1)\ \mathrm{d}\chi^{(*)}(f_2)\right\} \tag{1.14a}$$

$$= \mathrm{E}\left\{X(f_1)\, X^{(*)}(f_2)\right\}\ \mathrm{d}f_1\,\mathrm{d}f_2 \tag{1.14b}$$

and, accordingly with (1.8), we can formally write

$$\mathrm{E}\left\{x(t_1)\, x^{(*)}(t_2)\right\} = \int_{\mathbb{R}^2} \mathrm{E}\left\{X(f_1)\, X^{(*)}(f_2)\right\}\ e^{j2\pi[f_1t_1+(-)f_2t_2]}\,\mathrm{d}f_1\,\mathrm{d}f_2 \tag{1.15}$$

$$\mathrm{E}\left\{X(f_1)\, X^{(*)}(f_2)\right\} = \int_{\mathbb{R}^2} \mathrm{E}\left\{x(t_1)\, x^{(*)}(t_2)\right\} e^{-j2\pi[f_1t_1+(-)f_2t_2]}\,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{1.16}$$

A spectral characterization for nonstationary processes that resembles that for WSS processes (Section 1.1.4) can be obtained starting from the *time-averaged (conjugate) autocorrelation function*

$$\begin{aligned} R_{\boldsymbol{x}}(\tau) &\triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{E}\left\{x(t+\tau)\, x^{(*)}(t)\right\}\ \mathrm{d}t \\ &\equiv \left\langle \mathrm{E}\left\{x(t+\tau)\, x^{(*)}(t)\right\}\right\rangle_t \end{aligned} \tag{1.17}$$

when the limit exists. Its Fourier transform is called the *power spectrum*, is denoted by $S_{\boldsymbol{x}}(f)$, and represents the spectral density of the time-averaged power $R_{\boldsymbol{x}}(0)$ of the process. The time-averaged autocorrelation function and the power spectrum defined here for nonstationary processes exhibit the same properties of the autocorrelation function and power spectrum defined for wide-sense stationary processes (Wu and Lev-Ari 1997).

1.1.3 Time-Frequency Characterization

The Loève bifrequency spectrum (1.13) provides a description of the nonstationary behavior of the process $x(t)$ in the frequency domain. A description in terms of functions of time and frequency can be obtained by resorting to the time-variant spectrum, the Rihaczek distribution, and the Wigner-Ville spectrum.

The Fourier transform of the second-order moment (1.6) with respect to (w.r.t.) the lag parameter τ is the *time-variant spectrum*

$$\mathcal{S}_{\boldsymbol{x}}(t, f) \triangleq \int_{\mathbb{R}} \mathcal{R}_{\boldsymbol{x}}(t, \tau)\, e^{-j2\pi f\tau}\,\mathrm{d}\tau. \tag{1.18}$$

By substituting (1.6) into (1.18), interchanging the order of the expectation and Fourier-transform operators, and accounting for the formal relation $\mathrm{d}\chi(f) = X(f)\,\mathrm{d}f$, one obtains

$$\mathcal{S}_{\boldsymbol{x}}(t, f)\,\mathrm{d}f = \mathrm{E}\left\{ \mathrm{d}\chi(f)\, x^{(*)}(t) \right\}\, e^{j2\pi f t} \tag{1.19}$$

where the right-hand-side is referred to as the *(conjugate) Rihaczek distribution* of $x(t)$ (Scharf *et al.* 2005).

By the variable change $t' = t + \tau/2$ in (1.6) and Fourier transforming w.r.t. τ, we obtain a time-frequency representation in terms of *Wigner-Ville spectrum* for stochastic processes (Martin and Flandrin 1985)

$$\mathcal{W}_{\boldsymbol{x}}(t', f) \triangleq \int_{\mathbb{R}} \mathrm{E}\left\{ x(t' + \tau/2)\, x^{(*)}(t' - \tau/2) \right\}\, e^{-j2\pi f\tau}\,\mathrm{d}\tau \tag{1.20a}$$

$$= \int_{\mathbb{R}} \mathrm{E}\left\{ X(f + \nu/2)\, X^{(*)}(f - \nu/2) \right\}\, e^{j2\pi\nu t'}\,\mathrm{d}\nu \tag{1.20b}$$

where the second equality follows using (1.11).

Extensive treatments on time-frequency characterizations of nonstationary signals are given in (Amin 1992), (Boashash *et al.* 1995), (Cohen 1989, 1995), (Flandrin 1999), (Hlawatsch and Bourdeaux-Bartels 1992). Most of these references refer to finite-energy signals.

1.1.4 Wide-Sense Stationary Processes

Second-order nonstationary processes have (conjugate) autocorrelation function depending on both time t and lag parameter τ and the function defined in (1.6) is also called the time-lag (conjugate) autocorrelation function. Equivalently, their time-variant spectrum depends on both time t and frequency f. In contrast, second-order WSS processes are characterized by a (conjugate) autocorrelation and time-variant spectrum not depending on t. That is

$$\mathcal{R}_{\boldsymbol{x}}(t, \tau) = R_{\boldsymbol{x}}(\tau) \tag{1.21a}$$

$$\mathcal{S}_{\boldsymbol{x}}(t, f) = S_{\boldsymbol{x}}(f). \tag{1.21b}$$

In such a case, for $(*)$ present, the Fourier-transform (1.18) specializes into the Wiener-Khinchin relation that links the autocorrelation function and the *power spectrum* $S_{\boldsymbol{x}}(f)$ (Gardner 1985)

$$S_{\boldsymbol{x}}(f) = \int_{\mathbb{R}} R_{\boldsymbol{x}}(\tau)\, e^{-j2\pi f\tau}\,\mathrm{d}\tau. \tag{1.22}$$

Condition (1.21a) is equivalent to the fact that the time–time (conjugate) autocorrelation function (1.8) depends only on the time difference $t_1 - t_2$. This time dependence in the spectral domain corresponds to the property that the Loève bifrequency spectrum (1.13) is nonzero only on the diagonal $f_2 = -_{(-)} f_1$. That is,

$$\mathcal{S}_{\boldsymbol{x}}(f_1, f_2) = S_{\boldsymbol{x}}(f_1)\,\delta(f_2 +_{(-)} f_1) \tag{1.23}$$

where $\delta(\cdot)$ denotes Dirac delta. When $(*)$ is present, $S_{\boldsymbol{x}}(f_1)$ is the power spectrum of the process $x(t)$. From (1.23), it follows that for WSS processes distinct spectral component are uncorrelated. In contrast, the presence of spectral correlation outside the diagonal is evidence of nonstationarity in the process $x(t)$ (Loève 1963). Finally, for WSS processes the Wigner-Ville spectrum is independent of t' and is coincident with the power spectrum. That is, $\mathcal{W}_{\boldsymbol{x}}(t', f) = S_{\boldsymbol{x}}(f)$.

Extensive treatments on WSS processes are given in (Brillinger 1981), (Cramér 1940), (Doob 1953), (Grenander and Rosenblatt 1957), (Papoulis 1991), (Prohorov and Rozanov 1989), (Rosenblatt 1974, 1985).

1.1.5 Evolutionary Spectral Analysis

In (Priestley 1965), the class of zero-mean processes for which the autocovariance function admits the representation

$$\mathrm{E}\left\{x(t_1)\, x^*(t_2)\right\} = \int_{\mathbb{R}} \phi_{t_1}(\omega)\, \phi^*_{t_2}(\omega)\, \mathrm{d}\mu(\omega) \tag{1.24}$$

is considered, where $\{\phi_t(\omega)\}$ is a family of functions defined on the real line ($\omega \in \mathbb{R}$) indexed by the suffix t and $\mathrm{d}\mu(\omega)$ is a measure on the real line. In (Grenander and Rosenblatt 1957, paragraph 1.4), it is shown that if the autocovariance has the representation (1.24), then the process $x(t)$ admits the representation

$$x(t) = \int_{\mathbb{R}} \phi_t(\omega)\, \mathrm{d}Z(\omega) \tag{1.25}$$

where $Z(\omega)$ is an orthogonal process with

$$\mathrm{E}\left\{\mathrm{d}Z(\omega_1)\, \mathrm{d}Z^*(\omega_2)\right\} = \delta(\omega_1 - \omega_2)\, \mathrm{d}\mu(\omega_1). \tag{1.26}$$

In fact, we formally have

$$\begin{aligned}
\mathrm{E}\left\{x(t_1)\, x^*(t_2)\right\} &= \mathrm{E}\left\{\int_{\mathbb{R}} \phi_{t_1}(\omega_1)\, \mathrm{d}Z(\omega_1) \int_{\mathbb{R}} \phi^*_{t_2}(\omega_2)\, \mathrm{d}Z^*(\omega_2)\right\} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_{t_1}(\omega_1)\, \phi^*_{t_2}(\omega_2) \mathrm{E}\left\{\mathrm{d}Z(\omega_1)\, \mathrm{d}Z^*(\omega_2)\right\} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_{t_1}(\omega_1)\, \phi^*_{t_2}(\omega_2)\, \delta(\omega_1 - \omega_2)\, \mathrm{d}\mu(\omega_1) \\
&= \int_{\mathbb{R}} \phi_{t_1}(\omega_1)\, \phi^*_{t_2}(\omega_1)\, \mathrm{d}\mu(\omega_1)
\end{aligned} \tag{1.27}$$

where, in the last equality, the sampling property of the Dirac delta (Zemanian 1987, Section 1.7) is used.

When the process is second-order WSS, a valid choice for the family $\{\phi_t(\omega)\}$ is $\phi_t(\omega) = e^{j\omega t}$. The autocovariance is

$$\mathrm{E}\left\{x(t_1)\, x^*(t_2)\right\} = \int_{\mathbb{R}} e^{j\omega(t_1 - t_2)}\, \mathrm{d}\mu(\omega) \tag{1.28}$$

which is function of $t_1 - t_2$. The function $\mu(\omega)$ is the *integrated power spectrum*. If $\mu(\omega)$ is absolutely continuous or contains jumps and has zero singular component (Cramér 1940), then in the sense of distributions $\mathrm{d}\mu(\omega) = S(\omega)\, \mathrm{d}(\omega)$, where $S(\omega)$ is the power spectrum (with $\omega = 2\pi f$) which contains Dirac deltas in correspondence of the jumps in $\mu(\omega)$.