Dynamics of Structures

Patrick Paultre

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I dedicate this book

to the memory of my mother Anne Franchette Bérangère Wolff,

and

to my wife Solange and my daughters Geneviève and Catherine.

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Contents

Preface

Structural dynamics is a subject that traditionally figures in the curriculum of engineering schools. An introductory course in structural dynamics is often available as an elective in engineering programs, followed by a more advanced course during graduate work at the master's or doctoral level. The new standards and building codes promote the use of dynamic computation to determine the distribution of seismic forces when designing large or irregularly shaped buildings or, in some cases, as the method of choice for determining the effects of seismic forces. As a result, the importance of an introductory course in structural dynamics should be obvious. This book is intended for engineering students and practising engineers dealing with problems related to structural vibration and seismic design.

This volume has two parts. The first deals with single-DOF systems, which include complex systems that can be reduced to single-DOF systems. The second part looks at systems with multiple DOF that are solved using the finite-element method. This division could be viewed as the separation between an introductory course on structural dynamics for undergraduates and an advanced course for graduate students. That would not be a very profitable approach, since it would not include modal analysis, which is discussed in the second part of this book. The goal is to introduce modal analysis as part of an introductory course on structural dynamics analysis. Understanding the book's contents requires no more knowledge of mathematics and structural analysis than any engineering student would have. The book breaks down as follows.

Chapter 1 provides an introduction to structural dynamics. The first part of the book deals with single-degree-of-freedom (SDOF) systems. Chapter 2 provides the equations of motion for single-DOF systems. Chapter 3 develops conventional solutions for single-DOF systems, i.e. under the initial conditions imposed without dynamic loading. System response to harmonic loading is discussed in Chapter 4, which leads to damping and its experimental measurement, dealt with in Chapter 5. The Fourier decomposition of periodic loading is considered in Chapter 6, which shows that the response is the superimposition of a set of harmonic loadings. Chapter 7 shows how to calculate the response of a single-DOF system subjected to any kind of loading using Duhamel's integral. Chapter 8 introduces applying frequency-domain analysis to dynamics problems and calculating the response to any kind of loading using the Fourier transform. Chapter 9 provides an introduction to the direct numerical integration of equations of motion. The topics treated include an exact method for piecewise linear loading functions, the central difference method, and conventional Newmark methods. Chapter 10 considers computation of the response of nonlinear single-DOF systems using direct numerical integration combined with Newton's iterative method for error reduction. Chapter 11 focuses on systems that can be reduced to a single DOF using Rayleigh's method. The book's first part ends with an examination of single-DOF systems under earthquake action (Chapter 12).

Part 2 is devoted to discrete systems with multiple DOF. Chapter 13 establishes the equations of motion for multiple-degree-of-freedom (MDOF) systems and defines mass, damping and stiffness matrices based on a basic knowledge of structural matrix computations. Chapter 14 provides an introduction to the finite-element method so that the mass and rigidity matrices can be established more formally. The free response of conservative multiple-DOF systems is seen in Chapter 15, which provides for defining and computing the natural frequencies and associated mode shapes. Chapter 16 deals with the free vibration of discrete dissipative systems. Chapter 17 shows how to use modal superposition to compute the response of discrete systems for any load, whereas Chapter 18 deals with seismic loading. Chapter 19 looks at several properties of eigenvalues and eigenvectors required for a more in-depth study of their numerical determination. Chapter 20 presents several coordinate reduction methods, which are of prime importance in structural dynamics, and introduces Ritz analysis. Chapter 21 presents several classic methods for computing eigenvalues and the associated eigenvectors. Direct numerical integration methods to solve equations of motion for discrete multiple-DOF systems receive in-depth treatment in Chapter 22, including error and stability analysis of the different methods. Application of direct numerical integration methods to solve nonlinear problems is seen in Chapter 23.

The appendix provides some mathematical notions needed to understand the text. This book contains 88 examples illustrating application of the theories and methods discussed herein as well as 181 problems.

The contents can be used to develop a number of courses, including:

1) Introduction to Structural Dynamics: an introductory course for engineering students would cover Chapters 1 to 7, 9, 11 and part of 12, 13, 15 to 18.

2) Advanced Structural Dynamics: this course for graduate students who have taken the introductory course in structural dynamics would comprise Chapters 1, 8, 12 to 18, and 20 to 23, in part.

3) Computational Structural Dynamics: this advanced course would be reserved for graduate students who have already taken the advanced structural dynamics course, in which Chapters 1, 8 and 14 to 23 would be seen.

This text was used in delivering the structural dynamics course to senior students at the University of Sherbrooke. I take this opportunity to thank all my former students who, through attending lectures and their enthusiasm for solving weekly problems with CALWin, LAS and MATLAB, led me to write this book. Professor Jacky Mazars played an essential role in the process leading to this book, first by inviting me to give a course on structural dynamics at the École Normale Supérieure in Cachan to students at the DEA-MAISE and Laboratoire de Mécanique et Technologie for a number of years, and then by inviting me to publish it in the civil-engineering collection at ISTE and John Wiley & Sons. This text provided the foundation for an introductory course in structural dynamics given to Master's students in civil engineering and infrastructure at the Grenoble IUP as well as for an advanced computational structural dynamics course given to students at the doctoral school of Joseph Fourier University in Grenoble at the invitation of Professor Laurent Daudeville. Lastly, part of the book was presented in English to doctoral students attending ALERT sessions in Aussois, France.

I entered the text, performed the layout, and designed the artwork. Professor Najib Bouaanani read some of the chapters of the French version and made suggestions that, without a doubt, have improved the presentation and made the text clearer. In the final phase of writing the French version of the book, Dr. Benedikt Weber and Dr. Thien-Phu Le read all the chapters. Dr. Benedikt Weber played an essential role suggesting clarifications and developing solutions for several problems using MATLAB, whereas Éric Lapointe and myself developed all solutions with LAS. Olivier Gauron, Research Assistant at the University of Sherbrooke proofread the translation of the French version of the book into English, checked the solutions of the problems and coordinated the production of the artwork in English. His role was not limited to these tasks as he made valuable suggestions that helped clarify part of the book. The author is grateful to Sébastien Mousseau, Najib Bouaanani, Cédric Adagbe, Adamou Saidou Sanda, Danusa Tavares and Gustavo Siqueira for their dedication in meticulously and expertly preparing the drawings.

I am particularly grateful to my former Professor René Tinawi for initially piquing my interest in the subject. For a number of years, I taught a course on structural dynamics in parallel with a course taught by Professor Pierre Léger, first at McGill University and now at École Polytechnique in Montréal. I have fond memories of many discussions with Professor Léger about teaching approaches and the development of software for teaching structural dynamics. The same is true for Professor Jean Proulx who also wrote the first version of the CALWin program which is the ancestor of LAS. I wish to thank Éric Lapointe, Master's student at the University of Sherbrooke for the development of the LAS[1] program that runs under Windows. The LAS program is based on an earlier non-graphical basic version developed by Dr. Charles Carbonneau. Éric Lapointe's enthusiasm, technical knowledge, refined programming skills that allows him to put algorithms into code at a record speed led to completion of a project that was dear to me for a number of years. The program is available as freeware to anyone interested. LAS is a powerful program that can be used to quickly learn structural matrix computation methods, the finite-element method, structural dynamics and matrix computations. LAS software can be downloaded from http://www.civil.usherbrooke.ca/ppaultre/. Lastly, I would like to thank Professor Jean Proulx who was a great help in the translation of the book from French to English.

This book was typeset with LaTeX2ε. Donald Knuth cannot be thanked enough for TeX.

Patrick Paultre, Sherbrooke 2010

"Let no one say that I have said nothing new; the arrangement of the subject is new."

Blaise Pascal, Pensées 22-696

1. LAS is an acronym for **L**anguage for the **A**nalysis of **S**tructures which in French is **L**anguage pour l'**A**nalyse des **S**tructures.

Chapter 1

Introduction

The aim of this book is to study vibrations of structures caused by dynamic loadings that vary over time as opposed to static loadings. These dynamic loadings give rise to displacements, internal forces, reactions, and stresses that are time dependent. Hence, a unique solution does not exist as for a static problem. In a dynamic problem, it is necessary to calculate the displacements in time – collectively called *dynamic response* – before determining maximum values of forces, reactions, and stresses that are necessary for design purposes. It is, however, easy to conclude that time is the only difference between the dynamic and static analysis of a structure. This is obviously not true, because, on the one hand, a load is never applied statically and, on the other hand, the effects of a static load do vary in time due to the viscoelastic properties of the materials (creep, shrinkage, relaxation, etc.) forming the structures. The distinctive nature of a dynamic problem comes from the presence of *inertia forces*, $f_I(t)$, which oppose the motion generated by the applied dynamic loading, $p(t)$. The dynamic character of the problem is dominant if the inertia forces are large compared to the total applied forces. The problem can be treated as static if the motion generated by the applied load is so small that the inertia forces are negligible. Figure 1.1 illustrates the effects on the bending moment of a concentrated force applied dynamically and statically to the tip of a column.

A *dynamic load* has intensity, direction, and point of application that can vary in time. If it is a known function of time, the loading is said to be *prescribed dynamic loading*. The analysis of a structure under a prescribed dynamic loading is considered deterministic. If the variation in time of the loading function is unknown and can only be described in statistical terms, it is said to be *random dynamic loading*. *Random vibration analyses* study the response of a structure under random dynamic loadings.

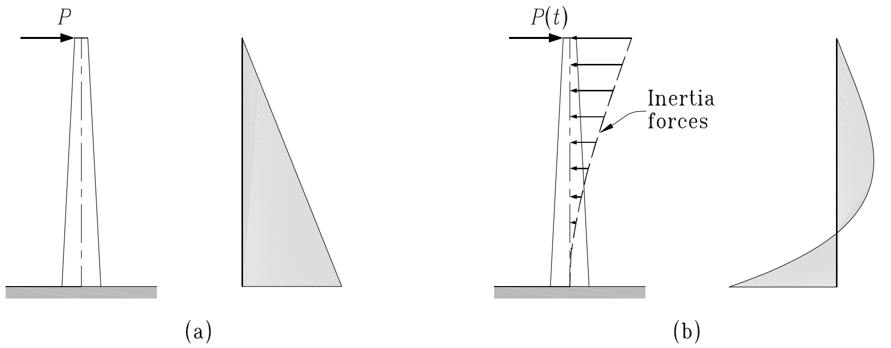

Figure 1.1. *Difference between static and dynamic loading: (a) static loading and corresponding bending moment diagram, (b) dynamic loading and corresponding bending moment diagram (not at scale)*

1.1. Dynamic response

The end result of the deterministic analysis of a structure excited by a given dynamic loading is the *dynamic response* expressing the displacements of the structure with time, which is also called the *displacement time history*. The strains, stresses, internal forces, and reactions are determined once the displacement time history is known (Figure 1.2). We recall that there is no uncertainty in expressing the loading function in a deterministic analysis.

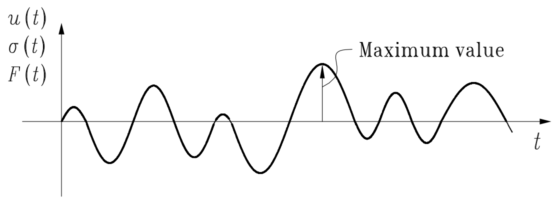

Figure 1.2. *Response time history: displacements, stresses or forces*

Dynamic response varies with time. However, for design or verification, all that is required is the *maximum dynamic response* which, for a linear system, can be added to the *maximum static response* to yield the *maximum total response*. For a nonlinear system, the static effects need to be calculated first and added to the dynamic effects to determine the total nonlinear response.

1.2. Dynamic loading

Dynamic loadings can be divided into *periodic loadings* and *non-periodic loadings*. Table 5.1 summarizes the different types of dynamic loadings that are encountered in civil engineering. Permanent and live loads that are applied slowly compared to the period of vibration of structures are generally considered static loadings, as are dead loads.

Sources of dynamic loadings			
Periodic		Non-periodic	
Simple harmonic	Arbitrary periodic	Arbitrary	Impulsive
Rotating machine	Reciprocating machine	Construction	Construction
	Walking, jogging	Wind	Impact
	Wind	Waves	Explosion
		Earthquakes	Loss of support
		Traffic	Rupture of an element

Table 1.1. *Types and sources of dynamic loadings*

1.2.1. ***Periodic loadings***

A periodic loading repeats itself after a regular time interval, T, called the period. Periodic loadings can be divided into simple harmonic loadings and arbitrary periodic loadings.

1.2.1.1. *Harmonic loadings*

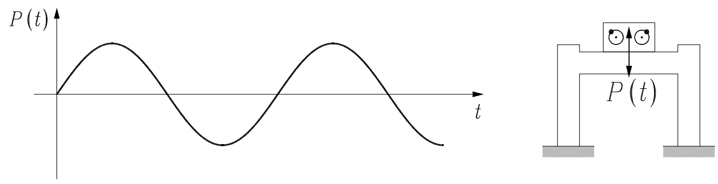

Figure 1.3. *Harmonic loading applied by a rotating machine*

The simplest periodic loading varies as a sinusoid and is called *simple harmonic loading* (Figure 1.3). This type of loading is generated by rotating machines and exciters with unbalanced masses and it gives rise to the resonance phenomenon when the excitation period matches the structure's natural period of vibration.

1.2.1.2. *Arbitrary periodic loadings*

Arbitrary periodic loadings repeat themselves at regular interval of time. This type of loading is generated by reciprocating machines, by walking or jogging by one or many persons crossing a pedestrian bridge (Figure 1.4), by rhythmic jumping and dancing by one or many persons on a floor, by hydrodynamic forces generated by the propeller of a boat, by waves, etc.

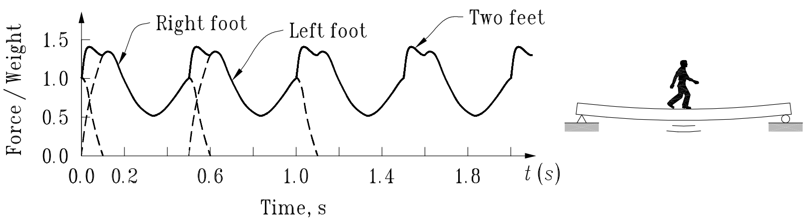

Figure 1.4. *Periodic loading caused by the steps of a person crossing a pedestrian bridge*

1.2.2. *Non-periodic loadings*

Non-periodic loadings vary arbitrarily in time without periodicity. Non-periodic loadings can be divided into impulsive short-duration loadings and arbitrary long-duration transient loadings.

1.2.2.1. *Impulse loadings*

Impulse loads have a very short duration with respect to the vibration period of the structures and are caused by explosions (Figure 1.5), shock, failure of structural elements, support failure, etc.

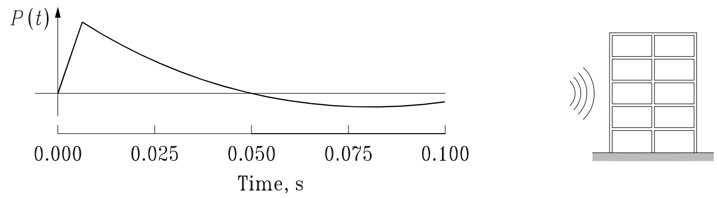

Figure 1.5. *Impulse loading caused by an explosion*

1.2.2.2. *Arbitrary loadings*

Arbitrary loads are of long duration and are caused by earthquakes, wind, waves, etc. Figure 1.6 shows the time variation of the acceleration that occurs at the base of a structure during an earthquake, giving rise to time-varying inertia forces over the structure's height.

1.3. Additional considerations

Additional considerations are needed for dynamic loads. These considerations are mostly related to the cyclic nature of the loading – which can lead to fatigue-related failure – and to the properties of specific materials whose behavior changes with the loading rate.

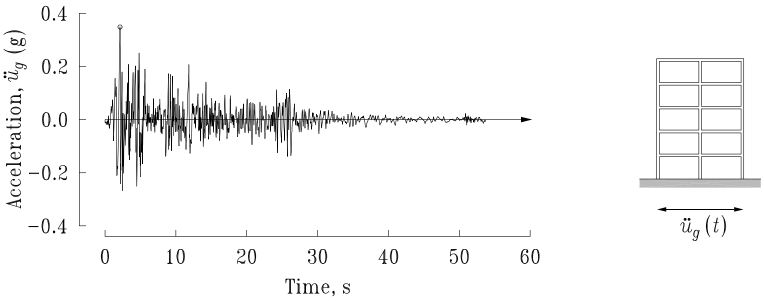

Figure 1.6. *Long-duration arbitrary load caused by an earthquake*

When the external loads are lower than the structure's elastic limit, fatigue-related rupture is caused by stress concentrations near defects where fatigue microcracks can begin to propagate. One such crack will dominate and propagate by cyclically opening and closing to a critical size that will lead to instability of the structural member. This failure depends on the difference between the maximum and minimum stress, and on the number of cycles during which this difference remains above a specific level.

The rate of loading also influences the stiffness and resistance characteristics of certain materials. The stiffness and resistance of such materials increase with the rate of loading. For example, the compressive strength of concrete can increase by close to 30% for strain rates of $0.05/s$, which is typical of the rates induced in a structure by earthquake loading.

1.4. Formulation of the equation of motion

In order to determine the dynamic response of a structural system, we need to write the equations of motion governing the dynamic displacement of the system. The solution of these equations provide the system's response as a function of time. Three methods will be used in this book to write the dynamic equations of motions, i.e. Newton's second law of motion, d'Alembert's principle, and the principle of virtual work, particularly the principle of virtual displacements. A variational approach using the notion of work and energy and leading to *Hamilton's principle* can also be used. Although very powerful and often leading to a more profound understanding of the dynamic phenomena, this formulation will not be used in this book.

1.4.1. *System with one mass particle*

1.4.1.1. *Newton's second law of motion*

Newton's[1] second law of motion states that the rate of change of momentum of a mass particle m is equal to the sum of forces acting onto it, that is

$$\mathbf{p}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right) \tag{1.1}$$

where $\mathbf{p}(t)$ is the sum or resultant of all forces acting on the mass particle m, $\mathbf{u}$ is its position vector and $m(\mathrm{d}\mathbf{u}/\mathrm{d}t)$ its momentum. Assuming the mass does not vary with time as is usually the case, equation [1.1] can be written as

$$\mathbf{p}(t) = m\frac{\mathrm{d}^2\mathbf{u}}{\mathrm{d}t^2} \tag{1.2}$$

which we will write

$$\mathbf{p}(t) = m\ddot{\mathbf{u}}(t) \tag{1.3}$$

where the dots represent differentiation with time. Equation [1.3] can be written in terms of the components of the vectors, that is

$$p_i(t) = m\ddot{u}_i(t), \qquad i = 1, 2, 3. \tag{1.4}$$

1.4.1.2. *D'Alembert's principle*

Transposing the right-hand side of equation [1.2] to the left, we obtain

$$\mathbf{p}(t) - m\ddot{\mathbf{u}}(t) = \mathbf{0} \tag{1.5}$$

or in component form

$$p_i(t) - m\ddot{u}_i(t) = 0, \qquad i = 1, 2, 3. \tag{1.6}$$

These equations are an expression of d'Alembert's[2] principle which states that *the sum of all applied force vectors and vector* $-m\ddot{\mathbf{u}}$ *for a dynamic system is equal to zero*. The vector $m\ddot{\mathbf{u}}$ whose magnitude is $m\ddot{u}$ and direction is opposite to the acceleration is called *inertia force vector*. In other words, this powerful principle states that an accelerating mass particle is equivalent to a static system in equilibrium when the

1. Isaac Newton, physicist, mathematician, and natural philosopher, born in Woolsthorpe, Lincolnshire, England on December 25, 1642; died in London, England on March 20, 1727.

2. Jean Le Rond d'Alembert, lawyer, mathematician, physicist, and philosopher, born on November 17, 1717 in Paris, France; died on October 29, 1783 in Paris, France.

inertial force is added. The mass particle is said to be in *dynamic equilibrium*. Note that the inertial force must act through the center of mass and in the case of a rotating mass, an inertial moment acting anywhere must also be considered. The sum of all applied forces includes all forces resulting from kinematic constraints opposing displacement, all viscous forces opposing velocities and all external applied forces. The application of d'Alembert's principle is in general the simplest way of writing the equations of motion of a dynamic system and will be used quite extensively in this book.

1.4.1.3. *Virtual work principle*

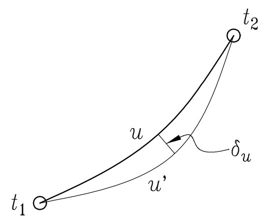

Figure 1.7. *Mass particle and virtual displacement*

Let us assume that the mass particle follows a path $\mathbf{u}$ from a given position $\mathbf{u}(t_1)$ at time t_1 to a final position at $\mathbf{u}(t_2)$ at time t_2 (Figure 1.7). Let us assume an arbitrary virtual path $\mathbf{u}'$ that has same position as $\mathbf{u}$ at time t_1 and t_2, i.e. $u_i'(t_1) = u_i(t_1)$ and $u_i'(t_2) = u_i(t_2)$. We define the components of a virtual displacement δu_i of the system at time $t_1 < t < t_2$ as

$$\delta u_i = u_i' - u_i, \qquad i = 1, 2, 3 \tag{1.7}$$

where u_i and u_i' are respectively the components of u and u' in direction 1, 2 and 3. The virtual displacement is arbitrary except for the following conditions:

$$\delta u_i(t_1) = \delta u_i(t_2), \qquad i = 1, 2, 3. \tag{1.8}$$

From equation [1.5], it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta u_i) = \frac{\mathrm{d}}{\mathrm{d}t}(u_i' - u_i) = \dot{u}_i' - \dot{u}_i = \delta \dot{u}_i \tag{1.9}$$

where it is seen that the symbol δ commutes with the first differential operator d. In fact, the symbol δ is more than an indicator of a virtual quantity but behaves like a *variational operator* obeying the rule of operation similar to the first differential operator d. If we multiply the dynamic equilibrium equations [1.6] by the corresponding virtual displacement and we take the sum of the components, we obtain

$$\sum_{i=1}^{3} (p_i(t) - m\ddot{u}_i(t))\, \delta u_i = 0 \tag{1.10}$$

which is the *principle of virtual displacements* – a particular case of the *principle of virtual work* – that can be stated as below.

THEOREM.– *The work done by the effective forces acting on a mass particle during a virtual displacement δu_i is equal to zero.*

1.4.1.4. *Constraints*

The position of a mass particle that is restricted to move in a plane can be described by two coordinates x and y or $x_i, i = 1, 2$. The system is said to have two DOF (DOFs).

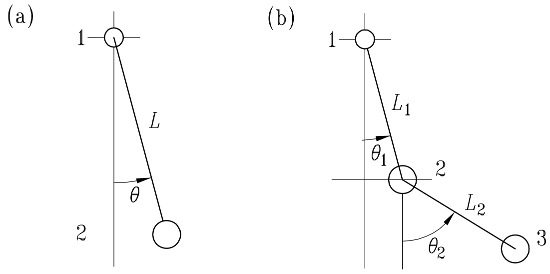

Figure 1.8. *Pendulum restricted to move on a plane: (a) simple pendulum, (b) double pendulum*

If the mass at coordinates x_1 and x_2 is attached to a frictionless hinge at position (0, 0) (Figure 1.8a) by a rigid massless bar with length L – this system is called a pendulum – a *constraint* is introduced which can be expressed by

$$x_1^2 + x_2^2 = L^2 \quad [1.11]$$

which is a *constraint equation*. The introduction of a constraint in this case reduce the number of DOFs by one. Either x_1 or x_2 or more often the angle θ between the pendulum and the vertical axis can be chosen as DOF. The constraint equation can be written as

$$f(x_1, x_2, x_3, t) = \text{const.} \quad [1.12]$$

Systems for which the constraint equation is a function of the coordinates and time are called *holonomic system* and the constraint equation is called a *holonomic constraint*. A holonomic system is further subdivided into *rheonomic* if time appears in the constraint equation or *scleronomic* otherwise. If the constraint equation is also a function of the derivatives of the coordinates with time such that

$$f(\dot{x}_1, \dot{x}_2, \dot{x}_3, x_1, x_2, x_3, t) = \text{const} \quad [1.13]$$

the system is called *non-holonomic*. We are concerned in this book with only holonomic systems.

1.4.2. *System with many mass particles*

If we have N mass particles, we will have $3N$ equations of dynamic equilibrium

$$p_{ik}(t) - m\ddot{u}_{ik} = 0, \qquad i = 1,2,3; \quad k = 1,2,\ldots,N \tag{1.14}$$

where p_{ik} are the components of all applied forces. In this case, the system is said to have $3N$ DOFs.

Let us define the virtual displacements which satisfy the kinematic conditions of the system such as

$$\delta u_{ik} = u'_{ik} - u_{ik}, \qquad i = 1,2,3; \quad k = 1,2,\ldots,N \tag{1.15}$$

with the conditions

$$\delta u_{ik}(t_1) = \delta u_{ik}(t_2), \qquad i = 1,2,3; \quad k = 1,2,\ldots,N \tag{1.16}$$

Equation [1.10] becomes

$$\sum_{k=1}^{N}\sum_{i=1}^{3} \left(p_{ik}(t) - m\ddot{u}_{ik}(t)\right) \delta u_{ik} = 0 \tag{1.17}$$

which can be stated as below.

THEOREM.– *A system of particles is in equilibrium if the total virtual work done for every virtual displacement is equal to zero.*

The position of a mass particle is described by three coordinates $x_i, i = 1,2,3$ in 3D space and has three DOFs. A system of N particles in space has $3N$ DOFs. The number of DOFs is reduced by one for every kinematic constraints that are introduced between the mass particles. Hence, the number of DOFs in 3D is given by

$$n = 3N - n_c \tag{1.18}$$

where n is the number of DOFs and n_c is the number of constraints. A double pendulum consisting of two masses m_1 and m_2 connected by massless rigid bars of length L_1 and L_2 and restricted to move in a plane has $n = 4 - 2 = 2$ DOFs (Figure 1.8b). The two constraint equations are

$$x_{11}^2 + x_{12}^2 = L_1^2 \quad \text{and} \quad x_{21}^2 + x_{22}^2 = L_2^2. \tag{1.19}$$

For many mass particles systems with constraints, the principle of virtual displacement can therefore be restated as below.

THEOREM.– *A mechanical system is in equilibrium if the total virtual work done for every virtual displacement consistent with the constraints is equal to zero.*

Mechanical systems in the previous theorem include rigid bodies with their mass and mass moment of inertia concentrated at their center of mass.

1.4.3. *System with deformable bodies*

No proof will be given, but the principle of virtual displacements can be stated as below.

THEOREM.– *A system is in equilibrium if the virtual work of external forces is equal to the virtual work of internal forces when it is subjected to a virtual displacement field that is consistent with the constraints.*

1.5. Dynamic degrees of freedom

From the preceding discussion, it can be stated that *the number of degrees of freedom (DOFs) of a structural system is the number of independent displacement coordinates or generalized coordinates that is necessary to completely and uniquely describe the displaced or deformed shape of a structure*. *Generalized coordinates* are Cartesian coordinates but can also be rotations or even amplitude of deflected shapes and Fourier series expansion as we shall observe. A simply supported beam has an infinite number of DOFs. Let us assume that two bending moments are applied to the ends of a simply supported beam. If we dispose of an analytical function relating the deflexion of the beam at any point along its length to the rotation at the ends of the beam, we need only two DOFs, namely these two rotations, to define the deformed shape of the beam. This definition applies to a static problem and needs some specialization for a dynamic problem. The generalized coordinates that must be considered in order to represent the effects of every important inertia forces on a structural system are called *dynamic DOFs*, and their number is the total number of DOFs in the system. In the case of a dynamic problem, the nodal displacements that control inertia forces are generally not significantly affected by local deformation variations. As a result, fewer DOFs are required for a dynamic model than for a static model. Let us illustrate the difference between a static and a dynamic problem with a simple example. Only basic knowledge of matrix structural analysis is required (see Chapter 13).

Consider the frame illustrated in Figure 1.9a, which consists of a beam supported by two columns fixed at their base. The beam and columns are modeled with linear beam elements and meet at points called nodes. Consider the case of static forces applied only at the nodes. In an elastic system that undergoes small displacements, the transverse displacement of the elements are uniquely related to the node displacements by cubic polynomials. If the six displacements u_1 to u_6 are known, the transverse displacements of any given point on an element can be determined. This structure therefore has six static DOFs, as shown in Figure 1.9b. If other forces are present or if the displacement of other points is sought, additional nodes must be added, which increases the number of static DOFs (three per additional node). A beam often consists of a web and a slab that is very rigid in the longitudinal axis. In this case, it can be considered as rigid in the longitudinal direction with respect to the column's flexural stiffness, which removes one DOF. Moreover, for low-rise structures, as is the case here, column longitudinal deformations can be neglected with very little impact on accuracy. Thus, there remain three static DOFs: the horizontal displacement u_1, and the rotations u_2 and u_3 (see Figure 1.9c). In a static problem, the stresses depend on the derivatives of the displacement. A more refined model improves the deformation gradient, thereby improving stress predictions.

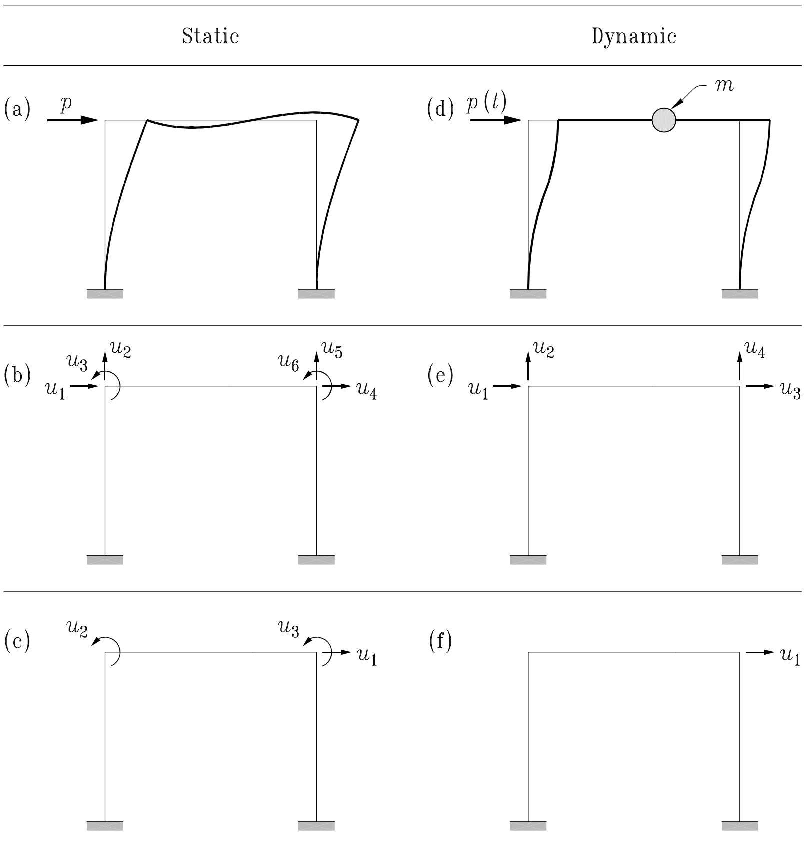

Figure 1.9. *Static and dynamic DOFs*

In the case of a dynamic load, as illustrated in Figure 1.9d, the effects of rotational inertia can be shown to be negligible. The system reduces to four translation DOFs u_1 to u_4 (see Figure 1.9e). The effects of longitudinal deformation are also negligible, as is the case in the static problem. We can also assume that the column masses are negligible with respect to the total structural mass, which is concentrated at the roof level. DOFs u_2 to u_4 can therefore be eliminated from the preceding model. The structure is reduced to a single-degree-of-freedom (SDOF) system in the horizontal direction, as shown in Figure 1.9f.

1.6. Modeling a dynamic problem

We have explained that the inertia forces characterize a dynamic problem. These forces must therefore be well defined in any model. For continuous systems such as a beam, the mass is distributed along its entire length, which means that accelerations and displacements should be defined for each point on the beam. Analysis of a beam, for example, leads to simultaneous partial differential equations that are a function of the position x along the beam and time t. It is almost impossible to solve these differential equations analytically, except with very simple structures and load cases. Discretization techniques are generally used to formulate and solve equations for dynamic problems. These techniques can be simple mass concentrations or more sophisticated coordinate-reduction methods such as Rayleigh[3] and Ritz[4] methods or the widely used finite element method. In structural dynamics, the finite element method is very often used for the spatial discretization of structures, combined with the finite difference method for time discretization. These methods are briefly described below.

1.6.1. *Mass concentration*

Important simplifications can be achieved by concentrating the masses on a given number of points. The inertia forces can only be developed at these points, and the response parameters are only defined at these locations. Figure 1.10 represents a three-span bridge with variable inertia. The bridge is modeled as a discrete system in which the mass is concentrated (or lumped) at seven specific points. Neglecting the longitudinal deformations and rotational inertia results in a model with seven dynamic DOFs. This type of modeling generally leads to an n DOFs system. The problem is determining n in order to represent the inertia forces as accurately as

3. John William Strutt Lord Rayleigh, mathematician and physicist, born on November 12, 1842 in Langford Grove, Essex, UK, died on June 30, 1919 in Terling Place, Essex, UK.

4. Walter Ritz, physicist, born on February 22, 1878 in Sion, Switzerland, died on July 7, 1909 in Göttingen, Germany.