IEEE Press
445 Hoes Lane
Piscataway, NJ 08854

IEEE Press Editorial Board 2013
John Anderson, Editor in Chief

Linda Shafer
George W. Arnold
Ekram Hossain
Om P. Malik

Saeid Nahavandi
David Jacobson
Mary Lanzerootti

George Zobrist
Tariq Samad
Dmitry Goldgof

Kenneth Moore, Director of IEEE Book and Information Services (BIS)
NUMERICAL ANALYSIS WITH APPLICATIONS IN MECHANICS AND ENGINEERING

PETRE TEODORESCU
NICOLAE-DORU STĂNESCU
NICOLAE PANDREA

IEEE PRESS

WILEY
CONTENTS

Preface xi

1 Errors in Numerical Analysis 1
   1.1 Enter Data Errors, 1
   1.2 Approximation Errors, 2
   1.3 Round-Off Errors, 3
   1.4 Propagation of Errors, 3
      1.4.1 Addition, 3
      1.4.2 Multiplication, 5
      1.4.3 Inversion of a Number, 7
      1.4.4 Division of Two Numbers, 7
      1.4.5 Raising to a Negative Entire Power, 7
      1.4.6 Taking the Root of $p$th Order, 7
      1.4.7 Subtraction, 8
      1.4.8 Computation of Functions, 8
   1.5 Applications, 8
      Further Reading, 14

2 Solution of Equations 17
   2.1 The Bipartition (Bisection) Method, 17
   2.2 The Chord (Secant) Method, 20
   2.3 The Tangent Method (Newton), 26
   2.4 The Contraction Method, 37
   2.5 The Newton–Kantorovich Method, 42
   2.6 Numerical Examples, 46
   2.7 Applications, 49
      Further Reading, 52
3 Solution of Algebraic Equations

3.1 Determination of Limits of the Roots of Polynomials, 55
3.2 Separation of Roots, 60
3.3 Lagrange’s Method, 69
3.4 The Lobachevski–Graeffe Method, 72
   3.4.1 The Case of Distinct Real Roots, 72
   3.4.2 The Case of a Pair of Complex Conjugate Roots, 74
   3.4.3 The Case of Two Pairs of Complex Conjugate Roots, 75
3.5 The Bernoulli Method, 76
3.6 The Bierge–Viète Method, 79
3.7 Lin Methods, 79
3.8 Numerical Examples, 82
3.9 Applications, 94
Further Reading, 109

4 Linear Algebra

4.1 Calculation of Determinants, 111
   4.1.1 Use of Definition, 111
   4.1.2 Use of Equivalent Matrices, 112
4.2 Calculation of the Rank, 113
4.3 Norm of a Matrix, 114
4.4 Inversion of Matrices, 123
   4.4.1 Direct Inversion, 123
   4.4.2 The Gauss–Jordan Method, 124
   4.4.3 The Determination of the Inverse Matrix by its Partition, 125
   4.4.4 Schur’s Method of Inversion of Matrices, 127
   4.4.5 The Iterative Method (Schulz), 128
   4.4.6 Inversion by Means of the Characteristic Polynomial, 131
   4.4.7 The Frame–Fadeev Method, 131
4.5 Solution of Linear Algebraic Systems of Equations, 132
   4.5.1 Cramer’s Rule, 132
   4.5.2 Gauss’s Method, 133
   4.5.3 The Gauss–Jordan Method, 134
   4.5.4 The LU Factorization, 135
   4.5.5 The Schur Method of Solving Systems of Linear Equations, 137
   4.5.6 The Iteration Method (Jacobi), 142
   4.5.7 The Gauss–Seidel Method, 147
   4.5.8 The Relaxation Method, 149
   4.5.9 The Monte Carlo Method, 150
   4.5.10 Infinite Systems of Linear Equations, 152
4.6 Determination of Eigenvalues and Eigenvectors, 153
   4.6.1 Introduction, 153
   4.6.2 Krylov’s Method, 155
   4.6.3 Danilevski’s Method, 157
   4.6.4 The Direct Power Method, 160
   4.6.5 The Inverse Power Method, 165
   4.6.6 The Displacement Method, 166
   4.6.7 Leverrier’s Method, 166
4.6.8 The L–R (Left–Right) Method, 166
4.6.9 The Rotation Method, 168
4.7 QR Decomposition, 169
4.8 The Singular Value Decomposition (SVD), 172
4.9 Use of the Least Squares Method in Solving the Linear Overdetermined Systems, 174
4.10 The Pseudo-Inverse of a Matrix, 177
4.11 Solving of the Underdetermined Linear Systems, 178
4.12 Numerical Examples, 178
4.13 Applications, 211
Further Reading, 269

5 Solution of Systems of Nonlinear Equations 273
5.1 The Iteration Method (Jacobi), 273
5.2 Newton’s Method, 275
5.3 The Modified Newton’s Method, 276
5.4 The Newton–Raphson Method, 277
5.5 The Gradient Method, 277
5.6 The Method of Entire Series, 280
5.7 Numerical Example, 281
5.8 Applications, 287
Further Reading, 304

6 Interpolation and Approximation of Functions 307
6.1 Lagrange’s Interpolation Polynomial, 307
6.2 Taylor Polynomials, 311
6.3 Finite Differences: Generalized Power, 312
6.4 Newton’s Interpolation Polynomials, 317
6.5 Central Differences: Gauss’s Formulae, Stirling’s Formula, Bessel’s Formula, Everett’s Formulae, 322
6.6 Divided Differences, 327
6.7 Newton-Type Formula with Divided Differences, 331
6.8 Inverse Interpolation, 332
6.9 Determination of the Roots of an Equation by Inverse Interpolation, 333
6.10 Interpolation by Spline Functions, 335
6.11 Hermite’s Interpolation, 339
6.12 Chebyshev’s Polynomials, 340
6.13 Mini–Max Approximation of Functions, 344
6.14 Almost Mini–Max Approximation of Functions, 345
6.15 Approximation of Functions by Trigonometric Functions (Fourier), 346
6.16 Approximation of Functions by the Least Squares, 352
6.17 Other Methods of Interpolation, 354
6.17.1 Interpolation with Rational Functions, 354
6.17.2 The Method of Least Squares with Rational Functions, 355
6.17.3 Interpolation with Exponentials, 355
6.18 Numerical Examples, 356
6.19 Applications, 363
Further Reading, 374
7 Numerical Differentiation and Integration

7.1 Introduction, 377
7.2 Numerical Differentiation by Means of an Expansion into a Taylor Series, 377
7.3 Numerical Differentiation by Means of Interpolation Polynomials, 380
7.4 Introduction to Numerical Integration, 382
7.5 The Newton–Côtes Quadrature Formulae, 384
7.6 The Trapezoid Formula, 386
7.7 Simpson’s Formula, 389
7.8 Euler’s and Gregory’s Formulae, 393
7.9 Romberg’s Formula, 396
7.10 Chebyshev’s Quadrature Formulae, 398
7.11 Legendre’s Polynomials, 400
7.12 Gauss’s Quadrature Formulae, 405
7.13 Orthogonal Polynomials, 406
  7.13.1 Legendre Polynomials, 407
  7.13.2 Chebyshev Polynomials, 407
  7.13.3 Jacobi Polynomials, 408
  7.13.4 Hermite Polynomials, 408
  7.13.5 Laguerre Polynomials, 409
  7.13.6 General Properties of the Orthogonal Polynomials, 410
7.14 Quadrature Formulae of Gauss Type Obtained by Orthogonal Polynomials, 412
  7.14.1 Gauss–Jacobi Quadrature Formulae, 413
  7.14.2 Gauss–Hermite Quadrature Formulae, 414
  7.14.3 Gauss–Laguerre Quadrature Formulae, 415
7.15 Other Quadrature Formulae, 417
  7.15.1 Gauss Formulae with Imposed Points, 417
  7.15.2 Gauss Formulae in which the Derivatives of the Function Also Appear, 418
7.16 Calculation of Improper Integrals, 420
7.17 Kantorovich’s Method, 422
7.18 The Monte Carlo Method for Calculation of Definite Integrals, 423
  7.18.1 The One-Dimensional Case, 423
  7.18.2 The Multidimensional Case, 425
7.19 Numerical Examples, 427
7.20 Applications, 435

Further Reading, 447

8 Integration of Ordinary Differential Equations and of Systems of Ordinary Differential Equations

8.1 State of the Problem, 451
8.2 Euler’s Method, 454
8.3 Taylor Method, 457
8.4 The Runge–Kutta Methods, 458
8.5 Multistep Methods, 462
8.6 Adams’s Method, 463
8.7 The Adams–Bashforth Methods, 465
8.8 The Adams–Moulton Methods, 467
8.9 Predictor–Corrector Methods, 469
  8.9.1 Euler’s Predictor–Corrector Method, 469
8.9.2 Adams’s Predictor–Corrector Methods, 469
8.9.3 Milne’s Fourth-Order Predictor–Corrector Method, 470
8.9.4 Hamming’s Predictor–Corrector Method, 470
8.10 The Linear Equivalence Method (LEM), 471
8.11 Considerations about the Errors, 473
8.12 Numerical Example, 474
8.13 Applications, 480
Further Reading, 525

9 Integration of Partial Differential Equations and of
Systems of Partial Differential Equations 529

9.1 Introduction, 529
9.2 Partial Differential Equations of First Order, 529
  9.2.1 Numerical Integration by Means of Explicit Schemata, 531
  9.2.2 Numerical Integration by Means of Implicit Schemata, 533
9.3 Partial Differential Equations of Second Order, 534
9.4 Partial Differential Equations of Second Order of Elliptic Type, 534
9.5 Partial Differential Equations of Second Order of Parabolic Type, 538
9.6 Partial Differential Equations of Second Order of Hyperbolic Type, 543
9.7 Point Matching Method, 546
9.8 Variational Methods, 547
  9.8.1 Ritz’s Method, 549
  9.8.2 Galerkin’s Method, 551
  9.8.3 Method of the Least Squares, 553
9.9 Numerical Examples, 554
9.10 Applications, 562
Further Reading, 575

10 Optimizations 577

10.1 Introduction, 577
10.2 Minimization Along a Direction, 578
  10.2.1 Localization of the Minimum, 579
  10.2.2 Determination of the Minimum, 580
10.3 Conjugate Directions, 583
10.4 Powell’s Algorithm, 585
10.5 Methods of Gradient Type, 585
  10.5.1 The Gradient Method, 585
  10.5.2 The Conjugate Gradient Method, 587
  10.5.3 Solution of Systems of Linear Equations by Means of Methods of
        Gradient Type, 589
10.6 Methods of Newton Type, 590
  10.6.1 Newton’s Method, 590
  10.6.2 Quasi-Newton Method, 592
10.7 Linear Programming: The Simplex Algorithm, 593
  10.7.1 Introduction, 593
  10.7.2 Formulation of the Problem of Linear Programming, 595
  10.7.3 Geometrical Interpretation, 597
  10.7.4 The Primal Simplex Algorithm, 597
  10.7.5 The Dual Simplex Algorithm, 599
CONTENTS

10.8 Convex Programming, 600
10.9 Numerical Methods for Problems of Convex Programming, 602
   10.9.1 Method of Conditional Gradient, 602
   10.9.2 Method of Gradient’s Projection, 602
   10.9.3 Method of Possible Directions, 603
   10.9.4 Method of Penalizing Functions, 603
10.10 Quadratic Programming, 603
10.11 Dynamic Programming, 605
10.12 Pontryagin’s Principle of Maximum, 607
10.13 Problems of Extremum, 609
10.14 Numerical Examples, 611
10.15 Applications, 623
   Further Reading, 626

Index 629
In writing this book, it is the authors’ wish to create a bridge between mathematical and technical disciplines, which requires knowledge of strong mathematical tools in the area of numerical analysis. Unlike other books in this area, this interdisciplinary work links the applicative part of numerical methods, where mathematical results are used without understanding their proof, to the theoretical part of these methods, where each statement is rigorously demonstrated.

Each chapter is followed by problems of mechanics, physics, or engineering. The problem is first stated in its mechanical or technical form. Then the mathematical model is set up, emphasizing the physical magnitudes playing the part of unknown functions and the laws that lead to the mathematical problem. The solution is then obtained by specifying the mathematical methods described in the corresponding theoretical presentation. Finally, a mechanical, physical, and technical interpretation of the solution is provided, giving rise to complete knowledge of the studied phenomenon.

The book is organized into 10 chapters. Each of them begins with a theoretical presentation, which is based on practical computation—the “know-how” of the mathematical method—and ends with a range of applications.

The book contains some personal results of the authors, which have been found to be beneficial to readers.

The authors are grateful to Mrs. Eng. Ariadna–Carmen Stan for her valuable help in the presentation of this book. The excellent cooperation from the team of John Wiley & Sons, Hoboken, USA, is gratefully acknowledged.

The prerequisites of this book are courses in elementary analysis and algebra, acquired by a student in a technical university. The book is addressed to a broad audience—to all those interested in using mathematical models and methods in various fields such as mechanics, physics, and civil and mechanical engineering; people involved in teaching, research, or design; as well as students.

PETRE TEODORESCU
NICOLAE-DORU STĂNESCU
NICOLAE PANDREA
1 ERRORS IN NUMERICAL ANALYSIS

In this chapter, we deal with the most encountered errors in numerical analysis, that is, enter data errors, approximation errors, round-off errors, and propagation of errors.

1.1 ENTER DATA ERRORS

Enter data errors appear, usually, if the enter data are obtained from measurements or experiments. In such a case, the errors corresponding to the estimation of the enter data are propagated, by means of the calculation algorithm, to the exit data.

We define in what follows the notion of stability of errors.

**Definition 1.1** A calculation process \( P \) is stable to errors if, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if for any two sets \( I_1 \) and \( I_2 \) of enter data we have \( \| I_1 - I_2 \|_i < \delta \), then the two exit sets \( S_1 \) and \( S_2 \), corresponding to \( I_1 \) and \( I_2 \), respectively, verify the relation \( \| S_1 - S_2 \|_e < \varepsilon \).

**Observation 1.1** The two norms \( \| \|_i \) and \( \| \|_e \) of the enter and exit quantities, respectively, which occur in Definition 1.1, depend on the process considered.

Intuitively, according to Definition 1.1, the calculation process is stable if, for small variations of the enter data, we obtain small variations of the exit data.

Hence, we must characterize the stable calculation process. Let us consider that the calculation process \( P \) is characterized by a family \( f_k \) of functions defined on a set of enter data with values in a set of exit data. We consider such a vector function \( f_k \) of vector variable \( f_k : \mathcal{D} \to \mathbb{R}^n \), where \( \mathcal{D} \) is a domain in \( \mathbb{R}^m \) (we propose to have \( m \) enter data and \( n \) exit data).

**Definition 1.2** \( f : \mathcal{D} \to \mathbb{R}^n \) is a Lipschitz function (has the Lipschitz property) if there exists \( m > 0 \), constant, so as to have \( \| f(x) - f(y) \| < m \| x - y \| \) for any \( x, y \in \mathcal{D} \) (the first norm is in \( \mathbb{R}^n \) and the second one in \( \mathbb{R}^m \)).
2 ERRORS IN NUMERICAL ANALYSIS

It is easy to see that a calculation process characterized by Lipschitz functions is stable.

In addition, a function with the Lipschitz property is continuous (even uniform continuous) but the converse does not hold; for example, the function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \), \( f(x) = \sqrt{x} \), is continuous but it is not Lipschitz. Indeed, let us suppose that \( f(x) = \sqrt{x} \) is Lipschitz, hence that it has a positive constant \( m > 0 \) such that

\[
|f(x) - f(y)| < m|x - y|, \quad (\forall)x, y \in \mathbb{R}_+.
\]

Let us choose \( x \) and \( y \) such that \( 0 < y < x < 1/4m^2 \). Expression (1.1) leads to

\[
\sqrt{x} - \sqrt{y} < m(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}),
\]

from which we get

\[
1 < m(\sqrt{x} + \sqrt{y}).
\]

From the choice of \( x \) and \( y \), it follows that

\[
\sqrt{x} + \sqrt{y} < \sqrt{\frac{1}{4m^2}} + \sqrt{\frac{1}{4m^2}} = \frac{1}{m},
\]

so that relations (1.3) and (1.4) lead to

\[
1 < m \frac{1}{m} = 1,
\]

which is absurd. Hence, the continuous function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \), \( f(x) = \sqrt{x} \) is not a Lipschitz one.

1.2 APPROXIMATION ERRORS

The approximation errors have to be accepted by the conception of the algorithms because of various objective considerations.

Let us determine the limit of a sequence using a computer; it is supposed that the sequence is convergent. Let the sequence \( \{x_n\}_{n \in \mathbb{N}} \) be defined by the relation

\[
x_{n+1} = \frac{1}{1 + x_n^2}, \quad n \in \mathbb{N}, \quad x_0 \in \mathbb{R}.
\]

We observe that the terms of the sequence are positive, excepting eventually \( x_0 \). The limit of this sequence, denoted by \( \bar{x} \), is the positive root of the equation

\[
x = \frac{1}{1 + x^2}.
\]

If we wish to determine \( \bar{x} \) with two exact decimal digits, then we take an arbitrary value of \( x_0 \), for example, \( x_0 = 0 \), and calculate the successive terms of the sequence \( \{x_n\} \) (Table 1.1).
TABLE 1.1 Calculation of $\bar{x}$ with Two Exact Decimal Digits

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$n$</th>
<th>$x_n$</th>
<th>$n$</th>
<th>$x_n$</th>
<th>$n$</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0.6028</td>
<td>8</td>
<td>0.6705</td>
<td>12</td>
<td>0.6804</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
<td>0.7290</td>
<td>9</td>
<td>0.6899</td>
<td>13</td>
<td>0.6836</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>6</td>
<td>0.6530</td>
<td>10</td>
<td>0.6775</td>
<td>14</td>
<td>0.6815</td>
</tr>
<tr>
<td>3</td>
<td>0.8</td>
<td>7</td>
<td>0.7011</td>
<td>11</td>
<td>0.6854</td>
<td>15</td>
<td>0.6828</td>
</tr>
</tbody>
</table>

We obtain $\bar{x} = 0.68 \ldots$

1.3 ROUND-OFF ERRORS

Round-off errors are due to the mode of representation of the data in the computer. For instance, the number 0.8125 in base 10 is represented in base 2 in the form $0.1101_2$ and the number 0.75 in the form $0.112_2$. Let us suppose that we have a computer that works with three significant digits. The sum $0.8125 + 0.75$ becomes

$$1.5625 = 0.1101_2 + 0.112_2 \approx 0.1101_2 + 0.1100_2 = 1.1002_2 = 1.5.$$  \hspace{1cm} (1.8)

Such errors may also appear because of the choice of inadequate types of data in the programming realized on the computer.

1.4 PROPAGATION OF ERRORS

Let us consider the number $x$ and let $\bar{x}$ be an approximation of it.

Definition 1.3

(i) We call absolute error the expression

$$E = x - \bar{x},$$  \hspace{1cm} (1.9)

(ii) We call relative error the expression

$$R = \frac{x - \bar{x}}{x}.$$  \hspace{1cm} (1.10)

1.4.1 Addition

Let $x_1, x_2, \ldots, x_n$ be the numbers for which the relative errors are $R_1, R_2, \ldots, R_n$, while their absolute errors read $E_1, E_2, \ldots, E_n$.

The relative error of the sum is

$$R \left( \sum_{i=1}^{n} x_i \right) = \frac{\sum_{i=1}^{n} E_i}{\sum_{i=1}^{n} x_i},$$  \hspace{1cm} (1.11)

and we may write the relation

$$\min_{i=1}^{n} |R_i| \leq \left| R \left( \sum_{i=1}^{n} x_i \right) \right| \leq \max_{i=1}^{n} |R_i|,$$  \hspace{1cm} (1.12)
that is, the modulus of the relative error of the sum is contained between the lower and the higher values in the modulus of the relative errors of the component members.

Thus, if the terms \( x_1, x_2, \ldots, x_n \) are positive and of the same order of magnitude,

\[
\max_{i=1, n} x_i < 10, \quad \min_{i=1, n} x_i < 10,
\]

then we must take the same number of significant digits for each term \( x_i, i = 1, n \), the same number of significant digits occurring in the sum too.

If the numbers \( x_1, x_2, \ldots, x_n \) are much different among them, then the number of the significant digits after the comma is given by the greatest number \( x_i \) (we suppose that \( x_i > 0, i = 1, n \)). For instance, if we have to add the numbers

\[
x_1 = 100.32, \quad x_2 = 0.57381,
\]

both numbers having five significant digits, then we will round off \( x_2 \) to two digits (as \( x_1 \)) and write

\[
x_1 + x_2 = 100.32 + 0.57 = 100.89.
\]

It is observed that addition may result in a compensation of the errors, in the sense that the absolute error of the sum is, in general, smaller than the sum of the absolute error of each term.

We consider that the absolute error has a Gauss distribution for each of the terms \( x_i, i = 1, n \), given by the distribution density

\[
\phi(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}},
\]

from which we obtain the distribution function

\[
\Phi(x) = \int_{-\infty}^{x} \phi(t) dt,
\]

with the properties

\[
\Phi(-\infty) = 0, \quad \Phi(\infty) = 1, \quad \Phi(x) \in (0, 1), \quad -\infty < x < \infty.
\]

The probability that \( x \) is contained between \(-x_0\) and \( x_0\), with \( x_0 > 0\) is

\[
P(|x| < x_0) = \Phi(x_0) - \Phi(-x_0) = \int_{-x_0}^{x_0} \phi(t) dt = \frac{\sqrt{2}}{\sigma \sqrt{\pi}} \int_{0}^{\frac{x_0^2}{2\sigma^2}} e^{-t} dt.
\]

Because \( \phi(x) \) is an even function, it follows that the mean value of a variable with a normal Gauss distribution is

\[
x_{\text{med}} = \int_{-\infty}^{\infty} x \phi(x) dx = 0,
\]

while its mean square deviation reads

\[
(x^2)_{\text{max}} = \int_{-\infty}^{\infty} x^2 \phi(x) dx = \sigma^2.
\]

Usually, we choose \( \sigma \) as being the mean square root

\[
\sigma = \sigma_{\text{RMS}} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} R_i^2},
\]
1.4.2 Multiplication

Let us consider two numbers \( x_1, x_2 \) for which the relative errors are \( R_1, R_2 \), while the approximations are \( \bar{x}_1, \bar{x}_2 \), respectively. We have

\[
\bar{x}_1 \bar{x}_2 = x_1 (1 + R_1) x_2 (1 + R_2) = x_1 x_2 (1 + R_1 + R_2 + R_1 R_2).
\] (1.23)

Because \( R_1 \) and \( R_2 \) are small, we may consider \( R_1 R_2 \approx 0 \), hence

\[
\bar{x}_1 \bar{x}_2 = x_1 x_2 (1 + R_1 + R_2),
\] (1.24)

so that the relative error of the product of the two numbers reads

\[
R(x_1 x_2) = R_1 + R_2.
\] (1.25)

Similarly, for \( n \) numbers \( x_1, x_2, \ldots, x_n \), of relative errors \( R_1, R_2, \ldots, R_n \), we have

\[
R \left( \prod_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} R_i.
\] (1.26)

Let \( x \) be a number that may be written in the form

\[
x = x^* \times 10^r, \quad 1 \leq x^* < 10, \quad 10^r \leq x < 10^{r+1}, \quad x^* \in \mathbb{Z}.
\] (1.27)

The absolute error is

\[
|E| \leq 10^{r-n},
\] (1.28)

while the relative one is

\[
|R| = \frac{|E|}{x} \leq \frac{10^{r-n+1}}{x^*} \times 10^r = \frac{10^{r-n+1}}{x^*} \leq 10^{r-n+1},
\] (1.29)

where we have supposed that \( x \) has \( n \) significant digits.

If \( \bar{x} \) is the round-off of \( x \) at \( n \) significant digits, then

\[
|E| \leq 5 \times 10^{r-n}, \quad |R| \leq \frac{5}{\bar{x}} \times 10^{r-n}.
\] (1.30)

The error of the last significant digit, the \( n \)th, is

\[
\varepsilon = \frac{E}{10^{r-n+1}} = \frac{xR}{10^{r-n+1}} = x^* R \times 10^{n-1}.
\] (1.31)

Let \( x_1, x_2 \) now be two numbers of relative errors \( R_1, R_2 \) and let \( R \) be the relative error of the product \( x_1 x_2 \). We have

\[
R = \frac{x_1 x_2 - \bar{x}_1 \bar{x}_2}{x_1 x_2} = R_1 + R_2 - R_1 R_2.
\] (1.32)

Moreover, \( |R| \) takes its greatest value if \( R_1 \) and \( R_2 \) are negative; hence, we may write

\[
|R| \leq 5 \left( \frac{1}{x_1} + \frac{1}{x_2} \right) \times 10^{-n} + \frac{25}{x_1 x_2} \times 10^{-2n}.
\] (1.33)
where the error of the digit on the \( n \)th position is
\[
|\varepsilon(x_1, x_2)| \leq \frac{(x_1 x_2)^*}{2} \left( \frac{1}{x_1^*} + \frac{1}{x_2^*} \right) + \frac{5}{2} \frac{(x_1 x_2)^*}{x_1^* x_2^*} \times 10^{-n}. \tag{1.34}
\]

On the other hand,
\[
(x_1 x_2)^* = x_1^* x_2^* \times 10^{-p}, \tag{1.35}
\]
where \( p = 0 \) or \( p = 1 \), the most disadvantageous case being that described by \( p = 0 \).

The function
\[
\phi(x_1^*, x_2^*) = \frac{10^p}{2} (x_1^* + x_2^* + 5 \times 10^{-n}) \tag{1.36}
\]
defined for \( 1 \leq x_1^* < 10, 1 \leq x_2^* < 10, 1 \leq x_1^* x_2^* < 10 \) will attain its maximum on the frontier of the above domain, that is, for \( x_1^* = 1, x_2^* = 10 \) or \( x_1^* = 10, x_2^* = 1 \). It follows that
\[
\phi(x_1^*, x_2^*) \leq \frac{10^p}{2} (11 + 5 \times 10^{-n}), \tag{1.37}
\]
and hence
\[
|\varepsilon(x_1, x_2)| \leq \frac{11}{2} + \frac{5}{2} \times 10^{-n} < 6, \tag{1.38}
\]
so that the error of the \( n \)th digit of the response will have at the most six units.

If \( x_1 = x_2 = x \), then the most disadvantageous case is given by
\[
(x^*)^2 = (x^2)^* = 10 \tag{1.39}
\]
when
\[
|\varepsilon(x^2)| \leq \frac{5}{2} \times 10^{-n} < 4, \tag{1.40}
\]
that is, the \( n \)th digit of \( x^2 \) is given by an approximation of four units.

Let \( x_1, \ldots, x_m \) now be \( m \) numbers; then
\[
|\varepsilon \left( \prod_{i=1}^{m} x_i \right) | \leq \frac{(x_1 \cdots x_m)^*}{2 \times 5 \times 10^{-n}} \left[ \prod_{i=1}^{m} \left( 1 + \frac{5 \times 10^{-n}}{x_i^*} \right) - 1 \right], \tag{1.41}
\]

the most disadvantageous case being that in which \( m - 1 \) numbers \( x_i^* \) are equal to 1, while one number is equal 10. In this case, we have
\[
|\varepsilon \left( \prod_{i=1}^{m} x_i \right) | \leq \frac{5}{5 \times 10^{-n}} \left[ (1 + 5 \times 10^{-n})^m - 1 \right]. \tag{1.42}
\]

If all the \( m \) numbers are equal, \( x_i = x, \ i = 1, m \), then the most disadvantageous situation appears for \( (x^*)^m = (x^m)^* = 10 \), and hence it follows that
\[
|\varepsilon(x^m)| \leq \frac{5}{5 \times 10^{-n}} \left[ \left( 1 + \frac{5 \times 10^{-m}}{10} \right)^m - 1 \right]. \tag{1.43}
\]
1.4.3 Inversion of a Number

Let \( x \) be a number, \( \bar{x} \) its approximation, and \( R \) its relative error. We may write

\[
\frac{1}{\bar{x}} = \frac{1}{x(1 + R)} = \frac{1}{x} \left(1 - R + R^2 - R^3 + \cdots\right) \approx \frac{1}{x} (1 - R),
\]

hence

\[
R \left( \frac{1}{x} \right) = -\frac{x - \bar{x}}{x} = R,
\]

so that the relative error remains the same.

In general,

\[
E \left( \frac{1}{x} \right) = -\frac{E}{x^2}.
\]

1.4.4 Division of Two Numbers

We may imagine the division of \( x_1 \) by \( x_2 \) as the multiplication of \( x_1 \) by \( \frac{1}{x_2} \), so that

\[
R \left( \frac{x_1}{x_2} \right) = R(x_1) + R(x_2);
\]

hence, the relative errors are summed up.

1.4.5 Raising to a Negative Entire Power

We may write

\[
R \left( \frac{1}{x^n} \right) = R \left( \frac{\frac{1}{x}}{\frac{1}{x}} \cdots \frac{1}{\frac{1}{x}} \right) = \sum_{i=1}^{m} R \left( \frac{1}{x} \right) = \sum_{i=1}^{m} R(x), \quad m \in \mathbb{N}, \ m \neq 0,
\]

so that the relative errors are summed up.

1.4.6 Taking the Root of \( p \)th Order

We have, successively,

\[
\sqrt[p]{x + R} = \sqrt[p]{\bar{x} + \bar{R}} = \sqrt[p]{\bar{x} + R} \\
= \sqrt[p]{x} \left[1 + \frac{R}{p} \left(1 + \frac{1}{p-1} \frac{R^2}{2!} + \frac{1}{p-1} \frac{1}{p-2} \frac{R^3}{3!} + \cdots\right)\right],
\]

\[
R \left( \sqrt[p]{x} \right) = \frac{\sqrt[p]{x} - \sqrt[p]{\bar{x}}}{\sqrt[p]{\bar{x}}} \approx -\frac{R}{\bar{x}}.
\]

The maximum error for the \( n \)th digit is now obtained for \( x = 10^{(k-m)/m}, \ x^* = 1, \ (x^*)^m = 10^{1-m}, \ m = 1/p, \ k \) entire, and is given by

\[
\left| \varepsilon (x^*)^\frac{1}{p} \right| \leq \frac{10^{1-m}}{2 \times 5 \times 10^{-n}} [(1 + 5 \times 10^{-n})^m - 1] = 10^{5-m} [(1 + 5 \times 10^{-n})^m - 1].
\]
1.4.7 Subtraction

Subtraction is the most disadvantageous operation if the result is small with respect to the minuend and the subtrahend.

Let us consider the subtraction $20.003 - 19.998$ in which the first four digits of each number are known with precision; concerning the fifth digit, we can say that it is determined with a precision of 1 unit. It follows that for $20.003$ the relative error is

$$R_1 \leq \frac{10^{-3}}{20.003} < 5 \times 10^{-5},$$

while for $19.998$ the relative error is

$$R_1 \leq \frac{10^{-3}}{19.998} < 5.1 \times 10^{-5}.$$  

The result of the subtraction operation is $5 \times 10^{-3}$, while the last digit may be wrong with two units, so that the relative error of the difference is

$$R = \frac{2 \times 10^{-3}}{5 \times 10^{-3}} = 400 \times 10^{-3},$$

that is, a relative error that is approximately 8000 times greater than $R_1$ or $R_2$.

It follows the rule that the difference of two quantities must be directly calculated, without previously calculating the two quantities.

1.4.8 Computation of Functions

Starting from Taylor’s relation

$$f(x) - f(\xi) = (x - \xi)f'(\xi),$$

where $\xi$ is a point situated between $x$ and $\xi$, it follows that the absolute error is

$$|E(f)| \leq |E| \sup_{\xi \in \text{Int}(x, \xi)} |f'(\xi)|,$$

while the relative error reads

$$|R(f)| \leq \frac{|E|}{|f(x)|} \sup_{\xi \in \text{Int}(x, \xi)} |f'(\xi)|,$$

where $\text{Int}(x, \xi)$ defines the real interval of ends $x$ and $\xi$.

1.5 APPLICATIONS

Problem 1.1

Let us consider the sequence of integrals

$$I_n = \int_0^1 x^n e^x \, dx, \quad n \in \mathbb{N}.$$  

(i) Determine a recurrence formula for $\{I_n\}_{n \in \mathbb{N}}$.  

Solution: To calculate $I_n, \ n \geq 1$, we use integration by parts and have

$$I_n = \int_0^1 x^n e^x \, dx = x^n e^x \bigg|_0^1 - n \int_0^0 x^{n-1} e^x \, dx = e - I_{n-1}. \quad (1.59)$$

(ii) Show that $\lim_{n \to \infty} I_n$ does exist.

Solution: For $x \in [0, 1]$ we have

$$x^{n+1}e^x \leq x^n e^x, \quad (1.60)$$

hence $I_{n+1} \leq I_n$ for any $n \in \mathbb{N}$. It follows that $\{I_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of real numbers.

On the other hand,

$$x^n e^x \geq 0, \ x \in [0, 1], \ n \in \mathbb{N}, \quad (1.61)$$

so that $\{I_n\}_{n \in \mathbb{N}}$ is a positive sequence of real numbers.

We get

$$0 \leq \cdots \leq I_{n+1} \leq I_n \leq \cdots \leq I_1 \leq I_0, \quad (1.62)$$

so that $\{I_n\}_{n \in \mathbb{N}}$ is convergent and, moreover,

$$0 \leq \lim_{n \to \infty} I_n \leq I_0 = \int_0^1 e^x \, dx = e - 1. \quad (1.63)$$

(iii) Calculate $I_{13}$.

Solution: To calculate the integral we have two methods.

Method 1.

$$I_0 = \int_0^1 e^x \, dx = e \bigg|_0^1 = e - 1, \quad (1.64)$$

$$I_1 = e - 1I_0 = 1, \quad (1.65)$$

$$I_2 = e - 2I_1 = e - 2, \quad (1.66)$$

$$I_3 = e - 3I_2 = 6 - 2e, \quad (1.67)$$

$$I_4 = e - 4I_3 = 9e - 24, \quad (1.68)$$

$$I_5 = e - 5I_4 = 120 - 44e, \quad (1.69)$$

$$I_6 = e - 6I_5 = 265e - 720, \quad (1.70)$$

$$I_7 = e - 7I_6 = 5040 - 1854e, \quad (1.71)$$

$$I_8 = e - 8I_7 = 14833e - 40320, \quad (1.72)$$

$$I_9 = e - 9I_8 = 362880 - 133496e, \quad (1.73)$$

$$I_{10} = e - 10I_9 = 1334961e - 3628800, \quad (1.74)$$

$$I_{11} = e - 11I_{10} = 39916800 - 14684570e, \quad (1.75)$$

$$I_{12} = e - 12I_{11} = 176214841e - 479001600, \quad (1.76)$$

$$I_{13} = e - 13I_{12} = 6227020800 - 229079932e. \quad (1.77)$$

It follows that

$$I_{13} = 0.1798. \quad (1.78)$$
Method 2. In this case, we replace directly the calculated values, thus obtaining

\[
I_0 = e - 1 = 1.718281828, \\
I_1 = e - I_0 = 1, \\
I_2 = e - 2I_0 = 0.718281828, \\
I_3 = e - 3I_2 = 0.563436344, \\
I_4 = e - 4I_3 = 0.464536452, \\
I_5 = e - 5I_4 = 0.395599568, \\
I_6 = e - 6I_5 = 0.34468442, \\
I_7 = e - 7I_6 = 0.305490888, \\
I_8 = e - 8I_7 = 0.274354724, \\
I_9 = e - 9I_8 = 0.249089312, \\
I_{10} = e - 10I_9 = 0.227388708, \\
I_{11} = e - 11I_{10} = 0.21700604, \\
I_{12} = e - 12I_{11} = 0.114209348, \\
I_{13} = e - 13I_{12} = 1.233560304.
\]

We observe that, because of the propagation of errors, the second method cannot be used to calculate \( I_n \), \( n \geq 12 \).

**Problem 1.2**

Let the sequences \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) be defined recursively:

\[
x_{n+1} = \frac{1}{2} \left( x_n + \frac{0.5}{x_n} \right), \quad x_0 = 1, \\
y_{n+1} = y_n - \lambda(y_n^2 - 0.5), \quad y_0 = 1.
\]

(i) Calculate \( x_1, x_2, \ldots, x_7 \).

**Solution:** We have, successively,

\[
x_1 = \frac{1}{2} \left( x_0 + \frac{0.5}{x_0} \right) = \frac{3}{4}, \\
x_2 = \frac{1}{2} \left( x_1 + \frac{0.5}{x_1} \right) = \frac{17}{24}, \\
x_3 = \frac{1}{2} \left( x_2 + \frac{0.5}{x_2} \right) = \frac{577}{816}, \\
x_4 = \frac{1}{2} \left( x_3 + \frac{0.5}{x_3} \right) = 0.707107, \\
x_5 = \frac{1}{2} \left( x_4 + \frac{0.5}{x_4} \right) = 0.707107.
\]
and standard deviations $\sigma$.

Solution: There result the values

(iii) Calculate $y_i, y_2, \ldots, y_7$ for $\lambda = 0.49$.

Solution: In this case, we obtain the values

$y_1 = y_0 - 0.49(y_0^2 - 0.5) = 0.755,$

$y_2 = y_1 - 0.49(y_1^2 - 0.5) = 0.720688,$

$y_3 = y_2 - 0.49(y_2^2 - 0.5) = 0.711186,$

$y_4 = y_3 - 0.49(y_3^2 - 0.5) = 0.708351,$

$y_5 = y_4 - 0.49(y_4^2 - 0.5) = 0.707488,$

$y_6 = y_5 - 0.49(y_5^2 - 0.5) = 0.707224,$

$y_7 = y_6 - 0.49(y_6^2 - 0.5) = 0.707143.$

(ii) Calculate $y_1, y_2, \ldots, y_7$ for $\lambda = 0.49$.

(iii) Calculate $y_1, y_2, \ldots, y_7$ for $\lambda = 49$.

Solution: In this case, we obtain the values

$y_1 = y_0 - 49(y_0^2 - 0.5) = -23.5,$

$y_2 = y_1 - 49(y_1^2 - 0.5) = -27059.25,$

$y_3 = y_2 - 49(y_2^2 - 0.5) = -3.587797 \times 10^{10},$

$y_4 = y_3 - 49(y_3^2 - 0.5) = -6.307422 \times 10^{32},$

$y_5 = y_4 - 49(y_4^2 - 0.5) = -1.949395 \times 10^{47},$

$y_6 = y_5 - 49(y_5^2 - 0.5) = -1.862070 \times 10^{96},$

$y_7 = y_6 - 49(y_6^2 - 0.5) = -1.698979 \times 10^{194}.$

We observe that the sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ converge to $\sqrt{\lambda} = 0.707107$ for $\lambda = 0.49$, while the sequence $\{y_n\}_{n \in \mathbb{N}}$ is divergent for $\lambda = 49$.

**Problem 1.3**

If the independent aleatory variables $X_1$ and $X_2$ have the density distributions $p_1(x)$ and $p_2(x)$, respectively, then the aleatory variable $X_1 + X_2$ has the density distribution

$$p(x) = \int_{-\infty}^{\infty} p_1(x - s) p_2(s) \, ds.$$  \hspace{1cm} (1.116)

(i) Demonstrate that if the aleatory variables $X_1$ and $X_2$ have a normal distribution by zero mean and standard deviations $\sigma_1$ and $\sigma_2$, then the aleatory variable $X_1 + X_2$ has a normal distribution.

Solution: From equation (1.116) we have

$$p(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(x-s)^2}{2\sigma_1^2}} \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_2^2}} \, ds = \frac{1}{2\pi \sigma_1 \sigma_2} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2\sigma_1^2}} e^{-\frac{x^2}{2\sigma_2^2}} \, ds.$$ \hspace{1cm} (1.117)
We require the values $\lambda_1$, $\lambda_2$, and $a$ real, such that
\[
\frac{(x-s)^2}{2\sigma_1^2} + \frac{s^2}{2\sigma_2^2} = \frac{x^2}{\lambda_1^2} + \frac{(s-ax)^2}{\lambda_2^2},
\] (1.118)
from which
\[
\frac{x^2}{\sigma_1^2} = \frac{x^2}{\lambda_1^2} + \frac{a^2x^2}{\lambda_2^2}, \quad \frac{s^2}{\sigma_2^2} = \frac{s^2}{\lambda_1^2} - \frac{2xs}{\lambda_1^2} = -\frac{2ax}{\lambda_2^2},
\] (1.119)
with the solution
\[
\lambda_2^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \quad a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \quad \lambda_1^2 = \sigma_1^2 + \sigma_2^2.
\] (1.120)

We make the change of variable
\[
s - ax = \sqrt{2}\lambda_2t, \quad ds = \sqrt{2}\lambda_2 dt
\] (1.121)
and expression (1.118) becomes
\[
p(x) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \frac{x^2}{\sigma_1^2 + \sigma_2^2} e^{-\frac{x^2}{2(\sigma_1^2 + \sigma_2^2)}} dt = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} e^{-\frac{x^2}{2(\sigma_1^2 + \sigma_2^2)}}.
\] (1.122)

(ii) Calculate the mean and the standard deviation of the aleatory variable $X_1 + X_2$ of point (i).

Solution: We calculate
\[
\int_{-\infty}^{\infty} xp(x)dx = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2(\sigma_1^2 + \sigma_2^2)}} dx = 0,
\] (1.123)
\[
\int_{-\infty}^{\infty} x^2 p(x)dx = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2(\sigma_1^2 + \sigma_2^2)}} dx
\]
\[
= \left. \left( -\frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{\sqrt{2\pi}x} e^{-\frac{x^2}{\sigma_1^2 + \sigma_2^2}} \right) \right|_{-\infty}^{\infty} + \sqrt{\sigma_1^2 + \sigma_2^2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(\sigma_1^2 + \sigma_2^2)}} dx
\]
\[
= \sigma_1^2 + \sigma_2^2.
\] (1.124)

(iii) Let $X$ be an aleatory variable with a normal distribution, a zero mean, and standard deviation $\sigma$. Calculate
\[
I_1 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx
\] (1.125)
and
\[
I_2 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}} du.
\] (1.126)

Solution: Through the change of variable
\[
x = \sigma\sqrt{2}u, \quad dx = \sigma\sqrt{2} du,
\] (1.127)
it follows that
\[ I_1 = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \sigma \sqrt{2} du = 1. \quad (1.128) \]

Similarly, we have
\[ I_2 = \int_{-\sigma}^{\sigma} e^{-u^2} du. \quad (1.129) \]

On the other hand,
\[ \int_{-\sigma}^{\sigma} e^{-u^2} du = \sqrt{2\pi} \int_{0}^{\infty} e^{-\rho^2} \rho d\rho = \sqrt{\pi(1 - e^{-\sigma^2})}. \quad (1.130) \]

so that
\[ I_2 = \sqrt{1 - e^{-\sigma^2}}. \quad (1.131) \]

(iv) Let \( 0 < \varepsilon < 1 \), fixed. Determine \( R > 0 \) so that
\[ \frac{1}{\sqrt{\pi}} \int_{-R}^{R} e^{-x^2} dx < \varepsilon. \quad (1.132) \]

Solution: Proceeding as with point (iii), it follows that
\[ \int_{-R}^{R} e^{-x^2} dx = \sqrt{\pi(1 - e^{-R^2})}, \quad (1.133) \]

so that we obtain the inequality
\[ \sqrt{1 - e^{-R^2}} < \varepsilon, \quad (1.134) \]

from which
\[ R < \sqrt{-\ln(1 - \varepsilon^2)}. \quad (1.135) \]

(v) Calculate
\[ I_3 = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{R} e^{-x^2/2\sigma^2} dx \quad (1.136) \]

and
\[ I_4 = \frac{1}{\sigma \sqrt{2\pi}} \int_{R}^{\infty} e^{-x^2/2\sigma^2} dx \quad (1.137) \]

Solution: We again make the change of variable (1.127) and obtain
\[ I_3 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{R} e^{-u^2/2\sigma^2} du. \quad (1.138) \]

Point (ii) shows that
\[ \int_{-A}^{A} e^{-x^2} dx = \sqrt{\pi(1 - e^{-A^2})}, \quad A > 0; \quad (1.139) \]

hence, it follows that
\[ I_3 = \sqrt{1 - e^{-\sigma^2/2\pi^2}}. \quad (1.140) \]
14 ERRORS IN NUMERICAL ANALYSIS

On the other hand, we have seen that \( I_1 = 1 \) and we may write

\[
I_1 = \frac{1}{\sigma \sqrt{2\pi}} \left( 2 \int_R^\infty e^{-\frac{x^2}{2\sigma^2}} \, dx + \int_{-\infty}^R e^{-\frac{x^2}{2\sigma^2}} \, dx \right) = 2I_4 + I_3. \tag{1.141}
\]

Immediately, it follows that

\[
I_4 = \frac{I_1 - I_3}{2} = \frac{1 - \sqrt{1 - e^{-\frac{\epsilon^2}{4\sigma^2}}}}{2}. \tag{1.142}
\]

(vi) Let \( X_1 \) and \( X_2 \) be two aleatory variables with a normal distribution, a zero mean, and standard deviation \( \sigma \). Determine the density distribution of the aleatory variable \( X_1 + X_2 \), as well as its mean and standard deviation.

Solution: It is a particular case of points (i) and (ii); hence, we obtain

\[
p(x) = \frac{1}{2\sigma \sqrt{\pi}} e^{-\frac{x^2}{4\sigma^2}}, \tag{1.143}
\]

that is, a normal aleatory variable of zero mean and standard deviation \( \sigma \sqrt{2} \).

(vii) Let \( N_1 \) and \( N_2 \) be numbers estimated with errors \( \epsilon_1 \) and \( \epsilon_2 \), respectively, considered to be aleatory variables with normal distribution, zero mean, and standard deviation \( \sigma \). Calculate the sum \( N_1 + N_2 \) so that the error is less than a value \( \epsilon > 0 \).

Solution: The requested probability is given by

\[
I = \int_0^{\epsilon} \frac{1}{2\sigma \sqrt{\pi}} e^{-\frac{x^2}{4\sigma^2}} \, dx = \int_{-\infty}^{\epsilon} \frac{1}{2\sigma \sqrt{\pi}} e^{-\frac{x^2}{4\sigma^2}} \, dx + \int_{-\epsilon}^0 \frac{1}{2\sigma \sqrt{\pi}} e^{-\frac{x^2}{4\sigma^2}} \, dx. \tag{1.144}
\]

Taking into account the previous results, we obtain

\[
\int_{-\infty}^{\epsilon} \frac{1}{2\sigma \sqrt{\pi}} e^{-\frac{x^2}{4\sigma^2}} \, dx = \frac{1 - \sqrt{1 - e^{-\frac{\epsilon^2}{4\sigma^2}}}}{2}, \tag{1.145}
\]

\[
\int_{-\epsilon}^0 \frac{1}{2\sigma \sqrt{\pi}} e^{-\frac{x^2}{4\sigma^2}} \, dx = \sqrt{1 - e^{-\frac{\epsilon^2}{4\sigma^2}}}, \tag{1.146}
\]

so that

\[
I = \frac{1}{2} \left( 1 + \sqrt{1 - e^{-\frac{\epsilon^2}{4\sigma^2}}} \right). \tag{1.147}
\]

FURTHER READING


