ANALYSIS AND MODELLING OF NON-STEADY FLOW IN PIPE AND CHANNEL NETWORKS

Vinko Jović
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Preface

This book deals with flows in pipes and channel networks from both the standpoint of hydraulics and of modelling techniques and methods. These classical engineering problems occur in the course of the design and construction of hydroenergy plants, water-supplies, and other systems. The author presents his experience in solving these problems from the early 1970s to the present. During this period new methods of solving hydraulic problems have evolved, primarily due to the development of electronic (analog and digital) computers, that is the development of numerical methods. The publication of this book is closely connected to the history and impact of the author’s software package for solving non-steady pipe flow using the finite element method, called Simpip which is an abbreviation of simulation of pipe flow. Initially, the program was intended for solving flows in pipe networks; however, it was soon expanded to flows in channels (see paper\(^1\)), though the name was retained. This program package can be found at www.wiley.com/go/jovic and has been used and is currently used for the solution of a great number of engineering problems in Croatia. It also has international references (it has been in the international market since 1992, but was withdrawn from the market for the author’s private reasons).

Chapter 1 – Hydraulic Networks. Many numerical methods result from the property of the scalar product of functions in Hilbert space, that is from the fundamental lemma of the variation calculus. This class of numerical methods includes the finite element method or – more precisely – the finite element technique. This means that the introduction of localized coordinate functions, both for the base of an approximate solution and for the test space, leads to the finite elements and topological properties which are connected into a network.

By assembling finite elements into a union which forms an entire domain, it is possible to assemble a global system of equations, which determines the modeled problem by using the same topological properties from the elemental equations. The finite element technique does not depend upon the mathematical method used for deriving the element equations.

It can be noted that hydraulic networks possess the same topological properties of the finite elements mesh and they are predetermined for problem solving using the finite element technique. These are unified networks. Unified networks can include various types of hydraulic branches such as pipes, valves, pumps, and other elements from the pressure systems, natural and artificial channels/canals, rivers, underground natural channels, and all other elements of channel systems. Unified hydraulic networks enable modelling of superficially quite different flows, such as modelling the water hammer with simultaneous modelling of the wave phenomena in the channel. The basis for solving unified networks is the numerical interpretation of the basic physical laws of mass and energy conservation. The first chapter presents a universal procedure for developing a matrix and vectors of the finite element from the elemental equations which are assembled into a global matrix and a vector of the equations system.

This is a fundamental system of equations which cannot be solved since the global matrix is singular so that the system has to be completed by natural and essential boundary conditions.

Chapter 2 – Modelling of Incompressible Fluid Flow. This chapter presents the derivation of a matrix and vector of a pipe finite element and a typical example for modelling the steady flow using the finite elements technique. The solution is iterative using the Newton–Raphson method in the assembled banded matrix of the system. It also states the drawbacks of the chosen method for assembling and solving the system of equations and it introduces a frontal technique which eliminates the unknowns already in the assembling phase. Since the frontal technique is “natural” for several reasons, and since it has been adopted as the basis for the program solution SimpipCore (which can be found at www.wiley.com/go/jovic), all the program phases of modelling the steady flow of incompressible fluid have been explained.

use GlobalVars
if(OpenSimpipInputOutputFiles()) then
if Input() then
  if BuildMesh() then ; finite element mesh
    if Steady(t0) then ; solve steady solution, t=t0
      call Output
    endif
  endif
endif
endif
endif

Modelling non-steady incompressible fluid flow is a logical continuation of modelling the steady flow by expanding it with a time loop, within which a respective matrix and vector of the non-steady flow of the pipe finite element are recalled. The initial conditions of the non-steady flow are the previously computed steady flows. Non-steady flow of the incompressible fluid can be divided into a quasi non-steady (temporally gradually varying) and non-steady (rigid) flow.

Matrices and vectors of the finite element of the quasi non-steady and rigid flows can be easily obtained from the basic laws of mass and energy conservation.

Chapter 3 – Natural Boundary Conditions Objects. In the fundamental system, the external nodal discharge is a natural condition which completes the nodal equation. It is a natural communication of the hydraulic network with the other systems, which is realized by using objects such as various valves, water tanks, vessels, surge tanks, and other objects. Generally, the external discharge depends upon the nodal piezometric height so that both a vector and matrix of the fundamental system are updated with a respective derivation. Special attention is paid to modelling the surge tanks as complex structures in a hydroenergetic system.

Chapter 4 – Water Hammer – Classic Theory. Modelling of non-steady phenomena cannot be imagined without a respective physical interpretation of the phenomenon, in this case the classic theory of the water hammer. Special attention has been paid to the relative motion of the water hammer and its phases as well as to the sudden acceleration and column separation of the water body. This chapter presents some classical computation methods and the principle of protection from the water hammer. By using the kinematic characteristics of wave front and linear relations of the water hammer (superposition principle) it is possible to obtain the wave functions and a general solution of the water hammer determined by a classic theory.

Chapter 5 – Equations of Non-steady Flow in Pipes. The beginning of this chapter presents the equation of the state of matter in the form of a p-V-T surface and in the form of phase projections, followed by equations of the water state under various flow conditions. Subsequently, the differential equations of

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a one-dimensional non-steady pipe flow are derived in a less typical way beginning with the principle of the mechanics of a material point. These are the continuity equation and the dynamic equation which follow from the law of mass conservation and the mechanical energy of the fluid particle. It has been shown that a precise analysis of a one-dimensional flow is not possible without simplification of the members resulting from the kinetic energy flow; this should be remembered when explaining the results of numerical modelling. Furthermore, various models of steady and non-steady flows of compressible and incompressible fluids in elastic and rigid pipelines are considered. Thus, it was possible to obtain equations of characteristics for the flow of elastic liquid by the simple transformation of the continuity equation and a dynamic equation; however, a more general, R. Courant and K.O. Friedrichs, procedure was employed for the compressible fluid. Finally, some analytical solutions of linearized equations for the non-steady water flow are presented.

Chapter 6 – Modelling of Non-steady Flow of Compressible Liquid. This chapter presents the numerical solution of the pipe flow with and without friction by employing a method of characteristics using discrete coordinates of the spatial and temporal variable. The solution can also be expressed as discrete values of primitive variables $p$, $v$ or wave functions $\Gamma^+$, $\Gamma^-$. The computation uses recursion. It is interesting that for the pipe flow without friction a simple recursive program can be made without a mesh of characteristics. For modelling the non-steady flow of the elastic fluid in hydraulic networks, matrices and vectors of the pipe finite element have been developed as follows: by the direct numerical interpretation of the continuity equation and dynamic equation and by applying the method of characteristics. The program solution SimpipCore (available at www.wiley.com/go/jovic) enables an optional choice of one or two procedures for integrating a matrix and vector of the pipe finite element.

Chapter 7 – Valves and Joints. Valves and various transient objects are relatively short branches which are used as joint elements connecting other branches in a hydraulic network. These are also finite elements used in modelling the hydraulic network, and therefore respective matrices and vectors have been determined for each valve type or connecting object. A non-return valve, either open or closed, should be treated as a special case wherein the valve status for the steady flow is given in advance, whereas for the non-steady flow it is computed from the hydraulic state of the system.

Chapter 8 – Pumping Units. Pumping units are a branch of the hydraulic network which consist of a pump and an asynchronous electromotor. This chapter presents the elements of turbo machines. Successful modelling of a non-steady flow with the functioning of pumping units is possible if we know the detailed characteristics of the pumps, i.e. the four-quadrant characteristics. The producers of pumps deliver pumps of serial production after performing standard tests (part of the first quadrant) while complete characteristics (four quadrants) are made only for special orders. In order to model abnormal operating conditions of pumping units it is necessary to reconstruct the detailed characteristic of the pump from normal characteristics. Furthermore, the program SimpipCore (available at www.wiley.com/go/jovic) reconstructs an approximate momentum characteristic of the electromotor according to the type of the declared pump working point and the number of rotations (for an electromotor 50 Hz).

The optional parameter SpeedTransients, with a false default, controls the nominal rotation velocity of the pumping units. The finite element matrix and vector are derived from three equations: the continuity equation, dynamic equation, and dynamic equation of the machine rotation, that is the computations of the discharge, manometric height, and the angular velocity of the pumping units, depending upon the SpeedTransients status and the operating variables of the voltage and frequency.

Chapter 9 – Open Channel Flow. Modelling of the non-steady channel flow is exceptionally complex since the flow can be subcritical or supercritical, that is during the modelling phases it can change from a subcritical to a supercritical state, or vice versa. Consequently, the program solution SimpipCore is restricted to modelling phases with an advance-determined flow state. The flow and the channel type are declared in the input phase, that is in the initialization of the channel section. The channel stretch is a branch of the hydraulic network which consists of a series of channel finite elements. The spatial position is defined by the coordinates of the points of the flow axes in which the cross-sectional profiles
of the riverbed have been assembled. The continuity equation and the dynamic equation of the channel flow are formally equal to equations obtained for the pipe flow since they result from the same laws of mass and energy conservation.

The matrix and the vector of the channel finite element, obtained by the integration of the continuity equation and the dynamic (energy) equation, retain all the properties of the mass and energy conservation law both for the channel stretch and the entire hydraulic network, which is a necessary condition for acceptable engineering modelling. Generally, it is possible to form the finite element matrix and vector using the interpolation of the boundary characteristics in a similar way as in pipe finite elements. However, experience in modelling has shown that this seriously threatens the energy and mass conservation law for the channel stretch, which can be explained by the errors caused by necessary interpolations.

**Chapter 10 – Numerical Modelling in Karst.** Approximately 50% of the soil in the Republic of Croatia is covered by Dinaric karst and significant karst terrains are densely populated, especially the coastal areas. The circulation of the groundwater in the Dinaric karst takes place within a well-developed channel system. Precipitation rapidly sinks underground through a system of fractures, flows through underground channels, and is drained in karst springs and submerged springs. This chapter is written according to the PhD. thesis of Davor Bojanić: Hydrodynamic modelling of karst aquifers (Faculty of Civil Engineering-Architecture and Surveying, University of Split), in 2011, which developed a matrix and vector of a karst channel finite element, a karst channel stretch and an elementary catchment – surface “Karst” for collecting effective rainfall and for defining spatial porosity. Thus, the karst channel stretch is one of the branches of a hydraulic network in the SimpipCore program solution (available at www.wiley.com/go/jovic).

**Chapter 11 – Convective-Dispersive Flows.** In unified hydraulic networks, apart from primary flows, other secondary processes can be coupled or not coupled to the basic flow. Secondary processes behave according to the law of extensive field conservation, for example the transfer of heat and substances. This chapter presents a solution for a convective-dispersive heat transfer in the hydraulic network; however, the derivation of the finite element matrix and vector in the first chapter is universal and is also valid for modelling other secondary processes.

**Chapter 12 – Hydraulic Vibrations in Networks.** Forced vibrations are a well developed harmonic state in the hydraulic network resulting from harmonic excitation. The vibration excitation can be any source in the hydraulic network which periodically changes the pressure or the discharge during its normal functioning. Vibration modelling is solved in the frequency domain, that is in the complex area of numbers.

**Appendix A – Program solutions.** The Appendix can be found at www.wiley.com/go/jovic and presents the program solution SimpleSteady – a typical educational program for modelling steady flow which uses a banded matrix for solving the system of equations. Furthermore, it includes the sources of the Fortran modulus ODE for solving ordinary differential equations with initial conditions and the program tests for the surge tank and a vessel.

**Appendix B – SimpipCore.** This appendix can also be found at www.wiley.com/go/jovic and presents the SimpipCore project which was developed by an integrated developmental environment Microsoft Developer Studio and Compaq Visual Fortran Version 6.6.

The accompanying website www.wiley.com/go/jovic contains, apart from the SimpipCore project (the Fortran sources and Project Workspace, that is the makefile), an independent Windows installation, a user manual, and examples with a series of Fortran sources and tests.
1

Hydraulic Networks

1.1 Finite element technique

1.1.1 Functional approximations

Let us observe a class of methods that can be generated from the property of the scalar product of functions in a Hilbert\(^1\) space

\[
(\varepsilon, w) = \int_{\Omega} \varepsilon(x)w(x)d\Omega. \tag{1.1}
\]

The following lemma is a direct consequence of the scalar product (1.1) property: if, for a continuous function \(\varepsilon: \Omega \rightarrow \mathbb{R}\) and for each continuous function \(w: \Omega \rightarrow \mathbb{R}, w \neq 0; \Omega \subset \mathbb{R}^n\)

\[
\int_{\Omega} \varepsilon(x)w(x)d\Omega = 0; \quad x \in \Omega, \tag{1.2}
\]

then \(\varepsilon(x) \equiv 0\) for each \(x \in \Omega\). The lemma (1.2) is often called the fundamental lemma of the variational calculus. The fundamental lemma will not be proved here since its validity is intuitive. The following can be considered; since \(\varepsilon(x)\) and \(w(x)\) are the vectors while Eq. (1.2) is a scalar product of vectors, a scalar product of any vector \(w(x)\) and vector \(\varepsilon(x)\) will always be equal to zero only if vector \(\varepsilon(x)\) is the null vector.

The fundamental lemma is widely applied in numerical analysis. Procedures derived from the fundamental lemma can be either approximate\(^2\) or exact.\(^3\) Approximate procedures arise from the meaning of the functional approximation regardless of whether the function was set directly or as a solution of differential equations.

An approximation of a function is sought in the form of the \(n\)-dimensional vector. Let the

\[
f(x): \Omega \rightarrow \mathbb{R} \tag{1.3}
\]

---

\(^1\)David Hilbert, German mathematician (1862–1943).

\(^2\)Approximate in the analytical sense.

\(^3\)Exact in the analytical sense.
be a function for which approximation is sought in the form of a vector $\tilde{f}(x)$. If an $n$-dimensional vector space is selected, then $\tilde{f}(x)$ is sought in the form of a linear combination of basis vectors

$$\tilde{f}(x): \sum_{i=1}^{n} \alpha_i \varphi_i(x); \quad \varphi_i(x): \Omega \to \mathbb{R},$$

where $\varphi_i$ are the basis or coordinate vectors, that is linearly independent functions, while $\alpha_i$ are the unknown parameters of a linear combination.

One meaning of an approximation is illustrated in Figure 1.1. Furthermore, the sum sign will be omitted because the Einstein\textsuperscript{4} summation convention and other rules of indices will be applied. The difference between the function and its approximation is called a residual

$$\varepsilon(x) = f(x) - \tilde{f}(x).$$

There is a question of the criteria for calculation of the unknown parameters of a linear combination in terms of the error minimization $\varepsilon(x)$. If fundamental lemma is applied, then

$$\int_{\Omega} (f - \alpha_i \varphi_i) w_j d\Omega = 0 \quad i, j \leq 1, 2, 3, \ldots n$$

Should Eq. (1.6) be valid for each continuous function, then the function $f(x)$ will be developed in a convergent series (1.4) for base $\varphi_i$. However, in order to calculate $n$ unknown parameters of a linear combination, it will be enough to set $n$ independent conditions, which is achieved by selection of $n$ linearly independent $w$ functions. If $n$ is a finite number, then the fundamental lemma will be satisfied in an approximate sense while functions will be developed with $n$ members from the convergent series.

Since, according to Eq. (1.5), vector $\varepsilon(x)$ is expressed by $n$ coordinate vectors, that is an approximation base $\varphi_i(x)$, it is also obvious that $n$ linearly independent functions $w_j(x)$ will form a base of a certain space, which is called the test space, while vector $w_j(x)$ is a coordinate vector of the test space.

The unknown approximation parameters $\alpha_i$ are obtained by developing Eq. (1.6):

$$\alpha_i \int_{\Omega} \varphi_i w_j d\Omega = \int_{\Omega} f w_j d\Omega \quad i, j \leq 1, 2, 3, \ldots n$$

---

\(4\) Albert Einstein, world famous physicist (1879–1955).
that is, following the calculation of integrals, from the equation system

\[ a_{ij} \alpha_i = b_j \]

\[ i, j = 1, 2, 3, \ldots n \]

(1.8)

where

\[ a_{ij} = \int_{\Omega} \varphi_i w_j d\Omega; \quad b_j = \int_{\Omega} f w_j d\Omega. \]

(1.9)

Since the approximation base and the test base can be selected from the wide range of functions, in general there are certain dilemmas regarding that selection. Details are given in Jovic (1993)\(^0\), while the most important approximation methods will be listed hereinafter:

- least squares integral method \( w_j(x) = \varphi_j(x) \),
- approximation with orthogonal basis (Legendre\(^5\) and Chebyshev\(^6\) polynomials, harmonic functions),
- the collocation method, algebraic, in particular Lagrange\(^7\) polynomials,
- the finite element method and spline approximations,
- a transfinite mapping method.

### 1.1.2 Discretization, finite element mesh

**Discretization.** A problem of function approximation, with the area \( \Omega \) discretized into sufficiently small finite elements \( e \) according to the one-dimensional concept, will be analyzed, see Figure 1.2. A finite element is selected so the function can be approximated by simple functions such as polynomials. A finite element mesh forms a compatible configuration, refer to Figure 1.2b, which provides a union without overlapping

\[ \Omega = \bigcup_{i=1}^{m} e_i. \]

(1.10)

**Table of element connections.** A connection between finite elements to form a compatible configuration is written in the table of element connections such as the following:

<table>
<thead>
<tr>
<th>Global nodes</th>
<th>Element e</th>
<th>1 local</th>
<th>2 local</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
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<td>2</td>
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<tr>
<td>6</td>
<td>\ldots</td>
<td>\ldots</td>
<td></td>
</tr>
</tbody>
</table>

where information is written for each finite element \( e \) regarding the correspondence between the local nodes and the global ones. A table of element connections is the basic topologic feature of the finite element mesh.

\(^5\)Adrien-Marie Legendre, French mathematician (1752–1833).
\(^6\)Pafnuty Lvovich Chebyshev, Russian mathematician (1821–1894).
\(^7\)Joseph-Louis Lagrange, mathematician and astronomer (1736–1813).
Shape or interpolation functions. Over each finite element $e = 1, 2, 3, \ldots, m$ a function is approximated by interpolation functions, which form a base of solutions over a finite element. These functions shape the solution over an element and are, therefore, called the shape or interpolation functions. They are either normalized polynomials of the Lagrange class, or Hermite\(^8\) polynomials attached to the nodes or some other interpolation polynomials. Figure 1.2c shows a two-nodal finite element with the shape functions $\Phi_1$ and $\Phi_2$ attached to local nodes 1 and 2. The shape functions are the basis vectors of the finite element. These basis vectors are generating a linear form of the function over a finite element, while approximation of a function over an area $\Gamma$ will be a polygonal one. Higher order approximation is achieved by elements with more nodes.

Localized global functions. A localized base, as shown in Figures 1.2d and 1.12e for the nodes $j$ and $j + 1$, is built from the shape functions of the adjacent elements. Hence, one localized base $\psi_i$, with the definition area $\Omega$, is appointed to each global node $i$. A localized base is intensive within a limited area, that is elements that contain the respective node, while outside it is equal to zero.

A sought approximation of the function is a linear combination of localized coordinate functions $\tilde{f}(x) = \alpha_i \psi_i(x)$. The value of the localized base equals 1 in the nodes that it is appointed to. Thus, the unknown parameters become equal to the nodal values of an approximate solution $\alpha_i = \tilde{f}_i$.

Continuity of an approximation. Linear shape functions secure continuity of a function, though not its derivations. Thus, it is said that an approximation belongs to the class $C^0$.

---

\(^8\)Charles Hermite, French mathematician (1822–1901).
n,m ; no nodes and elements
connect(m,2) ; element connectivities
Ag(n,n),Bg(n) ; global matrix and vector
Ae(2,2),Be(2) ; elemental matrix and vector
; loop over finite elements:
for e = 1 to m do
    call matrix(Ae,Be) ; compute matrix and vector
    for k = 1 to 2
        r = connect(e,1) ; first global node
        Bg(r)=Bg(r)+Be(k)
        For l = 1 to 2
            s = connect(e,2) ; second global node
            Ag(r,s)=Ag(r,s)+Ae(k,l)
        End loop l
    end loop k
end loop e

Figure 1.3 Assembling algorithm.

Finite element matrix and vector. If the least squares integral approximation procedure is applied, with the test base equal to the approximations base \( w_j(x) = \varphi_j(x) \), the matrix and vector members are obtained

\[
a_{ij} = \int_{\Omega} \varphi_i \varphi_j dx = \sum_{e=1}^{m} \int_{e} \varphi_i \varphi_j dx \quad \text{and} \quad b_j = \int_{\Omega} f \varphi_j dx = \sum_{e=1}^{m} \int_{e} f \varphi_j dx. \tag{1.11}
\]

These are the global matrix and vector, which are integrated from the contribution from individual finite elements. If in integrals (1.11) global functions \( \varphi \) are replaced by local ones \( \Phi \), then the finite element matrix and vector are obtained\(^9\)

\[
a'_{ij} = \int_{e} \Phi_k \Phi_l de \quad b'_l = \int_{e} f \Phi_l de. \tag{1.12}
\]

System assembling. A procedure of global equation system generation will be presented using an algorithm written in the pseudo-language, as shown in Figure 1.3. The assembling procedure starts with an empty global matrix and an empty global vector.

For each element \( e \) the finite element matrix \( a'_{k,l} \), \( k, l = 1, 2 \) and vector \( b'_l \), \( l = k, 2 \) are calculated and superimposed into the global matrix and vector using the connectivity table, see Figure 1.4. Note that calculation over finite elements is independent and can be processed in parallel\(^{10}\). Also note that the element assembling schedule is irrelevant.

Figure 1.5 shows the global matrix and vector separately for each element and their final form.

---

\(^9\)It is often referred to as the stiffness matrix and the load vector with respect to physical features occurring in the solving of the problem of elastic body equilibrium.

\(^{10}\)This property is suitable for computers with parallel processors.
1.1.3 Approximate solution of differential equations

**Strong formulation**

Let us focus on differential equations obtained by description of natural phenomena and their approximate solutions. Generally, a mathematical model of a natural phenomenon is formally written in the following form

\[
F(X, U, D^k U) = 0,
\]

where \( X = (x_1, x_2, x_3, \ldots x_p) \) are the coordinates of the space where the phenomenon takes place, \( p \) is the space dimension, \( U = (u_1, u_2, u_3, \ldots u_s) \) is the intensive field describing the phenomenon – namely
solution of the equation, \( s \) is the degree of freedom of a system, for a vector function it can be 1, 2, or 3 depending on the spatial dimension, and \( D^k U \) is the generalized partial derivation of the \( k \)-th order.

A solution of Eq. (1.13) is sought for the initial and boundary conditions. It is understood that the solution exists; it is unique and can be expressed as a vector

\[
\tilde{U} = \alpha_i \varphi_i(X); \quad i = 1, 2, 3, \ldots n
\]  

(1.14)

with the unknown \( \alpha_i \) parameters and \( \varphi_i \) basis vectors, functions selected from a class that are derivable enough.

By introducing Eq. (1.14) into Eq. (1.13) and applying the fundamental lemma, \( n \) independent equations are obtained for defining the parameters\(^{11}\)

\[
\int_{\Omega} F(X, \alpha_i \varphi_i, D^k \alpha_i \varphi_i) W_j d\Omega = 0
\]

\[
i, j = 1, 2, 3, \ldots n
\]  

(1.15)

The aforementioned equation system does not have to be regular because a solution without the initial and boundary conditions is not unique. Initial conditions are the known values of the function at a certain time. Thus, satisfying the initial conditions refers to the problem of approximation of the set function. Let the boundary conditions be set forth by the expression

\[
G(X, U, D^k U) = 0; \quad X \in \Gamma,
\]  

(1.16)

where \( \Gamma \) is the boundary of the domain \( \Omega \). The order of partial derivations in Eq. (1.16) is, in general, lower than the partial derivations order in Eq. (1.15). Since an approximate solution shall also satisfy the boundary conditions, the fundamental lemma will be applied again to boundary conditions. Thus

\[
\int_{\Omega} F(X, \alpha_i \varphi_i, D^k \alpha_i \varphi_i) W_j d\Omega = \int_{\Gamma} G(X, \alpha_i \varphi_i, D^k \alpha_i \varphi_i) W_j d\Gamma
\]

\[
i, j = 1, 2, 3, \ldots n
\]  

(1.17)

A solution is sought as an approximation by linear combination of the basis functions, which are derivable enough and satisfy the boundary conditions. These are the strong conditions. Strong formulation procedures play an important role in engineering that shall also not be negligible in the future. An approximate solution is sought in a linear combination of the global basis functions that are derivable enough

\[
\tilde{U} = \alpha_k \varphi_k.
\]  

(1.18)

If an approximate solution shall satisfy the boundary conditions accurately, a procedure of boundary condition homogenization is applied by transformation

\[
U = \Psi + V
\]  

(1.19)

\(^{11}\)Which are the weight factors; this is often called the weighted residuals method, in particular in terms of differential equations’ approximate solutions.
so that, after introducing Eq. (1.19) into Eq. (1.15), the strong formulation becomes

\[
\int_{\Omega} F^{(1)}(X, \alpha, \varphi_i, D^k \alpha, \varphi_i) W_j d\Omega = \int_{\Omega} F^{(2)}(X, \Psi, D^k \Psi) W_j d\Gamma,
\]
\(i, j = 1, 2, 3, \ldots n\) (1.20)

where the right side in Eq. (1.17) is eliminated due to homogeneous boundary conditions. If the equation set by a linear operator is observed

\[
L(u) = 0
\]

with the solution sought based on the mixed boundary conditions: natural \(q = q(x); x \in \Gamma_1\) and essential \(u = g(x), x \in \Gamma_2\), by application of the fundamental lemma, the strong formulation can be written in the following form

\[
\int_{\Omega} L(\tilde{u}) w_j d\Omega = \int_{\Gamma_1} (\tilde{q} - q) w_j d\Gamma + \int_{\Gamma_2} (\tilde{u} - g) w_j d\Gamma,
\]

(1.22)

where the boundary integrals are divided into two parts. Methods for approximate solving of differential equations can be classified according to procedure:

- according to the selection of the test space base: the moment method, the point collocation method, the least squares method, the least squares collocation method, the subdomain method or the subdomain collocation method (Biezeno and Koch\(^{12}\)), the Galerkin\(^{13}\)–Bubnov\(^{14}\) method, and other methods;
- according to the selection of an approximate solution base (basis separation, basis localization, minimization of the solution variation leads to the Galerkin procedure);
- operator methods for discrete and continuous parameters.

**Weak formulation**

Unlike the strong formulation, the problem is solved by integral transformations to decrease the derivation order. An integral formulation is solved instead of a differential equation, and weaker conditions are set for an approximate solution. The finite element technique and the Galerkin method (variational procedures) are the most commonly used for calculation of matrices and vectors. The numerical form of conservation law is one of the very important weak formulations. It is also referred to as the finite volume method or method of subdomain.

For easier understanding of the weak formulation procedures, a typical solution of the Boussinesq\(^{15}\) equation will be presented. The Boussinesq equation is a parabolic equation used for the description of the heat conduction problem in physics, seepage problem in hydraulics, and other problems in electrical

\(^{12}\)C. B. Biezeno, J. J. Koch, Dutch engineers.

\(^{13}\)Boris Grigoryevich Galerkin, Russian/Soviet an engineer and mathematician (1871–1945).

\(^{14}\)Ivan Grigoryevich Bubnov, Russian marine engineer (1872–1919).

\(^{15}\)J. V. Boussinesq, French physicist and mathematician (1842–1929).
engineering and so on. The steady form of the equation has an elliptical form. Starting from the heat conservation law over a control volume $\Omega$, the following is obtained

$$\frac{\partial}{\partial t} \int_{\Omega} c u d\Omega + \int_{\Gamma} q_i n_i d\Gamma = \int_{\Omega} p d\Omega,$$  \hspace{1cm} (1.23)

where the first integral is the rate of change of heat inside the control volume, the second integral is the change occurring due to heat flux through the surface $\Gamma$ enclosing the control volume, while the third integral is the heat production within the control volume, see Figure 1.6.

Application of the GGO Theorem\textsuperscript{16} on the surface interval is written as

$$\frac{\partial}{\partial t} \int_{\Omega} c u d\Omega + \int_{\Omega} \frac{\partial q_i}{\partial x_i} d\Omega = \int_{\Omega} p d\Omega. \hspace{1cm} (1.24)$$

After grouping under one integral, the following is obtained

$$\int_{\Omega} \left( c \frac{\partial u}{\partial t} + \frac{\partial q_i}{\partial x_i} - p \right) d\Omega = 0. \hspace{1cm} (1.25)$$

This particular integral vanishes for each area, which means that the sub-integral function must be equal to zero. Then, the heat continuity equation is obtained in the form

$$c \frac{\partial u}{\partial t} + \frac{\partial q_i}{\partial x_i} - p = 0. \hspace{1cm} (1.26)$$

\textsuperscript{16}General integral transformation theorem using the projection for $\mathbb{R}^n$:

$$\int_{\Gamma} f n_i d\Gamma = \int_{\Omega} \frac{\partial f}{\partial x_i} d\Omega$$

discovered independently by Gauss, German mathematician and astronomer (1777–1855), Green, English mathematician and physicist (1793–1844), and Ostrogradski, Russian mathematician and physicist (1801–1862) and thus named after them.
A simple dynamic equation – generalized as Fourier’s\textsuperscript{17} law can be applied to the thermal conduction processes

\[ q_i = -k_{ij} \frac{\partial u}{\partial x_j}, \]  

(1.27)

where \( q_i \) is the thermal flux and \( k_{ij} \) is the thermal conduction tensor. Introducing Fourier’s law into the conservation law

\[ c \frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} k_{ij} \frac{\partial u}{\partial x_j} + p \]  

(1.28)

a Boussinesq equation is obtained. When the fundamental lemma is applied to the Boussinesq equation, an extended form is obtained

\[ \int_{\Omega} c \frac{\partial u}{\partial t} w d\Omega - \int_{\Omega} w \frac{\partial}{\partial x_i} k_{ij} \frac{\partial u}{\partial x_j} d\Omega - \int_{\Omega} pw d\Omega = 0. \]  

(1.29)

Partial integration\textsuperscript{18} will be applied to the second integral; thus, expression (1.29) will become

\[ \int_{\Omega} c \frac{\partial u}{\partial t} w d\Omega + \int_{\Omega} k_{ij} \frac{\partial c}{\partial x_i} \frac{\partial w}{\partial x_j} d\Omega = \int_{\Gamma} w k_{ij} \frac{\partial u}{\partial x_i} n_i d\Gamma + \int_{\Omega} pw d\Omega. \]  

(1.30)

Integral equation (1.30) is a weak formulation of the Boussinesq equation. An approximate solution will be sought in the form of a linear combination of basis functions

\[ u = u_r(t)\phi_r(x_i), \]  

(1.31)

where the values of linear combination of time-dependent function are the unknowns (nodal temperatures). They are determined from the equation system, with test functions \( w = \phi_s(x_i) \). If the finite element technique is applied (localized approximation base and test space) the following is obtained

\[ \frac{du_r}{dt} \int_{\Omega} c \phi_r \phi_s d\Omega + u_r \int_{\Omega} k_{ij} \frac{d\phi_r}{dx_j} \frac{d\phi_s}{dx_i} d\Omega = \int_{\Gamma_2} \phi_s q_n d\Gamma + \int_{\Omega} \phi_s p d\Omega, \]  

(1.32)

where integrals are matrices and vectors. A boundary \( \Gamma = \Gamma_1 \cup \Gamma_2 \) consists of two parts. The first is an integral over the boundary \( \Gamma_1 \), with the known value of solution \( u^0 \), which does not have to be calculated; and the integral over the boundary \( \Gamma_2 \) with the known prescribed discharge \( q_n \) in the direction of the normal. If the following marked

\[ \text{capacitive matrix:} \quad C_{rs} = \int_{\Omega} c \phi_r \phi_s d\Omega, \]  

(1.33)

\[ \text{divergence matrix:} \quad D_{rs} = \int_{\Omega} k_{ij} \frac{d\phi_r}{dx_j} \frac{d\phi_s}{dx_i} d\Omega, \]  

(1.34)

\textsuperscript{17}Fourier, French mathematician and physicist (1768–1830).

\textsuperscript{18}Partial integration \( \int_{\Omega} u \frac{\partial v}{\partial x_i} d\Omega = \int_{\Gamma} u v n_i d\Gamma - \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} d\Omega \) is obtained by the GGO transformation theorem.
vector of boundary thermal fluxes:

$$Q_s = \int_{d\Gamma} \varphi_s q_n d\Gamma, \quad (1.35)$$

heat production vector:

$$P_s = \int_{\Omega} \varphi_s p d\Omega, \quad (1.36)$$

where the expressions have a physical meaning, a \textit{discrete global system} is obtained in the form

$$C_r \frac{du_r}{dt} + D_r u_r = Q_s + P_s. \quad (1.37)$$

Ordinary differential equations are obtained with nodal functions to be solved.

Figure 1.7a shows a discrete system. As can be observed, the boundary nodal discharge $Q_s$ consists of the concentrated contributions of the adjacent elements. A production vector $P_s$ is interpreted similarly, as a contribution from the adjacent elements with the common node.

Each finite element can be observed separately as an isolated discrete system, see Figure 1.7b. Elemental discrete equations can also be applied to

$$C_r^e \frac{du_r^e}{dt} + D_r^e u_r^e = Q_s^e + P_s^e. \quad (1.38)$$

For steady flow, nodal discharges will be

$$Q_s^e = D_r^e u_r^e. \quad (1.39)$$

Note that, besides the table of finite element connections, there are other topologic properties of the finite element method.

Figure 1.8 shows the generation of finite element configuration around the node $s$.

The same assembling procedure can be applied to nodal continuity equations, because the thermal flux conservation law is valid for node $s$

$$\sum_{e=1}^{p} Q_s^e = 0 \quad (1.40)$$

\textbf{Figure 1.7} Discrete system (a) global system and (b) finite element.
that consists of nodal discharge $Q_e^s$ contributions from the adjacent elements. If the nodal discharge vector over an element can be expressed via a nodal temperature vector, the global nodal equation can be written in the following form

$$\sum_{e=1}^{p} Q_e^s = \sum_{e=1}^{p} D_{rs}^e u_r = 0, \quad (1.41)$$

where the matrix $D_{rs}^e$ is equal to the finite element matrix. That matrix can be generated using the thermal flux over an element that is, in the presented example, numerically completely equal to the finite element matrix obtained from the weak formulation. Numerical equivalence is completely understandable because a linear differential equation was analyzed; also, a weak formulation gives the thermal flux conservation law, as can be observed if test function values in Eq. (1.30) are adopted as constant.

**First example.** A steady thermal flux will be observed on a bar of the length $L$, see Figure 1.9a, which consists of three segments with different thermal conduction coefficients $k_e$ and the same length $\Delta L$.

**Figure 1.8** Generation of the finite element configuration and nodal equation.

**Figure 1.9** Steady thermal flux along the bar. (a) bar, (b) decomposition of bar on segments (finite elements), (c) nodal flux continuity.