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Phillip Griffiths
John Morgan

Rational Homotopy Theory and Differential Forms

Second Edition

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Phillip Griffiths
Institute for Advanced Study
Princeton University
Princeton, NJ, USA

John Morgan
Simons Center for Geometry and Physics
Stony Brook University
Stony Brook, NY, USA

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Preface to the First Edition

This monograph originated as a set of informal notes from a summer course taught by the present authors, together with Eric Friedlander, at the Istituto Matematico "Ulisse Dini" in Florence during the summer of 1972. Even though more formal expositions of Sullivan's theory have since appeared, including the major original source [26], there has been a steady continuing demand for the old Florence notes. Moreover, one of us (J.M.) has become involved in the subject again through a series of lectures given at the University of Utah in January, 1980, together with joint work in progress with James Carlson and Herb Clemens on a new type of application of the theory to algebraic geometry. Since the Florence notes represented an approach and point of view that does not appear in the literature, we decided to publish the present revised and corrected version.

The material in this monograph is outlined in the table of contents and is informally discussed in the introduction below. Here we should like to observe that the text roughly divides into two parts. The first seven chapters essentially constitute an introductory course in algebraic topology with emphasis on homotopy theory. The main prerequisite is some familiarity with simplicial homology, covering spaces, and CW complexes.

Chapters 9–15 cover the main topic of differential forms and homotopy theory, with emphasis on the homotopy-theoretic and functorial properties of differential graded algebras and minimal models, a topic that does not appear explicitly in detail in the literature. An extensive set of exercises, frequently with copious hints, forms an essential complement to the material in the text.

We would like to make several acknowledgements to colleagues whose help and advice have been invaluable. The first and foremost is to Dennis Sullivan. It was he who introduced us to the idea of relating homotopy theory and differential forms and who explained to us his theory around which these notes are built.

The second is to Francesco Gherardelli who organized the original summer course and to the Istituto Matematico "Ulisse Dini" and the city of Florence, which together provided excellent mathematical and cultural conditions for the initial preparation of the notes. While in Florence we benefited from conversations with Ngo Van Que, Jim Carlson, and Mark Green. Finally, Moishe Breiner prepared

a beautifully handwritten set of notes that constituted the original version of this monograph.

We would also like to thank the University of Utah and abovementioned coworkers of J.M. for providing support and motivation leading to the revision of the Florence notes.

Finally we would like to point out two predecessors of the present theory. The first is Whitney's book [27]. As explained to us by Sullivan, this book contains the genesis of the use of differential forms to solve the commutative cochain problem and thus get the homotopy type of the space. The main thing lacking at the time Whitney wrote the book was the $\mathbb{Q}$-structure. Secondly the relationship between differential forms and homotopy theory was anticipated by Chen [2].

Many of the results we find from a general viewpoint were established, frequently in stronger form, by him using the method of iterated integrals.

Preface to the Second Edition

Thirty years have passed since the publication of the first edition, and we felt this monograph deserved updating. The essential structure and presentation remains the same, but several major additions seemed appropriate. We have included in an appendix a proof of the correspondence between rational minimal models and rational Postnikov towers different, and we feel more intuitive, from the one presented in the first edition. This proof relies on a set of polynomial forms with a filtration whose Serre spectral sequence agrees with the usual one for a fibration. We have also added a chapter describing Quillen's approach to rational homotopy theory and comparing and contrasting it with Sullivan's. Lastly, we have added a chapter on operads and A_∞ algebras and indicated briefly how the commutative version, C_∞-algebras, gives another algebraic description of rational homotopy theory.

The second author thanks Mohammed Abouzaid and Bruno Vallette for their help with the chapter on A_∞-structures and Eric Malm for his help in preparing the diagrams and figures.

Princeton, NJ, USA — Phillip A. Griffiths
Stony Brook, NY, USA — John W. Morgan

Contents

Chapter 1
Introduction

The purpose of this course is to relate the C^∞-differential forms on a manifold to algebro-topological invariants. A model of results along these lines is deRham's theorem, which says that the cohomology of the differential graded algebra (DGA) of C^∞-forms is isomorphic to the singular cohomology with coefficients in $\mathbb{R}$, i.e.

$$H^*_{dR}(M) \cong H^*(M,\ \mathbb{R}) \quad (C^\infty \text{ deRham theorem}).$$

The main theorem of this course will be that from the DGA of C^∞-forms, it is possible to calculate all of the real algebro-topological invariants of the manifold. More precisely, we shall be able to use the forms to obtain the (Postnikov tower) tensored with $\mathbb{R}$ of the manifold.

In the next seven chapters of this book, we shall discuss the standard terminology, objects, and theorems of elementary homotopy theory, culminating in the description of the Postnikov tower of a space. We then define the *localization* of a CW complex at 0; this allows us to take a CW complex and replace it by one in which all torsion and divisibility phenomena have been removed (allowing one to focus on the $\mathbb{Q}$-information in the original space). When we compare the Postnikov tower of the original space with that of its localization, we see that all the relevant information (homotopy and homology groups, k-invariants) has been tensored with $\mathbb{Q}$.

Once we have established these basic facts, we turn to the main theorem as shown to us by Sullivan. First, we define the rational p.l. forms on a simplicial complex K. They form a DGA defined over $\mathbb{Q}$. By integration, these forms give $\mathbb{Q}$-valued simplicial cochains on K, and this integration process induces an isomorphism of the cohomology of the rational p.l. forms to the usual (simplicial or singular) rational cohomology of the space:

$$H^*_{p.l.}(K) \cong H^*(K, \mathbb{Q}) \quad (\text{p.l. deRham theorem}).$$

There are two very important points here. The first is that we are working over $\mathbb{Q}$ rather than $\mathbb{R}$, as we would be forced to do with C^∞-forms. The second is that the

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p.l. forms are a differential, graded-commutative algebra—the simplicial or singular cochains over $\mathbb{Q}$ are not commutative. Thus, the p.l. forms have a good property of ordinary cochains (they are defined over $\mathbb{Q}$) and a good property of C^∞-forms (they are graded commutative). Both these properties are essential.

Next, we turn to the homotopy theory of DGAs, which are always implicitly assumed to be associative and graded commutative. Given one such, $\mathcal{A}$, we show how to extract a *minimal model* for it. This is a DGA, $\mathfrak{M}_{\mathcal{A}}$, which satisfies some internal condition, together with a map of DGAs:

$$\rho_{\mathcal{A}}: \mathfrak{M}_{\mathcal{A}} \longrightarrow \mathcal{A}$$

which induces an isomorphism on cohomology.

In the case that $H^1(\mathcal{A}) = 0$, the internal properties that $\mathfrak{M}_{\mathcal{A}}$ is required to satisfy are:

1. It is free as a graded-commutative algebra with generators in degrees ≥ 2 only.
2. For all $x \in \mathfrak{M}_{\mathcal{A}}$, the element dx is decomposable.

It turns out that, given $\mathcal{A}$, these properties characterize $\mathfrak{M}_{\mathcal{A}}$ up to isomorphism.

We shall show, in addition, that when $\mathcal{A}$ is the algebra of p.l. forms on a simply connected, simplicial complex X, then $\mathfrak{M}_{\mathcal{A}}$ is dual to the rational Postnikov tower of X.

The duality between minimal models defined over $\mathbb{Q}$ and rational Postnikov towers is described in Chap. 12. Schematically we have a "commutative diagram":

$$\begin{tikzcd}
\{\text{Simplicial Complexes}\} \arrow[r, "\text{p.l. forms}"] \arrow[d, "\text{localization at } 0"'] & \{\text{DGAs}/\ \mathbb{Q}\} \arrow[dd, "\text{minimal model}"] \\
\{\mathbb{Q}\ \text{spaces}\} \arrow[d, "\text{Postnikov tower}"'] & \\
\{\text{Postnikov towers}/\mathbb{Q}\} \arrow[r, leftrightarrow] & \{\text{minimial DGAs}/\mathbb{Q}\}
\end{tikzcd}$$

Given a C^∞-manifold M, we can smoothly triangulate M. Let K be the simplicial complex of this triangulation. We have both the C^∞-forms on M and p.l. forms on K. These are both included in $A^*_{p.C^\infty}(M)$ the DGA of "piecewise C^∞-forms" on M, (i.e., the forms whose restriction to each simplex of the triangulation is smooth), and the inclusions

$$A^*_{C^\infty}(M) \longrightarrow A^*_{p.\ C^\infty}(M) \longleftarrow A^*_{p.l.}(K) \otimes_{\mathbb{Q}} \mathbb{R}$$

induce isomorphisms in cohomology. From this comparison theorem, it follows that the minimal models satisfy

$$\mathfrak{M}(\mathrm{A}^*_{\mathrm{C}\infty}(\mathrm{M})) \cong \mathfrak{M}(\mathrm{A}^*_{\mathrm{p.l.}}(\mathrm{M})) \otimes_{\mathbb{Q}} \mathbb{R}.$$

This is the precise statement that "the deRham complex contains all the real algebraic-topological information from the manifold M." Schematically the theory is arranged as follows:

$$\begin{tikzcd}
\{\text{Manifolds}\} \arrow[rrr, "C^{\infty}-\text{forms}"] & & & \{\text{DGAs}/\mathbb{R}\} \arrow[d, "="] \\
\left\{\begin{matrix}\text{triangulated}\\ \text{manifolds}\end{matrix}\right\} \arrow[u] \arrow[rrr, "\text{p. } C^{\infty}-\text{forms}"] \arrow[d] \arrow[dr] & & & \{\text{DGAs}/\mathbb{R}\} \arrow[dd, bend left] \\
\left\{\begin{matrix}\text{Simplicial}\\ \text{complexes}\end{matrix}\right\} \arrow[r, "\text{p.l. forms}"] \arrow[d] & \{\text{DGAs}/\mathbb{Q}\} \arrow[rr, "\otimes\mathbb{R}"] \arrow[d] & & \{\text{DGAs}/\mathbb{R}\} \arrow[u] \arrow[d] \\
\{\mathbb{Q}-\text{spaces}\} \arrow[r] \arrow[d] & \{\text{Minimal models}/\mathbb{Q}\} \arrow[rr, "\otimes\mathbb{R}"] \arrow[dl, leftrightarrow] & & \left\{\begin{matrix}\text{Minimal models}\\ \text{over } \mathbb{R}\end{matrix}\right\} \\
\{\text{Postnikov towers}/\mathbb{Q}\} & & &
\end{tikzcd}$$

Though these notes concentrate mainly on the case of simply connected spaces, there are generalizations to the nonsimply connected case. In purely algebraic terms, part of the theory of the nonsimply connected case is similar to the simply connected one. When we try to make comparisons with homotopy, the results are much weaker. The available information from the algebra of forms which is most meaningful in classical terms deals with the fundamental group. Chapter 13 discusses this.

Chapter 2
Basic Concepts

Here we give a brief introduction to the basics of CW complexes, homotopy theory, homology, and the algebraic topology of manifolds. Here are some more references for more details on the material in this chapter. For a good introduction to CW complexes, homology, and cohomology, consult Greenberg's book [7]. For a more encyclopedic treatise on algebraic topology which covers all the homotopy theory presented in this course, save localization, one should see Spanier's book [23]. For another account of some of the topics presented later in this course, such as obstruction theory, one should see Hu's book [9].

2.1 CW Complexes

It will suffice for the purposes of this course (and for most other situations, also) to do homotopy theory for a restricted class of spaces. These are the spaces which are homotopy equivalent to CW complexes. All naturally encountered spaces have this property (e.g., manifolds, algebraic varieties, loop spaces on CW complexes, $K(\pi, n)$s). Moreover for these spaces, the Whitehead theorem which states that ($f\colon X \to Y$ is a homotopy equivalence if and only if f_* is an isomorphism on homotopy groups—cf. Sect. 9.2 for a proof) is true. What this means is that the usual functors of homotopy theory are powerful enough to decide when two CW complexes are homotopically equivalent.

We begin with the definition of a CW complex. Let D^n denote the unit n-disk, namely,

$$D^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n\colon ||x||^2 \leq 1\}$$

and let S^{n-1} denote the unit $(n-1)$ sphere, i.e., the boundary ∂D^n of D^n. (Note: In these notes, we have also used the notation e^n, for n-cell, as another symbol for D^n.) Given X and a continuous map $f\colon S^{n-1} \to X$, we form the *adjunction space*

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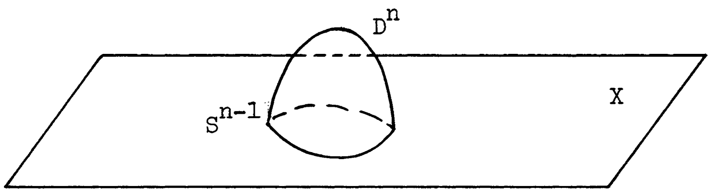
$$X \cup_f D^n$$

which is the quotient space of the disjoint union of X and D^n where every $a \in \partial D^n$ is identified with $f(a) \in X$. (Note: Above we required that f be a continuous map; usually we shall omit mention of continuity, with the understanding that map means continuous function.) Geometrically, what we have done is attach an n-cell to X



To give a space X, the structure of a CW complex means intuitively that X is obtained from a point by successively attaching cells. More precisely we have subspaces $X^{(i)}$ of X with

$$\phi = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \ldots, \quad X = \cup_{i=0}^{\infty} X^{(i)}$$

such that (1) $X^{(i+1)}$ is obtained from $X^{(i)}$ by attaching (i + 1)-cells and (2) if $X \neq X^{(n)}$ for any n (thus X is infinite dimensional), then X has the *weak topology* with respect to the $X^{(n)}$'s meaning that $U \subset X$ is an open set if and only if $U \cap X^{(n)}$ is open for all n. We call $X^{(n)}$ the n-skeleton of X. (Note: Infinite-dimensional CW complexes such as $\mathbb{C}P^{\infty}$, the infinite Grassmannians, the infinite sphere, etc. are very useful in homotopy theory. The weak topology means that in all cases, "∞" can be well approximated by "arbitrarily large n." Thus, for example, a map of a compact space into the infinite CW complex X is simply given by a map into X^N for large some N. Also, a map $f: X \to Y$ from a CW complex to a space is continuous if and only if its restriction to each skeleton $X^{(n)}$ is continuous.

Examples of CW Complexes

1. The n-sphere $S^n = \{pt.\} \cup_f D^n$ where $f: \partial D^n \to \{pt.\}$ is a degenerate attaching map.
2. The *complex projective space* $\mathbb{C}P^n$ is given a CW structure inductively by

$$\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_f D^{2n}$$

where

$$f: S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$$

is the *Hopf map*. More precisely, if we think of $\mathbb{C}P^n$ as the lines through the origin in $\mathbb{C}^{n+1}$, then taking D^{2n} to be the unit ball in $\mathbb{C}^n$, the attaching map $f\colon \partial D^{2n} \to \mathbb{C}P^{n-1}$ assigns to each point on the unit sphere in $\mathbb{C}^n$ the complex line joining through the origin containing that point 5.

3. $\mathbb{C}P^\infty = \lim_{n\to\infty}\mathbb{C}P^n$ is the infinite CW complex having one 2n-cell for each $n \in \mathbb{Z}^+$ and with the attaching maps given as above.
4. Any *simplicial complex* K has the natural structure of a CW complex. The n-cells of this CW structure are exactly the n-simplices. Conversely, if X is a CW complex, then there is a simplicial complex K and a homotopy equivalence from K to X (cf., Exercise (13)).
5. A *CW pair* (X, A) is a pair of spaces $A \subset X$ such that X is obtained from A by attaching cells. (It is not necessary that A itself be a CW complex.) If (X, A) is a CW pair, then we denote by $X^{(n)} \cup A$ the union of A with all cells of dimension $\leq$ n. Again, if X is obtained by attaching infinitely many cells to A, then X is given the limit (or weak) topology. If X is a CW complex and $A \subset X$ is a subcomplex, then (X, A) is a CW pair.

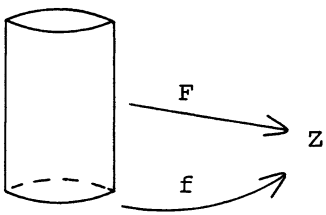
CW complexes are constructed so that, almost by definition, one works inductively up through the skeleton. As an example of this, we prove the *homotopy extension theorem* for CW pairs.

Theorem 2.1. *Given a CW pair* (Y, X)*, a map* $f\colon Y \to Z$*, and a homotopy* $F\colon X \times I \to Z$ *with* $F|_{X\times\{0\}} = f|_X$*, then there is an extension* $G\colon Y \times I \to Z$ *of* F *such that* $G(y, 0) = f(y)$.

Proof.

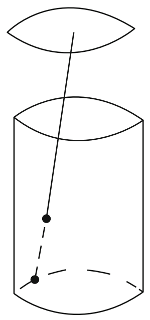
Step I: Given $f\colon D^n \to Z$ and $F\colon S^{n-1} \times I \to Z$ with $F|_{S^{n-1}\times\{0\}} = f|_{\partial D^n}$, find $G\colon D \times I \to Z$ extending F with $G|_{D^n\times\{0\}} = f$.

In the picture of $D^n \times I$



we are given a map G_0 on the union of the "bottom" ($D^n \times \{0\}$) and the "side" ($\partial D^n \times I$). We want to extend to a map on all of $D^n \times I$.

This is done taking the projection p of $D^n \times I$ onto $(D^n \times \{0\}) \cup (S^{n-1} \times I)$ from the point $\{(\text{middle of } D^n) \times \{2\}\}$



and defining $G(y,t) = G_0(p(y,t))$. Note: In this argument, as throughout homotopy theory, 99% of the proof is to find the correct “picture.” If this is done properly, no geometric argument will be difficult (although some algebraic computations may be messy).

Step II: Given $f\colon Y \to Z$ and $F\colon X \times I \to Z$, we shall inductively construct $G^{(i)}\colon (Y \times \{0\}) \cup [(X \cup Y^{(i)}) \times I\} \to Z$. Given $G^{(i-1)}$, consider any i-cell D^i_α and attaching map $\partial D^i_a \to Y^{(i-1)}$. Then we have

$$G^{(i-1)} \circ f_\alpha \times I\colon (S^{i-1} \times I) \cup (D^i_\alpha \times \{0\}) \longrightarrow Z$$

and we may use Step I to extend this map over $D^i_\alpha \times 1$ to a map $G^{(i)}_a$. Doing this over each i-cell and taking the union of the maps give $G^{(i)}$. Let $G = \cup_i G^{(i)}$ (i.e., $G|Y^{(i)} = G^{(i)}$). By the definition of the weak topology, G is continuous and gives the required extension of F, by the construction. ■

2.2 First Notions from Homotopy Theory

In homotopy theory, one always considers CW complexes modulo an equivalence relation, that of *homotopy equivalence*. Two maps $f_0, f_1\colon X \to Y$ are *homotopic* if there exists $F\colon X \times I \to Y$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. We may set $f_t(x) = F(z,t)$ and think of the f_t as giving a continuous deformation of f_0 into f_1. Maps $f\colon X \to Y$ and $g\colon Y \to X$ are *homotopy inverses* if $g \circ f \sim id_X$ and $f \circ g \sim id_Y$ (here, the notation “$\sim$” means “is homotopic to,” and $id_X\colon X \to X$ is the identity map). A map $f\colon X \to Y$ is a *homotopy equivalence* if it has a homotopy inverse; X and Y are *homotopy equivalent* if there is a homotopy equivalence $f\colon X \to Y$. Homotopy equivalence is an equivalence relation on the collection of CW complexes. In homotopy theory, one considers spaces as equivalent if they are homotopy equivalent (in particular, topological dimension is not defined). An equivalence class of homotopy equivalent spaces is said to be a *homotopy type*. For the rest of this course, space will mean CW complex, with the only exception that we shall speak of path spaces and loop spaces on CW complexes (definitions below), which are not CW complexes as they stand. However, they always have the homotopy type of a CW complex [17] and so may be unambiguously considered as "spaces."

While the category of CW complexes is quite flexible and easy to work with for many applications in homotopy theory, sometimes, for example, in defining an appropriate DGA of forms, it is better to work with simplicial complexes.Of course, as we have already remarked, any simplicial complex is a CW complex. The converse is true up to homotopy.

Lemma 2.2. *Let* X *be a CW complex. Then there is a simplicial complex* K *homotopy equivalent to* X.

Proof. The proof is by induction over the skeleta of X. Suppose inductively that for some $n \geq 0$, we have a simplicial complex K^n and a homotopy equivalence $\varphi_n\colon K^n \to X^{(n)}$. For each $(n+1)$-cell e_α of X, we have the attaching map $f_\alpha\colon S^n \to X^n$. There is a map $g_\alpha\colon \partial\Delta^{n+1} \to K^n$ with $\varphi_n \circ g_\alpha$ homotopic to f_α. By subdividing the standard triangulation of $\partial\Delta^{n+1}$ to produce a triangulation τ of $\partial\Delta^{n+1}$, we can arrange that g_α is a simplicial map (without subdividing K^n). Consider a collar neighborhood C of $\partial\Delta^{n+1}$ in Δ^{n+1} with $\partial\Delta^{n+1}$ corresponding to the 0-end. The product $\tau \times I$ defines a linear cell structure on C. We form the quotient $\bar{C}$ of C where we identify points in $\partial\Delta^{n+1} \subset C$ if they have the same image under g_α. Then the image in $\bar{C}$ of the product of each simplex of τ with I is a cell. Let B be the image of $\partial\Delta^{n+1}$ in $\bar{C}$. Now by induction on the simplices σ of the triangulation τ, we triangulate the image $\bar{\sigma}$ of $\sigma \times I$ in $\bar{C}$ without subdividing $\bar{\sigma} \cap B$. Suppose that we have done this for $(\overline{\partial\sigma})$. At the 1-end, take the cone to the barycenter of $\sigma \times \{1\}$ of the triangulation of $\partial\sigma \times \{1\}$. This produces a triangulation of the boundary of the cell $\bar{\sigma}$ in $\bar{C}$, a triangulation that does not subdivide the image of $\sigma \times \{0\}$ in $\bar{C}$. We then take the cone over this triangulation to $v_\sigma \times \{1/2\}$, where v_σ is the barycenter of σ. Then we extend the triangulation on the 1-end of C to a triangulation over the rest of Δ^{n+1}. The produces a simplicial complex K', containing K as a subcomplex, and there is an extension of the homotopy equivalence from K' to $X^{(n)} \cup_{f_\alpha} e_\alpha$ extending φ_n. Performing this construction simultaneously for every $(n+1)$-cell of X produces a simplicial complex K^{n+1} containing K^n and a homotopy equivalence $\varphi_{n+1}\colon K^{n+1} \to X^{(n)}$ extending φ_n. Taking the limit (i.e., increasing union) over all n establishes the result. ∎

In many problems in homotopy theory, we wish to make a construction relative to a map

$$f\colon X \longrightarrow Y.$$

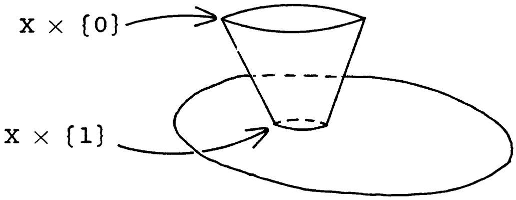
It is frequently easier to work with an inclusion rather than an arbitrary map. This is always possible up to homotopy equivalence.

Theorem 2.3. *Given* $f\colon X \to Y$*, there is a space* M_f*, the mapping cylinder of* f*, inclusions* $j\colon X \to M_f$ *and* $i\colon Y \to M_f$*, and a map* $\pi\colon M_f \to Y$

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & M_f \arrow[d, shift left, "\pi"] \\ & Y \arrow[u, shift left, "i"] \end{tikzcd}$$

where π *and* i *are homotopy inverses and* $\pi \circ j = f$. *Thus, we may replace* Y *by a homotopy equivalent space in which X is included.)*

Proof. Define $M_f = (X \times I) \cup_f Y$



where $(x, 1)$ is identified with $f(x) \in Y$. Then $\pi : M_f \to Y$ is given by $\pi(x, t) = f(x)$ for all $x \in X$ and $\pi(y) = y$ for all $y \in Y$ (this is consistent), and this gives a retraction of M_f onto Y. ■

Note: If X and Y are CW complexes and $f : X \to Y$ is a *cellular map* (i.e., $f(X^{(i)}) \subset Y^{(i)}$), it is easy to give M_f the structure of a CW complex. In Exercise (32), a proof of the fact that any f is homotopic to a cellular map f' is outlined, so that $M_f \sim M_{f'}$ which is a CW complex (the notation $A \sim B$ means that A and B are homotopy equivalent). Thus, we may consider any map as an inclusion without leaving the category of CW complexes.

Above we discussed the homotopy extension property (h. e. p.) and proved that a subcomplex of a CW complex always has the h. e. p. Now there is a dual property to the h. e. p. called the *homotopy lifting property* or *covering homotopy property*. Given spaces E, B, we say that $\pi : E \to B$ has the *homotopy lifting property* if given any space Y, a map $Y \xrightarrow{f} E$, and a homotopy g_t of $g = \pi \circ f$, there is a homotopy f_t of f such that $\pi \circ f_t = g_t$ (thus, the homotopy f_t "covers" or "lifts" the homotopy g_t).

Here f is said to be a *lifting* of g, and covering homotopy says that if a map g can be lifted, then any homotopy g_t of g can be lifted also. Not all maps have the homotopy lifting property; e.g., if B is connected, then π must be onto. If $\pi : E \to B$ has the h. l. p. (= homotopy lifting property), then it is said to be a *fibration*. For any $b \in B$, the fiber $F_b = \pi^{-1}(b)$ is the preimage of the point. In a fibration, any two fibers are homotopy equivalent provided that the base is path connected (cf. Chap. 4). We let F be any space having the homotopy type of F_b (F is called a typical fiber). We write the fibration as

$$
\begin{array}{ccc}
F & \longrightarrow & E \\
 & & \big\downarrow \pi \\
 & & B
\end{array}
$$

having in mind a picture like

$$
\begin{array}{ccc}
F_b & \longrightarrow & E \\
\big\downarrow & & \big\downarrow \\
\{b\} & \longrightarrow & B
\end{array}
$$

Examples of Fibrations

1. Locally trivial *fiber bundles*, *vector bundles*, and the associated *sphere bundles*, *covering spaces*, are all examples of fibrations (for a discussion of these, cf. [25]).
2. Let X be a space with a base point x_0. Define the *path space* based at $x_0 \in X$, $\mathcal{P}(X, x_0)$, to be the set of all paths given by maps $\omega\colon I \to X$, $\omega(0) = x_0$. The topology on $\mathcal{P}(X)$ is the compact-open topology. Thus, a sub-basis for the open sets in $\mathcal{P}(X)$ is given by taking $K \subset I$ a compact subset and $U \subset X$ an open set and letting $< K, U >$ be all maps $\omega\colon I \to X$ with $\omega(K) \subset U$. Define $\pi\colon \mathcal{P}(X) \to X$ by $\pi(\omega) = \omega(1)$. This is a fibration.

Homotopy Exact Sequence of a Fibration. Here is the statement:

Theorem 2.4. *Suppose that* $\pi\colon E \to B$ *is a fibration with* B *path connected. Fix* $b \in B$*, and let* F_b *be the fiber over* b *and* $i\colon F_b \to E$ *the inclusion. Finally, fix* $e \in F_b$. *Then we have a long exact sequence of homotopy groups:*

$$\longrightarrow \pi_n(F_b, e) \xrightarrow{i_\#} \pi_n(E, e) \xrightarrow{\pi_\#} \pi_n(B, b) \longrightarrow \pi_{n-1}(F_b, e) \longrightarrow .$$

Proof.
Exactness at $\pi_n(E, e)$. It is clear that $\pi_\# \circ i_\# = 0$. Suppose $a \in \pi_n(E, e)$ and $\pi_\#(a) = 0$. Represent a by a map $\varphi\colon (S^n, p) \to (E, e)$, where $p \in S^n$ is the base point. Since $\pi_\#(a) = 0$, there is a homotopy H from $\pi \circ \varphi\colon (S^n, p) \to (B, b)$ to the constant map at b, a homotopy that is constant on $\{p\} \times I$. Use the homotopy lifting property for the relative CW complex (S^n, p) to lift H to a homotopy $\tilde{H}\colon S^n \times I \to E$ beginning at φ and sending $\{p\} \times I$ to e. The map $H|_{S^n \times \{1\}}$ is a map $(S^n, p) \to (F_b, e)$ representing $a \in \pi_n(E, e)$. This proves that the image of $i_\#$ contains the kernel of $\pi_\#$, completing the proof of exactness at $\pi_n(E, e)$.
Exactness at $\pi_n(B, b)$. Let us first define the connecting homomorphism $\pi_n(B, b) \to \pi_{n-1}(F_b, e)$. Given an element, $\bar{a} \in \pi_n(B, b)$ represent it by a map $\psi\colon (S^n, p) \to (B, b)$. Let $p\colon (D^n, \partial D^n) \to (S^n, p)$ be the map collapsing the boundary

of the disk to the base point of the sphere. Use the fact that the disk is contractible and the homotopy lifting property to lift the composite of p followed by ψ to a map $\tilde{\psi}\colon(D^n, \partial D^n) \to (E, F_b)$ sending the base point (in ∂D^n) to e. Then the restriction of $\tilde{\psi}|_{\partial D^n}$ represents the image under the connecting homomorphism of a. We leave the proof that this is process determines well-defined homomorphism to the reader. It is clear from this construction that the composition of $\pi_\#$ followed by the connecting homomorphism is zero since we can take the lift of $\bar{a} = \pi_\#(a)$ to be the composition the collapsing map $p\colon (D^n, \partial D^n) \to (S^n, p)$ followed by a. It follows that the image of the connecting homomorphism applied to $\pi_\#(a) = 0$. Conversely, if the image of the connecting homomorphism applied to $\bar{a}$ is zero, then the lifted map $(D^n, \partial D^n) \to (E, F_b)$ has the property that its restriction to the boundary is homotopic in (F_b, e) to a point map. Homotopy extension allows us to lift the original map of S^n to B to a map of S^n to E, showing that the image of $\pi_\#$ contains the kernel of the connecting homomorphism.

Exactness at $\pi_n(F_b, e)$. From the construction, any based sphere in (F_b, e) coming from the connecting homomorphism bounds a disk in (E, e), showing that the composition of the connecting homomorphism followed by $i_\#$ is zero. Conversely, if an element $c \in \pi_n(F_b, e)$ is trivial in $\pi_n(E, e)$, then the sphere in (F_b, e) representing c bounds a disk in (E, e) whose image under $\pi_\#$ is a sphere of one higher dimension in (B, b) whose image under the connecting homomorphism is c. ■

The Loop Space

Proposition 2.5. *Let* $x_0 \in X$ *be given and let* $\mathcal{P}(X, x_0)$ *denote the space of paths in* X *beginning at* x_0. *Then* $\pi\colon \mathcal{P}(X, x_0) \to X$ *is a fibration.*

Proof. Given a path $g\colon I \to X$ and an element $\tilde{g}_0 \in \mathcal{P}(X)$ such that $\pi(\tilde{g}_0) = g(0)$ (i.e., given a path g in X and a path g_0 beginning at x_0 and ending at $g(0)$), we define $\tilde{g}_t \in \mathcal{P}(X, x_0)$ by

$$\tilde{g}_t(s) = \begin{cases} \tilde{g}_0(s(1+t)) & 0 \le s \le \frac{1}{1+t} \\ g(s(1+t)-1) & \frac{1}{1+t} \le s \le 1. \end{cases}$$

One sees easily that $\pi(\tilde{g}_t) = g_t$ and that $t \to \tilde{g}_t$ is a continuous mapping of I into $\mathcal{P}(X, x_0)$. This proves the homotopy lifting property for points. One checks that the construction varies continuously with the original data and hence gives the homotopy lifting property for all spaces. ■

Definition. The fiber $\pi^{-1}(x_0) \subset \mathcal{P}(X, x_0)$ is denoted $^{\cdot}(X, x_0)$ and is the *loop space of* X *based at* x_0.