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Stability of Functional Equations in Random Normed Spaces

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Stability of Functional Equations in Random Normed Spaces

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To our beloved family:
In-Suk Kang (wife) and Eun-Hong Cho (granddaughter)
Ninetta (wife) and Michael, Stamatina (children)
Laleh (wife) and Shahriar, Shirin (children)

Preface

The study of functional equations has a long history. In 1791 and 1809, Legendre [152] and Gauss [90] attempted to provide a solution of the following functional equation:

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, which is called the *Cauchy functional equation*. A function $f : \mathbb{R} \to \mathbb{R}$ is called an *additive function* if it satisfies the Cauchy functional equation. In 1821, Cauchy [30] first found the general solution of the Cauchy functional equation, that is, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous additive function, then f is linear, that is, $f(x) = mx$, where m is a constant. Further, we can consider the biadditive function on $\mathbb{R} \times \mathbb{R}$ as follows:

A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called an *biadditive function* if it is additive in each variable, that is,

$$f(x+y, z) = f(x, z) + f(y, z)$$

and

$$f(x, y+z) = f(x, y) + f(x, z)$$

for all $x, y, z \in \mathbb{R}$. It is well known that every continuous biadditive function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the form

$$f(x, y) = mxy$$

for all $x, y \in \mathbb{R}$, where m is a constant.

Since the time of Legendre and Gauss, several mathematicians had dealt with additive functional equations in their books [2–4, 126, 145] and a number of them have studied Lagrange's mean value theorem and related functional equations, Pompeiu's mean value theorem and associated functional equations, two-dimensional mean value theorem and functional equations as well as several kinds of functional equations. We know that the mean value theorems have been motivated to study the functional equations (see the book "Mean Value Theorems and Functional Equations" by Sahoo and Riedel, 1998 [239]).

In 1940, S.M. Ulam [250] proposed the following stability problem of functional equations:

Given a group G_1, a metric group G_2 with the metric $d(\cdot,\cdot)$ and a positive number ε, does there exist $\delta > 0$ such that, if a mapping $f : G_1 \to G_2$ satisfies

$$d\big(f(xy), f(x)f(y)\big) \le \delta$$

for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with

$$d\big(f(x), h(x)\big) \le \varepsilon$$

for all $x \in G_1$?

Since then, several mathematicians have dealt with special cases as well as generalizations of Ulam's problem.

In fact, in 1941, D.H. Hyers [107] provided a partial solution to Ulam's problem for the case of approximately additive mappings in which G_1 and G_2 are Banach spaces with $\delta = \varepsilon$ as follows:

Let X and Y be Banach spaces and let $\varepsilon > 0$. Then, for all $g : X \to Y$ with

$$\sup_{x,y\in X} \big\|g(x+y) - g(x) - g(y)\big\| \le \varepsilon,$$

there exists a unique mapping $f : X \to Y$ such that

$$\sup_{x\in X} \big\|g(x) - f(x)\big\| \le \varepsilon,$$

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

This proof remains unchanged if G_1 is an Abelian semigroup. Particularly, in 1968, it was proved by Forti (Proposition 1, [88]) that the following theorem can be proved.

Theorem F *Let $(S,+)$ be an arbitrary semigroup and E be a Banach space. Assume that $f : S \to E$ satisfies*

$$\big\|f(x+y) - f(x) - f(y)\big\| \le \varepsilon. \tag{A}$$

Then the limit

$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{B}$$

exists for all $x \in S$ and $g : S \to E$ is the unique function satisfying

$$\big\|f(x) - g(x)\big\| \le \varepsilon, \quad g(2x) = 2g(x).$$

Finally, if the semigroup S is Abelian, then G is additive.

Here, the proof method which generates the solution g by the formula like (B) is called the *direct method*.

If f is a mapping of a group or a semigroup $(S, \cdot)$ into a vector space E, then we call the following expression:

$$Cf(x, y) = f(x \cdot y) - f(x) - f(y)$$

the *Cauchy difference* of f on $S \times S$. In the case that E is a topological vector space, we call the equation of homomorphism *stable* if, whenever the Cauchy difference Cf is bounded on $S \times S$, there exists a homomorphism $g : S \to E$ such that $f - g$ is bounded on S.

In 1980, Rätz [230] generalized Theorem F as follows: Let $(X, *)$ be a power-associative groupoid, that is, X is a nonempty set with a binary relation $x_1 * x_2 \in X$ such that the left powers satisfy $x^{m+n} = x^m * x^n$ for all $m, n \geq 1$ and $x \in X$. Let $(Y, |\cdot|)$ be a topological vector space over the field $\mathbb{Q}$ of rational numbers with $\mathbb{Q}$ topologized by its usual absolute value $|\cdot|$.

Theorem R *Let V be a nonempty bounded $\mathbb{Q}$-convex subset of Y containing the origin and assume that Y is sequentially complete. Let $f : X \to Y$ satisfy the following conditions: for all $x_1, x_2 \in X$, there exist $k \geq 2$ such that*

$$f\left((x_1 * x_2)^{k^n}\right) = f\left(x_1^{k^n} * x_2^{k^n}\right) \tag{C}$$

for all $n \geq 1$ and

$$f(x_1) + f(x_2) - f(x_1 * x_2) \in V. \tag{D}$$

Then there exists a function $g : X \to Y$ such that $g(x_1) + g(x_2) = g(x_1 * x_2)$ and $f(x) - g(x) \in \overline{V}$, where $\overline{V}$ is the sequential closure of V for all $x \in X$. When Y is a Hausdorff space, then g is uniquely determined.

Note that the condition (C) is satisfied when X is commutative and it takes the place of the commutativity in proving the additivity of g. However, as Rätz pointed out in his paper, the condition

$$(x_1 * x_2)^{k^n} = x_1^{k^n} * x_2^{k^n}$$

for all $x_1, x_2 \in X$, where X is a semigroup, and, for all $k \geq 1$, does not imply the commutativity.

In the proofs of Theorem F and Theorem R, the completeness of the image space E and the sequential completeness of Y, respectively, were essential in proving the existence of the limit which defined the additive function g. The question arises whether the completeness is necessary for the existence of an odd additive function g such that $f - g$ is uniformly bounded, given that the Cauchy difference is bounded.

For this problem, in 1988, Schwaiger [240] proved the following:

Theorem S *Let E be a normed space with the property that, for each function $f : \mathbb{Z} \to E$, whose Cauchy difference $Cf = f(x+y) - f(x) - f(y)$ is bounded for all $x, y \in \mathbf{Z}$ and there exists an additive mapping $g : \mathbb{Z} \to E$ such that $f(x) - g(x)$ is bounded for all $x \in \mathbb{Z}$. Then E is complete.*

Corollary 1 *The statement of theorem S remains true if $\mathbb{Z}$ is replaced by any vector space over $\mathbb{Q}$.*

In 1950, T. Aoki [14] generalized Hyers' theorem as follows:

Theorem A *Let E_1 and E_2 be two Banach spaces. If there exist $K > 0$ and $0 \leq p < 1$ such that*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq K\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in E_1$, then there exists a unique additive mapping $g : E_1 \to E_2$ such that

$$\left\| f(x) - g(x) \right\| \leq \frac{2K}{2 - 2^p}\|x\|^p$$

for all $x \in E_1$.

In 1978, Th.M. Rassias [216] formulated and proved the stability theorem for the linear mapping between Banach spaces E_1 and E_2 subject to the continuity of $f(tx)$ with respect to $t \in \mathbb{R}$ for each fixed $x \in E_1$. Thus, Rassias' theorem implies Aoki's theorem as a special case. Later, in 1990, Th.M. Rassias [218] observed that the proof of his stability theorem also holds true for $p < 0$. In 1991, Gajda [89] showed that the proof of Rassias' theorem can be proved also for the case $p > 1$ by just replacing n by $-n$ in (B). These results are stated in a generalized form as follows (see Rassias and Šemrl [228]):

Theorem RS *Let $\beta(s,t)$ be nonnegative for all nonnegative real numbers s, t and positive homogeneous of degree p, where p is real and $p \neq 1$, that is, $\beta(\lambda s, \lambda t) = \lambda^p \beta(s,t)$ for all nonnegative λ, s, t. Given a normed space E_1 and a Banach space E_2, assume that $f : E_1 \to E_2$ satisfies the inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \beta\left(\|x\|, \|y\|\right)$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $g : E_1 \to E_2$ such that

$$\left\| f(x) - g(x) \right\| \leq \delta\|x\|^p$$

for all $x \in E_1$, where

$$\delta := \begin{cases} \frac{\beta(1,1)}{2-2^p}, & p < 1, \\ \frac{\beta(1,1)}{2-2^p}, & p > 1. \end{cases}$$

The proofs for the cases $p < 1$ and $p > 1$ were provided by applying the direct methods. For $p < 1$, the additive mapping g is given by (B), while in case $p > 1$ the formula is

$$g(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

Corollary 2 *Let $f : E_1 \to E_2$ be a mapping satisfying the hypotheses of Theorem RS and suppose that f is continuous at a single point $y \in E_1$, then the additive mapping g is continuous.*

Corollary 3 *If, under the hypotheses of Theorem RS, we assume that, for each fixed $x \in E_1$, the mapping $t \to f(tx)$ from $\mathbb{R}$ to E_2 is continuous, then the additive mapping g is linear.*

Remark 4 (1) For $p = 0$, Theorem RS, Corollaries 2 and 3 reduce to the results of Hyers in 1941. If we put $\beta(s,t) = \varepsilon(s^p + t^p)$, then we obtain the results of Rassias [216] in 1978 and Gajda [89] in 1991.

(2) The case $p = 1$ was excluded in Theorem RS. Simple counterexamples prove that one can not extend Rassias' Theorem when p takes the value one (see Z. Gajda [89], Rassias and Šemrl [228] and Hyers and Rassias [109] in 1992).

A further generalization of the Hyers-Ulam stability for a large class of mappings was obtained by Isac and Rassias [110] by introducing the following:

Definition 5 A mapping $f : E_1 \to E_2$ is said to be *ϕ-additive* if there exist $\Phi \geq 0$ and a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = 0$$

such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \Phi\left[\phi\left(\|x\|\right) + \phi\left(\|y\|\right)\right]$$

for all $x, y \in E_1$.

In [110], Isac and Rassias proved the following:

Theorem IR *Let E_1 be a real normed vector space and E_2 be a real Banach space. Let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed $x \in E_1$. If f is ϕ-additive and phi satisfies the following conditions:*

(a) *$\phi(ts) \leq \phi(t)\phi(s)$ for all $s, t \in \mathbb{R}$;*
(b) *$\phi(t) < t$ for all $t > 1$,*

then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\| f(x) - T(x)\| \leq \frac{2\theta}{2-\phi(2)}\phi(\|x\|)$$

for all $x \in E_1$.

Remark 4 (1) If $\phi(t) = t^p$ with $p < 1$, then, from Theorem IR, we obtain Rassias' theorem [216].

(2) If $p < 0$ and $\phi(t) = t^p$ with $t > 0$, then Theorem IR is implied by the result of Gajda in 1991.

Since the time the above stated results have been proven, several mathematicians (see [1, 5–13, 17, 18, 20–25, 28, 31–35, 37, 38, 42–44, 46, 47, 50–63, 67–99, 105, 108, 111–120, 124, 132–137, 144–159, 169–208, 212–215, 219–238, 243, 245–260, 262] and [263]) have extensively studied stability theorems for several kinds of functional equations in various spaces, for example, Banach spaces, 2-Banach spaces, Banach n-Lie algebras, quasi-Banach spaces, Banach ternary algebras, non-Archimedean normed and Banach spaces, metric and ultra metric spaces, Menger probabilistic normed spaces, probabilistic normed space, p-2-normed spaces, C^*-algebras, C^*-ternary algebras, Banach ternary algebras, Banach modules, inner product spaces, Heisenberg groups and others. Further, we have to pay attention to applications of the Hyers-Ulam-Rassias stability problems, for example, (partial) differential equations, Fréchet functional equations, Riccati differential equations, Volterra integral equations, group and ring theory and some kinds of equations (see [29, 114, 121–123, 128, 129, 142, 143, 153, 155, 157, 170–172, 209–211, 255, 257]). For more details on recent development in Ulam's type stability and its applications, see the papers of Brillouët-Belluot [19] and Ciepliński [41] in 2012.

The notion of random normed space goes back to Sherstnev [242] as well as the works published in [100, 101, 241] who were dulled from Menger [160], Schweizer and Sklar [241] works. After the pioneering works by several mathematicians including authors [9, 10, 148–150, 236] who focused at probabilistic functional analysis, Alsina [8] considered the stability of a functional equation in probabilistic normed spaces and, in 2008, Miheţ and Radu considered the stability of a Cauchy additive functional equation in random normed space via fixed point method [161].

The book provides a recent survey of both the latest and new results especially on the following topics:

(1) Basic theory of random normed spaces and related spaces;
(2) Stability theory for several new functional equations in random normed spaces via fixed point method, under the special t-norms as well as arbitrary t-norms;
(3) Stability theory of well known new functional equations in non-Archimedean random normed spaces;
(4) Applications in the class of fuzzy normed spaces.

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Contents

Acronyms

AQCQ	additive–quadratic–cubic–quartic
ACQ	additive–cubic–quartic
IRN-space	intuitionistic random normed space
IF-normed space	intuitionistic fuzzy normed space
LRN-space	lattice random normed space
RN-space	random normed space
QC	quadratic–cubic
Δ^+	space of all distribution functions
$\mathcal{L}$	complete lattice
$\mathcal{N}$	involutive negation
$(\mathbb{R}^n, \Phi, T)$	random Euclidean normed space
t-norm	triangular norm
T_L	Łukasiewicz t-norm
T_M	minimum t-norm
$(X, \mathcal{F}, T)$	Menger probabilistic metric space
(X, μ, T)	random normed space
τ	binary operation on Δ^+
$(X, M, *)$	fuzzy metric space
$(X, N, *)$	fuzzy normed space

Chapter 1
Preliminaries

In this chapter, we recall some definitions and results which will be used later on in the book.

1.1 Triangular Norms

Triangular norms first appeared in the framework of probabilistic metric spaces in the work of Menger [160]. It turns also out that this is a essential operation in several fields. Triangular norms are an indispensable tool for the interpretation of the conjunction in fuzzy logics [104] and, subsequently, for the intersection of fuzzy sets [261]. They are, however, interesting mathematical objects for themselves. We refer to some papers and books for further details (see [100, 138–141] and [241]).

Definition 1.1.1 A *triangular norm* (shortly, *t-norm*) is a binary operation on the unit interval $[0, 1]$, that is, a function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that, for all $a, b, c \in [0, 1]$, the following four axioms are satisfied:

(T1) $T(a, b) = T(b, a)$ (commutativity);
(T2) $T(a, (T(b, c))) = T(T(a, b), c)$ (associativity);
(T3) $T(a, 1) = a$ (boundary condition);
(T4) $T(a, b) \leq T(a, c)$ whenever $b \leq c$ (monotonicity).

The commutativity of (T1), the boundary condition (T3) and the monotonicity (T4) imply that, for each t-norm T and $x \in [0, 1]$, the following boundary conditions are also satisfied:

$$T(x, 1) = T(1, x) = x,$$
$$T(x, 0) = T(0, x) = 0,$$

and so all the t-norms coincide with the boundary of the unit square $[0, 1]^2$.

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The monotonicity of a t-norm T in the second component (T4) is, together with the commutativity (T1), equivalent to the (joint) monotonicity in both components, that is, to

$$T(x_1, y_1) \leq T(x_2, y_2) \tag{1.1.1}$$

whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

Basic examples are the Łukasiewicz t-norm T_L:

$$T_L(a, b) = \max\{a + b - 1, 0\}$$

for all $a, b \in [0, 1]$ and the t-norms T_P, T_M, T_D defined as follows:

$$T_P(a, b) := ab,$$

$$T_M(a, b) := \min\{a, b\},$$

$$T_D(a, b) := \begin{cases} \min\{a, b\}, & \text{if } \max\{a, b\} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

If, for any two t-norms T_1 and T_2, the inequality $T_1(x, y) \leq T_2(x, y)$ holds for all $(x, y) \in [0, 1]^2$, then we say that T_1 is *weaker* than T_2 or, equivalently, T_2 is *stronger* than T_2.

From (1.1.1), it follows that, for all $(x, y) \in [0, 1]^2$,

$$T(x, y) \leq T(x, 1) = x,$$

$$T(x, y) \leq T(1, y) = y.$$

Since $T(x, y) \geq 0 = T_D(x, y)$ for all $(x, y) \in (0, 1)^2$ holds trivially, for any t-norm T, we have

$$T_D \leq T \leq T_M,$$

that is, T_D is weaker and T_M is stronger than any others t-norms. Also, since $T_L < T_P$, we obtain the following ordering for four basic t-norms:

$$T_D < T_L < T_P < T_M.$$

Proposition 1.1.2 [100] (1) *The minimum T_M is the only t-norm satisfying $T(x, x) = x$ for all $x \in (0, 1)$.*

(2) *The weakest t-norm T_D is the only t-norm satisfying $T(x, x) = 0$ for all $x \in (0, 1)$.*

Proposition 1.1.3 [100] *A t-norm T is continuous if and only if it is continuous in its first component, i.e., for all $y \in [0, 1]$, if the one-place function*

$$T(\cdot, y) : [0, 1] \to [0, 1], \quad x \mapsto T(x, y),$$

is continuous.

For example, the minimum T_M and Łukasiewicz t-norm T_L are continuous, but the t-norm T^{Δ} defined by

$$T^{\Delta}(x, y) := \begin{cases} \frac{xy}{2}, & \text{if } \max\{x, y\} < 1; \\ xy, & \text{otherwise,} \end{cases}$$

for all $x, y \in [0, 1]$ is not continuous.

Definition 1.1.4 (1) A t-norm T is said to be *strictly monotone* if

$$T(x, y) < T(x, z)$$

whenever $x \in (0, 1)$ and $y < z$.

(2) A t-norm T is said to be *strict* if it is continuous and strictly monotone.

For example, the t-norm T^{Δ} is strictly monotone, but the minimum T_M and Łukasiewicz t-norm T_L are not strictly monotone.

Proposition 1.1.5 [100] *A t-norm T is strictly monotone if and only if*

$$T(x, y) = T(x, z), \quad x > 0 \quad \Longrightarrow \quad y = z.$$

If T is a t-norm, then $x_T^{(n)}$ for all $x \in [0, 1]$ and $n \geq 0$ is defined by 1 if $n = 0$ and $T(x_T^{(n-1)}, x)$ if $n \geq 1$.

Definition 1.1.6 A t-norm T is said to be *Archimedean* if, for all $(x, y) \in (0, 1)^2$, there exists an integer $n \geq 1$ such that

$$x_T^{(n)} < y.$$

Proposition 1.1.7 [100] *A t-norm T is Archimedean if and only if, for all $x \in (0, 1)$,*

$$\lim_{n \to \infty} x_T^{(n)} = 0.$$

Proposition 1.1.8 [100] *If t-norm T is Archimedean, then, for all $x \in (0, 1)$, we have*

$$T(x, x) < x.$$

For example, the product T_p, Łukasiewicz t-norm T_L and the weakest t-norm T_D are all Archimedean, but the minimum T_M is not an Archimedean t-norm.

A t-norm T is said to be *of Hadžić-type* (denoted by $T \in \mathcal{H}$) if the family $\{x_T^{(n)}\}$ is equicontinuous at $x = 1$, that is, for any $\varepsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that

$$x > 1 - \delta \quad \Longrightarrow \quad x_T^{(n)} > 1 - \varepsilon \tag{1.1.2}$$

for all $n \geq 1$.

The t-norm T_M is a trivial example of Hadžić type, but T_P is not of Hadžić type.

Proposition 1.1.9 [100] *If a continuous t-norm T is Archimedean, then it can not be a t-norm of Hadžić-type.*

Other important t-norms are as follows (see [102]):

(1) The *Sugeno–Weber family* $\{T_\lambda^{SW}\}_{\lambda\in[-1,\infty]}$ is defined by $T_{-1}^{SW} = T_D$, $T_\infty^{SW} = T_P$ and

$$T_\lambda^{SW}(x, y) = \max\left\{0, \frac{x + y - 1 + \lambda xy}{1 + \lambda}\right\}$$

if $\lambda \in (-1, \infty)$.

(2) The *Domby family* $\{T_\lambda^D\}_{\lambda\in[0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M if $\lambda = \infty$ and

$$T_\lambda^D(x, y) = \frac{1}{1 + ((\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda)^{1/\lambda}}$$

if $\lambda \in (0, \infty)$.

(3) The *Aczel–Alsina family* $\{T_\lambda^{AA}\}_{\lambda\in[0,\infty]}$ is defined by T_D, if $\lambda = 0$, T_M if $\lambda = \infty$ and

$$T_\lambda^{AA}(x, y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}}$$

if $\lambda \in (0, \infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking, for any $(x_1, \ldots, x_n) \in [0, 1]^n$, the value $T(x_1, \ldots, x_n)$ defined by

$$\mathrm{T}_{i=1}^0 x_i = 1, \qquad \mathrm{T}_{i=1}^n x_i = T\left(\mathrm{T}_{i=1}^{n-1} x_i, x_n\right) = T(x_1, \ldots, x_n).$$

The t-norm T can also be extended to a countable operation taking, for any sequence $\{x_n\}$ in $[0, 1]$, the value

$$\mathrm{T}_{i=1}^\infty x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^n x_i. \tag{1.1.3}$$

The limit on the right side of (1.1.3) exists since the sequence $\{\mathrm{T}_{i=1}^n x_i\}$ is non-increasing and bounded from below.

Proposition 1.1.10 [102] (1) *For $T \geq T_L$ the following implication holds:*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \iff \sum_{n=1}^\infty (1 - x_n) < \infty.$$

(2) *If T is of Hadžić-type, then we have*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1$$

for any sequence $\{x_n\}_{n\geq 1}$ in $[0, 1]$ such that $\lim_{n\to\infty} x_n = 1$.

(3) *If* $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, *then we have*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \iff \sum_{n=1}^{\infty}(1-x_n)^{\alpha} < \infty.$$

(4) *If* $T \in \{T_\lambda^{SW}\}_{\lambda\in[-1,\infty)}$, *then we have*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^{\infty} x_{n+i} = 1 \iff \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

Definition 1.1.11 Let T and T' be two continuous t-norms. Then we say that T' *dominates* T (denoted by $T' \gg T$) if, for all $x_1, x_2, y_1, y_2 \in [0, 1]$,

$$T\big[T'(x_1, x_2), T'(y_1, y_2)\big] \le T'\big[T(x_1, y_1), T(x_2, y_2)\big].$$

1.2 Triangular Norms on Lattices

Now, we extend definitions and results on the t-norm to lattices.

Let $\mathcal{L} = (L, \ge_L)$ be a complete lattice, that is, a *partially ordered set* in which every nonempty subset admits supremum, infimum and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$.

Definition 1.2.1 [49] A *t-norm* on L is a mapping $\mathcal{T} : L \times L \to L$ satisfying the following conditions:

(1) $\mathcal{T}(x, 1_{\mathcal{L}}) = x$ for all $x \in L$ (boundary condition);
(2) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ for all $x, y \in L$ (commutativity);
(3) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ for all $x, y, z \in L$ (associativity);
(4) $x \le_L x'$ and $y \le_L y'$ implies that $\mathcal{T}(x, y) \le_L \mathcal{T}(x', y')$ for all $x, x', y, y' \in L$ (monotonicity).

Let $\{x_n\}$ be a sequence in L convergent to $x \in L$ (equipped order topology). The t-norm $\mathcal{T}$ is said to be a *continuous t-norm* if

$$\lim_{n\to\infty} \mathcal{T}(x_n, y) = \mathcal{T}(x, y)$$

for each $y \in L$.

Now, we put $\mathcal{T} = T$ whenever $L = [0, 1]$.

Definition 1.2.2 [49] A continuous t-norm $\mathcal{T}$ on $L = [0, 1]^2$ is said to be *continuous t-representable* if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example, the following t-norms

$$\mathcal{T}(a,b) = \big(a_1 b_1, \min\{a_2 + b_2, 1\}\big)$$

and

$$\mathbf{M}(a,b) = \big(\min\{a_1, b_1\}, \max\{a_2, b_2\}\big)$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0, 1]^2$ are continuous t-representable.

Define the mapping $\mathcal{T}_\wedge : L^2 \to L$ by:

$$\mathcal{T}_\wedge(x,y) = \begin{cases} x, & \text{if } y \geq_L x, \\ y, & \text{if } x \geq_L y. \end{cases}$$

A *negation* on $\mathcal{L}$ is a decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_\mathcal{L}) = 1_\mathcal{L}$ and $\mathcal{N}(1_\mathcal{L}) = 0_\mathcal{L}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involutive negation*. In the following, $\mathcal{L}$ is endowed with a (fixed) negation $\mathcal{N}$.

1.3 Distribution Functions

Let Δ^+ denote the space of all distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ such that F is left-continuous, non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Example 1.3.1 The function $G(t)$ defined by

$$G(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1 - e^{-t}, & \text{if } t > 0, \end{cases}$$

is a distribution function. Since $\lim_{t \to \infty} G(t) = 1$, $G \in D^+$. Note that $G(t+s) \geq T_p(G(t), G(s))$ for each $t, s > 0$.

Example 1.3.2 The function $F(t)$ defined by

$$F(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ t, & \text{if } 0 \leq t \leq 1, \\ 1, & \text{if } 1 \leq t, \end{cases}$$

is a distribution function. Since $\lim_{t\to\infty} F(t) = 1$, $F \in D^+$. Note that $F(t+s) \geq T_M(F(t), F(s))$ for all $t, s > 0$.

Example 1.3.3 [9] The function $G_p(t)$ defined by

$$G_p(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \exp(-|p|^{1/2}), & \text{if } 0 < t < +\infty, \\ 1, & \text{if } t = +\infty, \end{cases}$$

is a distribution function. Since $\lim_{t\to\infty} G_p(t) \neq 1$, $G \in \Delta^+ \setminus D^+$. Note that $G_p(t+s) \geq T_M(G_p(t), G_p(s))$ for all $t, s > 0$.

Definition 1.3.4 A *non-measure distribution function* is a function $\nu : \mathbb{R} \to [0, 1]$ which is right continuous on $\mathbb{R}$, non-increasing and $\inf_{t\in\mathbb{R}} \nu(t) = 0$, $\sup_{t\in\mathbb{R}} \nu(t) = 1$.

We denote by B the family of all non-measure distribution functions and by G a special element of B defined by

$$G(t) = \begin{cases} 1, & \text{if } t \leq 0, \\ 0, & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, then $\nu : X \to B$ is called a *probabilistic non-measure* on X and $\nu(x)$ is denoted by ν_x.

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, that is, a partially ordered set in which every nonempty subset admits supremum, infimum and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$. The space of latticetic random distribution functions, denoted by Δ_L^+, is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to L$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0_{\mathcal{L}}$ and $F(+\infty) = 1_{\mathcal{L}}$.

$D_L^+ \subseteq \Delta_L^+$ is defined as $D_L^+ = \{F \in \Delta_L^+ : l^- F(+\infty) = 1_{\mathcal{L}}\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ_L^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ_L^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0_{\mathcal{L}}, & \text{if } t \leq 0, \\ 1_{\mathcal{L}}, & \text{if } t > 0. \end{cases}$$

1.4 Fuzzy Sets

In this section, we consider the definition of *fuzzy sets* and present some examples. For more details, see [264]. The first publication in fuzzy set theory by Zadeh [261] showed a generalization of the classical notation of a set. A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$ which can be finite, countable or uncountable. Each single element can either belong to or not belong to a set A, $A \subseteq X$. In the former case, the statement "x belong to A" is true, whereas, in the latter case, this statement is false.

Such a classical set can be described in different ways. One way is defined the member element by using the characteristic function, in which 1 indicates membership and 0 non-membership. For a fuzzy set, the characteristic function allows various degrees of membership for the elements of a given set.

Definition 1.4.1 If W is a collection of objects denoted generically by w, then a fuzzy set A in W is a set of ordered pairs:

$$A = \big\{\big(w, \lambda_A(w)\big) : w \in W\big\},$$

where $\lambda_A(w)$ is called the *membership function* or *grade of membership* of w in A which maps W to the membership space M.

Note that, when M contains only the two points 0 and 1, A is non-fuzzy and $\lambda_A(w)$ is identical to the characteristic function of a non-fuzzy set. The range of the membership function is [0, 1] or a complete lattice.

Example 1.4.2 Consider the following fuzzy set A which is real numbers considerably larger than 10:

$$A = \big\{\big(w, \lambda_A(w)\big) : w \in W\big\},$$

where

$$\lambda_A(w) = \begin{cases} 0, & \text{if } w < 10, \\ \frac{1}{1+(w-10)^{-2}}, & \text{if } w \geq 10. \end{cases}$$

Example 1.4.3 Consider the following fuzzy set A which is real numbers close to 10:

$$A = \big\{\big(w, \lambda_A(w)\big) : w \in W\big\},$$

where

$$\lambda_A(w) = \frac{1}{1 + (w - 10)^2}.$$

Note that, in this book, in short, we apply membership functions instead fuzzy sets.

Definition 1.4.4 [96] Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and U be a nonempty set called the *universe*. An *$\mathcal{L}$-fuzzy set* in U is defined as a mapping $\mathcal{A} : U \to L$. For each $u \in U$, $\mathcal{A}(u)$ represents the *degree* (in L) to which u is an element of $\mathcal{A}$.

Lemma 1.4.5 [49] *Consider the set L^* and the operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2,\ x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \quad \Longleftrightarrow \quad x_1 \leq y_1,\ x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 1.4.6 [16] An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ in the universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(u, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ for all $u \in U$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Example 1.4.7 Consider the following intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ which is real numbers considerably larger than 10 for the first place and real numbers close to 10 in the second place:

$$\mathcal{A}_{\zeta,\eta} = \{(w, \zeta_A(w), \eta_A(w)) \ :\ w \in W\},$$

where

$$(\zeta_A(w), \eta_A(w)) = \begin{cases} (0, \frac{1}{1+(w-10)^2}), & \text{if } w < 10, \\ (\frac{1}{1+(w-10)^{-2}}, \frac{1}{1+(w-10)^2}), & \text{if } w \geq 10. \end{cases}$$

As we said in the above, forward, we will use $\mathcal{A}_{\zeta,\eta}(w) = (\zeta_A(w), \eta_A(w))$ in the next chapters.

Chapter 2
Generalized Spaces

In this chapter, we present some generalized spaces and their properties for the main results in this chapter.

2.1 Random Normed Spaces

Random (probabilistic) normed spaces were introduced by Šerstnev in 1962 [242] by means of a definition that was closely modelled on the theory of (classical) normed spaces, and used to study the problem of best approximation in statistics. In the sequel, we shall adopt usual terminology, notation and conventions of the theory of random normed spaces, as in [9, 10, 148, 241].

Definition 2.1.1 A *Menger probabilistic metric space* (or *random metric spaces*) is a triple $(X, \mathcal{F}, T)$, where X is a nonempty set, T is a continuous t-norm and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at a point $(x, y) \in X \times X$, the following conditions hold: for all x, y, z in X,

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t + s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.1.2 [242] A *random normed space* (briefly, a RN-space) or *a Šerstnev (Sherstnev) probabilistic normed space* (briefly, a Šerstnev PN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$ (0 is the null vector in X);
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$ and $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$, where μ_x denotes the value of μ at a point $x \in X$.

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