Lecture Notes in Applied Mathematics and Mechanics 1

Erwin Stein *Editor*

The History of Theoretical, Material and Computational Mechanics - Mathematics Meets Mechanics and Engineering

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Lecture Notes in Applied Mathematics and Mechanics

Volume 1

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Potential contributors should contact the appropriate editor with a title, table of contents, and a sample chapter. Full manuscripts accepted by the editors will then be peer-reviewed.

Erwin Stein
Editor

The History of Theoretical, Material and Computational Mechanics - Mathematics Meets Mechanics and Engineering

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Foreword

In 2008 Professor Stein, the editor of this volume, applied for setting up a new section in the yearly GAMM conference related to the history of mechanics. This suggestion was approved by the Board of GAMM and the first session on history of mechanics started in 2010.

Lectures and contributions that were presented in these sessions are the backbone of this first volume of LAMM. There is no better way to start the series of GAMM lecture notes to reflect the history of the research field.

The contributions in this volume discuss different aspects of mechanics. They are related to solid and fluid mechanics in general and to specific problems in these areas including the development of numerical solution techniques. Thus this first addition of LAMM provides an overview on the field of mechanics and describes the wide area of applications within GAMM.

Finally I like to thank the editor, Professor Erwin Stein, for his continuous effort and his hard work to make this volume possible.

Hannover, May 2013

Peter Wriggers
Vice-President of GAMM

Preface

This collection of 23 articles is the output of lectures in special sessions on "The History of Theoretical, Material and Computational Mechanics" within the yearly conferences of the GAMM in the years 2010 in Karlsruhe, Germany, 2011 in Graz, Austria, and in 2012 in Darmstadt, Germany; GAMM is the "Association for Applied Mathematics and Mechanics", founded in 1922 by Ludwig Prandtl and Richard von Mises.

Guiding topics for the yearly sections were proposed and leading scientists invited as keynote-lecturers. This is reflected in the four parts of this book. In their sequence and in the total concept the published articles provide a certain completeness and logical consistency within the selected topics of theoretical, material, applied and computational mechanics.

I am indebted to the co-chairmen of the sections, Professor Oskar Mahrenholtz in 2010 and 2011, and Professor Lothar Gaul in 2013. It should be mentioned that each of the three sections had two sessions, each with about 150 attendees which shows the great interest of the conference participants.

The success of the new historical sections motivated the other authors and me to publish them in a book, also stimulated by Professor Peter Wriggers, President of GAMM in the period from 2008 to 2010. I also thank him for writing a foreword.

The rich history of theoretical, material, applied and computational mechanics of solids, structures and fluids should be of vivid interest for the community of mechanicians working in science and technology as well as of applied mathematicians. This is important for the self-conception of students and practitioners in order to know and realize on which shoulders we stand and how long it often took to arrive at simple-looking formulas for describing dominant effects in loading and deformation processes of engineering structures and in fluid flow processes, and moreover to derive rather general mathematical models – despite the ambitions and efforts of eminent scientists over decades and even centuries.

Following, the four parts of the book are briefly commented.

In Part I, the origins and developments of conservation principles in mechanics and related variational methods are treated together with challenging applications from the 17^{th} to the 20^{th} century.

Part II treats general as well as more specific aspects of material theories of deforming solid continua and porous soils, e.g. the foundation of classical theories of elastoplastic deformations, the development of theories and analysis for contact with friction and plastic deformations, as well as the formation and progress of fracture in brittle and ductile solid materials.

Part III presents important theoretical and engineering developments in fluid mechanics, beginning with remarkable inventions in the old Egypt, the dominating role of the Navier-Stokes PDEs for fluid flows and their complex solutions for a wide field of parameters as well as the invention of pumps and turbines in the 19^{th} and 20^{th} century.

And finally, Part IV gives a survey on the development of direct variational (numerical) methods – the Finite Element Method – in the 20^{th} century with many extensions and generalizations, requiring a strong coupling of engineering, mathematical and computer science aspects. These three articles are restricted to static and dynamic elastic continua, according to page limitations of the book.

One may ask whether the well-written historical essays on a period of about $3\,{}^1/_2$ centuries of research in mechanics can highlight overriding insight to the motivation, the connections, the progress and the setbacks of so many eminent scientists in the past. Additionally, it has to be regarded that a master plan for the contents of the book could only be realized roughly, viewing the open calls for contributions to the related historical sections of GAMM conferences.

Nevertheless, the structure and the contents of the book are above all characterized by the invited lectures (chapters) of well-known scientists in their fields.

However, in order to know the real genesis of the scientific truth, we would have to ask all those splendid researchers behind the huge work about their motivations and goals, which – of course – is not possible.

Instead, we reflected essential individual achievements as parts and driving forces of the integral subject "Mechanics" with their important and distinct positions in the whole framework of this discipline. Thus, each chapter can be widely understood independently from the others.

It is my pleasant duty to deeply thank all authors for elaborating their articles on a high standard and publishing them in this book. The friendly collaboration over nearly a year provided the nice feeling of partnership.

We are thankful to Wiley Publishing Company for admitting republications of five over-worked and extended articles published in the "GAMM-Mitteilungen", Vol. 34 (2011) (Issue 2) and an article published in "ZAMM", Vol. 92 (2012), pp. 683–708. Further, I thank the editors of the Polish "Journal of Computer Assisted Methods in Engineering Science" for permitting

publication of the abbreviated and revised article Vol. 19 (2012) No. 1, pp. 7–91.

The authors and the editor appreciate the publication of the book as Volume 1 of the new series “Lecture Notes in Applied Mathematics and Mechanics (LNAMM)”. We thank Dr. Thomas Ditzinger, Springer-Verlag, for his advice and helpful collaboration.

Hannover, May 2013 Erwin Stein, Editor

Contents

Part I
Mechanical Conservation Principles, Variational Calculus and Engineering Applications from the 17th to the 20th Century

The Origins of Mechanical Conservation Principles and Variational Calculus in the 17th Century

Erwin Stein

Abstract. The 17th century is considered as the cradle of modern natural sciences and technology as well as the begin of the age of enlightenment with the invention of analytical geometry by Descartes (1637), infinitesimal calculus by Newton (1668) and Leibniz (1674), and based on the rational mechanics by Newton (1687), initiated by Galilei (1638). In 1696, Johann Bernoulli posed the so-called brachistochrone problem in Acta Eruditorum, asking for solutions within a year's time. Seven solutions were submitted and published in 1697, the most famous one by his brother Jacob Bernoulli, anticipating Euler's idea of discrete equidistant support points and triangular test functions between three neighboured points, followed by the infinitesimal limit. Johann Bernoulli himself presented two intelligent solutions by joining geometrical and mechanical observations. Leibniz submitted a geometrical integration method for the differential equation of the cycloid and, what is important for this article, a short draft of a discrete or "direct variational" numerical approximation method, also using triangular test functions between neighboured support points with finite distances. This can be considered as a precursor of the finite element method. In connection with the brachistochrone, more general tautochrony problems were investigated, e.g. by Huygens and Newton. In conclusion many important developments of energy methods in mechanics using variational methods were already invented in the 17th century.

1 The 17th Century as the Cradle of Modern Natural Science and Mathematics

The late scholastic philosophy of the 16th century in central Europe, dominated by the catholic theology and based on the thinking of Aristoteles and Augustinus,

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imposed severe restrictions on the progress of natural science and the human inventive genius for creating new useful technical tools. The Italian Renaissance of the 15^{th} and 16^{th} century already brought a fundamental change of human identity, orientation and self-assured thinking with the claim that man – not the Gods or God – had invented and still were inventing helpful artifacts.

Inspired by the ancient Greek culture, especially based on the New-Platonism, an autonomous thinking and creative abilities became attractive, and a new typus of gifted craftsmen and artistic engineers created revolutionary experiments of living and inanimate nature, inspired by this insight they made spectacular technical inventions, among them Brunelleschi, the architect and engineer of the Duomo of Florence, and Leonardo da Vinci, whose fascinating technical inventions were far ahead of his time. The first technical patents were conferred to inventors in Florence in the 16^{th} century.

In central Europe Gutenberg, Paracelsus and especially Copernicus prepared the new age of natural science and technology. In 1620, Bacon, who has been called the father of empiricism, published his *Novum Organum*, [1], (addressing Aristoteles' Organum) in which he established inductive methodologies for scientific inquiring. He fought against prejudices and preconceived ideas.

And then, Descartes established the new mechanistic philosophy of rationalism and doubt with the dualism of the two different substances: matter (body) and mind. Later he asserted that these substances are not separated but build a single identity, Descartes (1637), [2]. Descartes marks the beginning of the philosophy of enlightenment, highlighted by his statement "*cogito ergo sum*".

Spinoza, a lense grinder, was active in the Dutch Jewish Community and developed his so-called pantheistic philosophy from a deep critical study of the Christian Bible, Spinoza (1670), [3]. He provided an alternative to materialism, atheism and deism, claiming the identity of spirit and nature, so to say a religion of nature, Spinoza (1677), [4]. Spinoza was heavily attacked by the Catholic Church; all his publications were indexed, and being called a *spinocist* at that time was comparable to an *atheist* with the consequence of persecution.

Leibniz was a multi-ingenious scholar in all branches of science at his time and highly interested in new technical inventions, and applications for practical use in his holistic and universal thinking and the postulates "*theoria cum praxi*" and "*commune bonum*", based on systematic collections of former scientific cognition and new findings in a universal frame, combined with new technical inventions and the improved production of goods in new manufactures. And he also contributed essentially to the new rational philosophy, guided by his postulates "*nihil sine ratione*", "*nihil fit sine causa sufficiente*", and "*the continuity principle*". He was highly motivated to smooth down and to settle controversial political and religious convictions and ideas in order to achieve piece in the European states and to unify the Christian churches as a "*pacidius*" (a peacemaker), as he conceived himself. In his quasi-axiomatic *monadology*, Leibniz (1714), [5], with 90 short paragraphs, his theology different from Descartes is framed by the conviction that God as the highest monade created the universe as the best of all thinkable ones in conjunction with optimal natural laws. Thus, the creation and the development of the universe relies on rationality and mathematical logic. He was sure that reasoning of natural science inevitably leads to metaphysics: *nihil est in intellektu quod non fuerat in sensu, excipe: nisi ipse intellectu*, Leibniz (1686), [6]. Thus, the discrepancy of body and soul can be overcome in this metaphysical draft of the universe. He created a new paradigm from Christian salvation history to apprenticeship of wisdom, thus overcoming the Christian stigmas of the Original Sin and the Last Judgement.

There is no doubt that Isaac Newton outshines all physicists in the 17$^{\text{th}}$ and the following two centuries by the creation of new natural science in his famous principia (1687) [7]. In the introduction, he claims: the old (Greek) developed the *mechanica practica* but I created the *mechanica rationalis* (rational mechanics) which was the origin and the *bible* for the 18$^{\text{th}}$ and 19$^{\text{th}}$ century. C. Truesdell wrote a remarkable appraisal on the ingenious work of Newton in [8].

About four years after Newton, in 1674, Leibniz independently invented the infinitesimal calculus in Paris, published in 1684, [9], and gave a much deeper understanding of the infinitesimal limit for integration, using already the later Riemannian sums from the 19$^{\text{th}}$ century, Leibniz (1676), [10], unpublished until the 20$^{\text{th}}$ century.

Moreover he falsificated Descartes's findings for the "*true measure of the living force*", who assumed erroneously that the product of mass and velocity of a moved body (which is not a scalar) ought to be a conservation quantity, and he discovered the kinetic energy $1/2m \cdot v^2$, Leibniz (1686), [11], first without the factor $1/2$, as the wanted conservation quantity of a straight on moved body with mass m and velocity v in quasi-static state.

With his important *continuity principle* he investigated short times before and after the impact of two bodies and thus found the error in Descartes' assumptions for his impact laws, see Szabo (1987), [12], also applying Galilei's finding of the velocity $v = \sqrt{2gh}$ of a falling or frictionless down gliding body due to gravity, according to the potential property of the kinetic energy of this mass.

Leibniz had the teleological vision that the physical laws of nature fulfil extremal principles for certain (scalar) conservation quantities, according to his postulate of ours as the best of all possible worlds.

The very first conservation principle in mechanics, the principle of minimum potential energy was established by Torricelli, secretary to Galilei, about 1630. He postulated that the gravity centre of an assembly of masses with arbitrary connections and boundary conditions finds its stable static equilibrium in a configuration for which the gravity centre takes the deepest possible position, published only in 1919, [13].

The birth of *variational calculus* can be dated with Jacob Bernoulli's ingenious solution of the brachistochrone problem in 1697, [14], first stated and approximately solved by Galilei in his *Discorsi* from 1638, [15], then again posed by Jacob's younger brother Johann Bernoulli in *Acta Eruditorum* in 1696, and 7 solutions were published in this journal by Leibniz in 1697, [14], [16]. Therein Jacob Bernoulli anticipated Eulers' idea of reducing the variational problem of an extremum of a functional of the requested extremal function first into a finite number of equidistant discrete problems of infinitesimal calculus for functions, Euler (1744), [17], also using triangular test functions (with value 1 at the considered discrete point and 0 at the neighbouring points) and then performing the transition to infinitely many time steps. Six other solutions of this problem were submitted, among them by Leibniz which will be treated later in this article; but Leibniz did not recognize that the variational calculus was a new important branch of infinitesimal calculus which he liked to dominate in Europe, and he did not contribute further to this new important branch of mathematics.

A second article on the brachistochrone problem was published by Johann Bernoulli with his own solution also in 1697, [16].

2 Snell's Law of Light Refraction and Fermat's Principle of Least Time for the Optical Path Length

Snellius published in 1621 the law of light refraction which was better reasoned later by Huygens with the principle that every point of a wave is the source of a new wave. This law reads

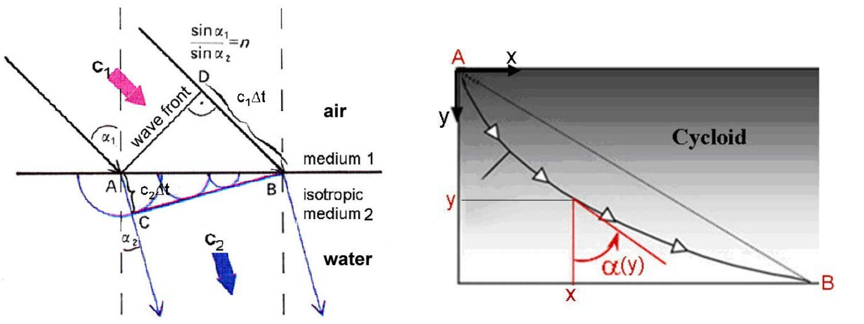
$$\sin\alpha_1 = \frac{c_1 \Delta t}{AB}\,; \quad \sin\alpha_2 = \frac{c_2 \Delta t}{AB}, (1a) \tag{1}$$

yielding

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{c_1}{c_2} = n\,; \quad \frac{\sin\alpha_1}{c_1} = \frac{\sin\alpha_2}{c_2} = \text{const}, (1b) \tag{2}$$

n the refraction coefficient.

Fermat established the principle of the light path in minimal time using Cartesian coordinates: $T = s_1/c_1 + s_2/c_2 = n_1/s_1 + n_2/s_2$,
$T = n_1\left[(x-x_1)^2 + y_1^2\right]^{1/2} + n_2\left[(x_2-x)^2 + y_2^2\right]^{1/2}$. The stationarity condition written with the derivative in the later formulation by Newton and Leibniz reads
$\mathrm{d}t/\mathrm{d}x(P_1, P_2; x) = 0 \;\rightarrow\; n_1(x-x_1)/s_1 - n_2(x_2-x)/s_2 = 0$ or



a) b)

Fig. 1 a) Refraction of light at the transition from a less dense medium (air) to a denser medium (water) with the velocities c_1 and c_2; b) The common cycloid as the light path at least time in a medium with linearly varying density

$$\sin\alpha_1/c_1 = \sin\alpha_2/c_2. \tag{2}$$

In case of linearly varying density from a more to a less dense medium, Fig. 1b, the light path fulfills the optimality (stationarity) condition

$$T_{AB} = \int \mathrm{d}s/c(y)\sqrt{y} = \min;\ \mathrm{d}t = \mathrm{d}s/c(y),\ \text{yelding } \sin\alpha(y)/c(y) = \text{const} \tag{3}$$

which describes a common cycloid.

Fermat's principle had significant influence on the finding of conservation principles for physical problems in the 17$^{\text{th}}$ and 18$^{\text{th}}$ century. It will be shown in Sect. 3 that the famous problem of a guided frictionless down-gliding mass due to gravity in shortest time, first posed by Galilei, [15], also has the solution of the common cycloid.

3 The Mechanical Properties of the Catenary Curve

In the 17$^{\text{th}}$ century Galilei, Huygens, Leibniz as well as the brothers Bernoulli were searching for the catenary or funicular curve. In 1690 Jacob Bernoulli called for the "solution problematio funicularis" in Acta Eruditorum.

Leibniz discovered the symmetric exponential function as the catenary function

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right) = a\cosh(x/a); \quad y(x=0) = a \tag{4}$$

with the equal normal and curvature radius

$$n(y) = R(y) = y^2/a. \tag{5}$$

He also gave a representation of the exponential function by the sum of the catenary curve and its derivative.

Furthermore, Leibniz found the catenary curve by a counter clock wise rolling of a parabola on the horizontal axis, Fig. 2a, with the positions $Y, Y', Y'', Y''', \ldots,$ where the normals are points of the catenary curve. And finally, Leibniz constructed the logarithmus function from the catenary function, Fig. 2b.

The first term of the power expansion of the catenary function with respect to the parameter f/ℓ, Fig. 2a, yields the quadratic parabola $y = fx(\ell - x)/4\ell^2$. This is the first approximation of the $\cosh(x/a)$-function for small f/ℓ. In this case the vertical line load is constant along the x-axis, whereas the line load of the catenary function caused by dead weight obviously grows from the middle point to the edges.

Another important property of the catenary curve is related to the principle of minimum potential energy of arbitrary connected masses, as outlined in section 1. One gets the function of the hanging catenary curve by postulating the deepest possible position of the gravity centre, and one can show that it has minimal length. Furthermore, the rotational surface of this catenary curve, the catenoid, has minimal surface and is a solution of the Plateau problem.

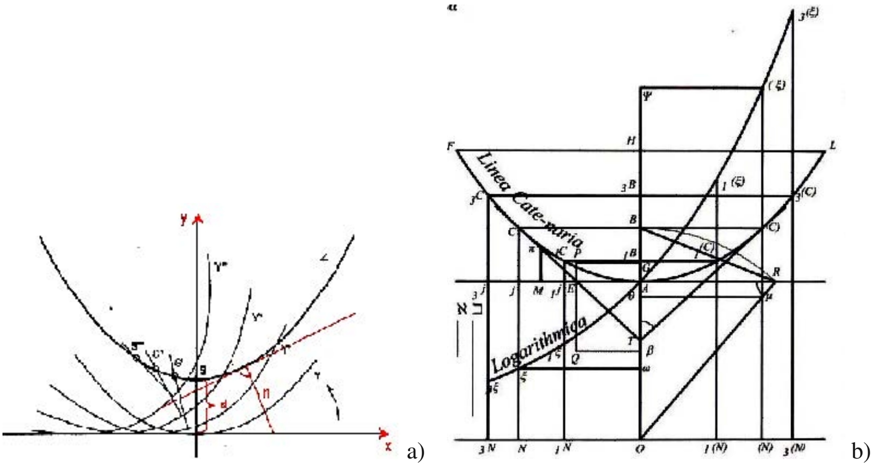


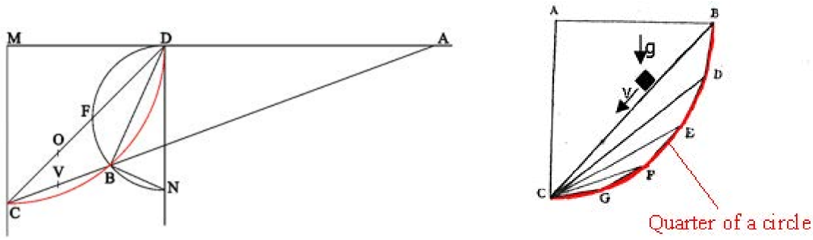
Fig. 2 a) Construction of the funicular function by a rolling parabola; b) construction of the natural logarithmus function from the catenary function

The first scientific application of catenary curves in civil engineering was realised for the restoration of the dome of the St. Peter's Cathedral in Rome which had meridional cracks. Pope Benedikt XIV commissioned the Venecian monks Le Seur, Boscovich and Poleni for a restoration proposal which was submitted in 1742 as the "Parere di tre mathematici", [18], see also Szabo (1987), [19]. They had the ingenious idea to model the dome experimentally by mirroring it with chains and hangings weights (imaging the real dead loads) with respect to the horizontal plane. The meridian of the catenoid surface represents the pure membrane state of internal forces, and the distances of this meridian with respect to the meridian curve of the dome represent the lever arms of the meridian forces, yielding the bending moments in the dome which caused the cracks. Therefore, the restoration was realized by two iron stiffening rings at positions with the largest lever arms.

4 The So-Called Brachistochrone Problem of a Mass Gliding Down Frictionless in Shortest Time

4.1 Galileo Galilei's First Formulation and Approximated Solution

Galilei was the first to pose this famous problem of optimization and variational calculus in his "Discorsi" in 1638, [15]. Not only the position and the value of a function with extremal property is requested but the whole function under the condition of an extremum of an integral of this function and its derivatives within a given domain.



a) b)

Fig. 3 a) Construction of a polygon point B for a down gliding mass assuming the quarter of a circle as the optimal curve; b) comparative polygons with the quarter of the circle as the approximative optimal solution

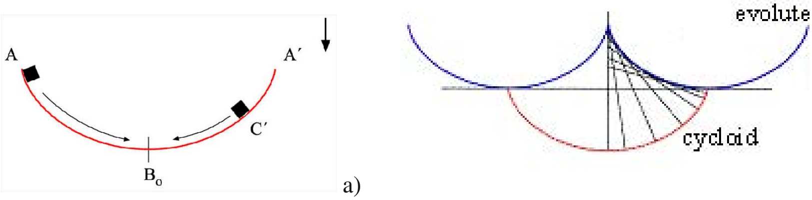
Galilei got the experimental results for the gliding times:
$t(BC) > t(BDC) > t(BDEC) > t(BDEFC) > t(BDEFGC)$, with the quarter of a circle as the hull.

About 60 years later the problem was solved analytically with different challenging methods by Johann Bernoulli et al., subsections 4.6 – 4.9, yielding the common cycloid as the solution of the problem.

4.2 Tautochrony or Isochrony Property of the Cycloid

Another access to the extremal properties of the cycloid was provided by Huygens and Newton with the so-called tautochrony or isochrony property, Huygens (1673), [20], as shown in Fig. 4.

With equal gliding times T_{ABo} and $T_{C'Bo}$ for arbitrary starting points A and C′ a remarkable property of the cycloid was found. This property is related to Fig. 4b and was also used by Huygens in his famous physical cycloid pendulum, yielding a constant frequency for arbitrary amplitudes, realized by two cycloids on both sides of the pendulum, at which the thread of the pendulum tangentially touches the cycloids, thus reducing the free length of the thread.



a) b)

Fig. 4 a) The tautochrony property of the cycloid for a down-gliding mass due to gravity: the times T_{ABo} and $T_{C'Bo}$ are equal; b) The evolute of the cycloid is a congruent cycloid

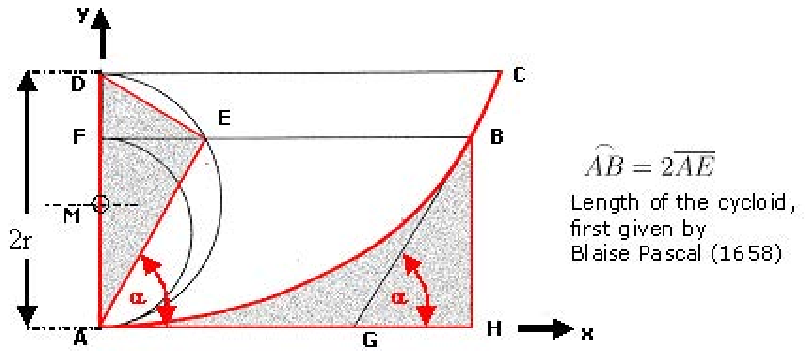


Fig. 5 Proof of the tautochrony of the cycloid using equal area segments

The proof can be found in [20]; it uses the equality of two area segments, Fig. 5,

$$\mathscr{A}_{ADE} = \mathscr{A}_{ABH} \text{ due to } \tan\alpha = HB/GH = AF/FE. \tag{6}$$

4.3 Analysis of the Common Cycloid

For better understanding of the following sections it is helpful to treat first the analysis of the cycloid in a condensed format, Fig. 5 and 6.

The first calculation of the area and the arc length were published by Cavalieri (1629).

Parametric representation:

coordinates — normal and curvature radius — 1^{st} derivative

$$\begin{aligned} x &= r(\alpha - \sin\alpha) \\ y &= r(1-\cos\alpha) \end{aligned} \quad ; \quad \begin{aligned} \overline{SK} &= \quad n \\ \overline{SM} &= \rho = 2n \end{aligned} \quad ; \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \cot\frac{\alpha}{2} \tag{7}$$

Ordinary differential equation of the cycloid, Leibniz (1686):

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \sqrt{\frac{x}{c-x}}\,;\; c = 2r;\; n = 2r\sin\frac{\alpha}{2}\,;\; \rho = 2n = 4r\sin\frac{\alpha}{2}\,;\; \widehat{\mathrm{OS}} = 4r \tag{8}$$

Area of the cycloid, Pascal (1659):

$$2\cdot\mathscr{A}\,OSR = \int\limits_{x=0}^{x=2\pi r} y\,\mathrm{d}x = \int\limits_{0}^{x=2\pi} y\dot{x}\,\mathrm{d}t = r^2\int\limits_{0}^{2\pi}(1-\cos t)^2\,\mathrm{d}t = 3\pi r^2 \tag{9}$$

Scaled gliding time:

$$T_{OS} = \frac{1}{\sqrt{2g}} = \int\limits_{\alpha_O=0}^{\alpha_S}\left(\frac{\sin^2\alpha + 1 - 2\cos\alpha + \cos^2\alpha}{r(1-\cos\alpha)\sin^2\alpha}\right)^{\frac{1}{2}} r\sin\alpha\,\mathrm{d}\alpha = \sqrt{\frac{r}{g}}\pi \tag{10}$$

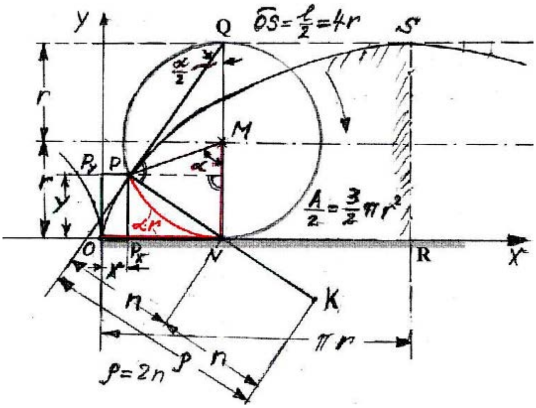


Fig. 6 Cycloid as the rolling or wheel curve on a plane

4.4 *Today's Variational Formulation of the Brachistochrone Problem*

Due to the singularity at $y = 0$ for $t = 0$, the representation $x = f(y)$ is adequate. The stationarity condition for the wanted extremal function reads, Fig. 7,

$$T_{AB} = \int_{t_A=0}^{t_B} \mathrm{d}t(y) = \min, \tag{11}$$

with the velocity, Galilei (1638),

$$v(y) = \sqrt{2gy}. \tag{12}$$

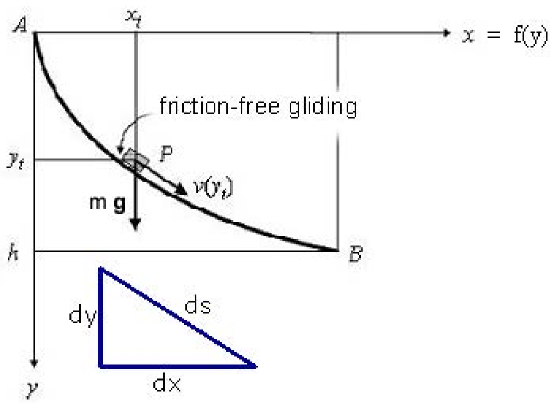


Fig. 7 The brachistochrone problem

With the relations

$$v(y) = \frac{\mathrm{d}s(y)}{\mathrm{d}t}; \quad \mathrm{d}t = \frac{\mathrm{d}s(y)}{v(y)}; \quad \mathrm{d}s(y) = \sqrt{1+(x'(y))^2}\mathrm{d}y \tag{13}$$

the extremal principle for the non-linear implicit time functional reads

$$T_{AB} = \int_{t_A=0}^{t_B} \frac{\sqrt{1+(x'(y))^2}}{\sqrt{2gy}}\,\mathrm{d}y \quad \to \ \min. \tag{14}$$

Its 1st variation has to be zero according to the stationarity condition

$$\delta T = \frac{1}{\sqrt{2g}} \int_{y_0}^{y_1} \frac{x'}{\sqrt{y(1+(x')^2)}}\,\delta x' \mathrm{d}y \overset{!}{=} 0 \tag{15}$$

and yields the ODE of the cycloid. The 2nd variation, the Hesse matrix (analytical tangent), is the basis for a finite element method by using, e.g., linear trial and test functions for the vertical convective coordinate $\eta = y/h$.

$$\delta^2 T = \frac{1}{\sqrt{2g}} \int_{y_0}^{y_1} \delta\bar{x}' \left[\frac{1}{\sqrt{y(1+(x')^2)}} - \frac{(x')^2}{\sqrt{y(1+(x')^2)^3}} \right] \delta x' \mathrm{d}y \tag{16}$$

The positive definit Hesse matrix assigns a minimum of the functional T_{AB}.

It should be remarked that a change of the independent variable, i.e. $y = \tilde{f}(x)$ instead of $x = f(y)$, leads to a variational problem with a more complicated differential equation because of the singularity at the origin.

After these pre-informations from the point of view of the variational calculus of today we turn back to the first solutions in the late 17th century.

4.5 Solutions of the Brachistochrone Problem After the Call of Johann Bernoulli in Acta Eruditorum in 1696

Johann Bernoulli introduced the denotation "brachistochrone" (Greek, means curve of shortest time) for Galilei's problem and called for solutions in one years time in Acta Eruditorum 1696. A total of seven solutions was submitted and published in May 1697 in Acta Eruditorum, [14]; Johann and Jacob Bernoulli had conceived that after the development of infinitesimal calculus this problem required a new branch of analysis. The following 7 solutions were published in [14] and [16], see also Funk (1970), [21], Stein, Wiechmann (2003), [22]:

- by Jacob Bernoulli: the mathematically most important one with the first development of variational calculus;
- two by Johann Bernoulli himself: using ingenious geometrical and analytical insight;

- two by Leibniz: the concept for an approximated discrete solution and a geometric integration of the ODE of the cycloid;
- by Newton anonymously, without proof, provided only in 1724;
- by L'Hôpital and Tschirnhaus: analytical ansatz with incomplete proof.

4.6 Jacob Bernoulli's Ingenious Derivation of the ODE for the Cycloid through a Variational Problem

In anticipation of Euler's idea of piecewise discrete triangular test functions for the derivation of the 1st variation of a functional from 1743, Jacob Bernoulli introduced the same idea for the first time in 1796/97. Starting from the known condition for a minimum or maximum value of a function $y = f(x)$ at a certain point, i.e. $\mathrm{d}y/\mathrm{d}x = 0$, he first follows this idea for the extremum of the functional $T_{AB} = \int F[y, y']\mathrm{d}x = \min$ by dividing the time domain, parametrized by the coordinates $y_A = 0$ and $y_B = 2r$, into a set of equidistant support points $y_i - h$, y_i, $y_i + h$ and choosing triangular test functions for the wanted extremal function, Fig. 8.

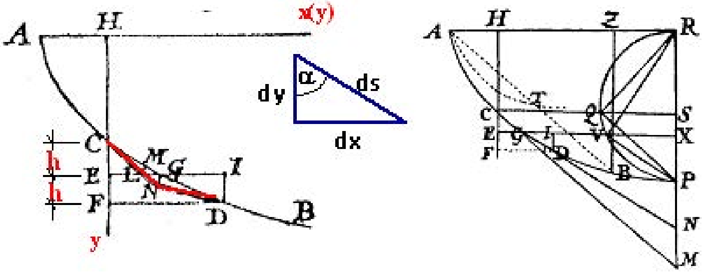


Fig. 8 Jacob Bernoulli's original figures for the variational derivation of the cycloid, published in Acta Eruditorum, May 1697

The obvious discrete stationarity condition for the extremal time consists in the equality of gliding times for the searched extremal curve and the neighboured test curve for all intervals as

$$t_{CG} + t_{GD} \stackrel{!}{=} t_{CL} + t_{LD}, \tag{17}$$

yielding the differential equation of the cycloid for the limit case $h \to 0$ as

$$\frac{\mathrm{d}s}{\mathrm{d}x} \sim \frac{k}{\sqrt{y}}; \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \tan\alpha = \sqrt{\frac{y}{k^2 - y}}; \quad k^2 = 2r, \tag{18}$$

where r is the radius of the rolling wheel.

In Euler's derivation of the differential equation for the extremal function of an isoparametric variational problem from 1744, [17], he used the same discrete method with equidistant support points for the minimum problem

$$\int_{x=A}^{Z} F[y(x), y'(x), x]\mathrm{d}x = \min, \tag{19}$$

with the required extremal function $y = f(x)$. Euler got the following wellknown ODE for the general problem

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'} = 0 \quad \text{or} \quad F_x - \frac{\mathrm{d}}{\mathrm{d}y}F_{y'} = 0 \text{ for } x = f(y), \tag{20}$$

whereas Jacob Bernoulli directly derived the differential equation of the cycloid as the extremal function of the brachistochrone problem.

4.7 Johann Bernoulli's Two Solutions

Johann Bernoulli presented two very tricky solutions, combining geometrical and analytical experience and deep knowledge of function theory. They are both treated in [14], see also [20], [21]. The first solution is got for the substitute problem of the lightway in shortest time through a medium with linearly changing density, Fig. 1b and equ. (3). From the related stationarity condition

$$\frac{\sin\varphi_i(x_i)}{v_i(x_i)} = \text{const.}; \quad i = 1, 2, \ldots, (n+1), \tag{21}$$

where α (in equ. (3)) is replaced by φ and v is the velocity, two coordinate substitutions yield directly the coordinates of the cycloid, equ. (7), in implicit form.

The second solution treats the brachistochrone problem, postulating the stationarity condition as follows: the gliding time for a path increment of the stationary solution must be equal to an infinitesimally varied path increment, Fig. 9, which in principle is the same condition as Jacob Bernoulli's criterion.

The stationarity condition reads

$$\begin{aligned} \Delta t' - \Delta t &= \frac{1}{\sqrt{2g}}\left(\frac{\Delta s(y+\Delta y)}{\sqrt{y+\Delta y}} - \frac{\Delta s(y)}{\sqrt{y}}\right) \overset{!}{=} 0 \\ &= \frac{1}{\sqrt{2g}}\left(\frac{(\rho+\Delta\rho)\Delta\varphi}{\sqrt{y+\Delta y}} - \frac{\rho\Delta\varphi}{\sqrt{y}}\right); \quad \Delta y = \Delta\rho\cdot\sin\varphi, \end{aligned} \tag{22}$$

with
$$\frac{(\rho+\Delta\rho)\Delta\varphi}{\sqrt{y+\Delta y}} = \frac{\rho+\Delta\rho}{\sqrt{y}}\left(1 - \frac{1}{2}\frac{\Delta y}{y} + \ldots\right)\Delta\varphi \tag{23}$$

$$\Delta t' - \Delta t = \frac{1}{\sqrt{2gy}}\underbrace{\left(1 - \frac{\rho\cdot\sin\varphi}{2y}\right)}_{\overset{!}{=}0}\Delta\rho\Delta\varphi \overset{!}{=} 0. \tag{24}$$

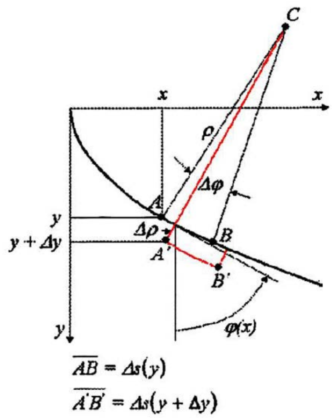


Fig. 9 Johann Bernoulli's 2nd solution of the brachistochrone problem, postulating equal incremental time steps for the stationary solution and a neighboured test function

This yields

$$\rho \stackrel{!}{=} \frac{2y}{\sin\varphi} \quad \text{and} \quad n = \frac{y}{\sin\varphi} = \frac{\rho}{2} \text{ with } \varphi = \frac{\alpha}{2}, \tag{25}$$

which is a special property of the cycloid. Thus the wanted solution was determined.

4.8 Leibniz' Draft of a Discrete Approximation Called "Tachystoptota"

Leibniz contributed two very different solutions to the brachistochrone problem in [14]. The first one is a discrete geometrical integration of the ODE of the cycloid by means of the so-called "quadratrix", [10], based on his transmutation theorem.

The second one treated here in more detail is a short draft of a direct numerical method which is of considerable interest as a predecessor of a finite element method, Fig. 10.

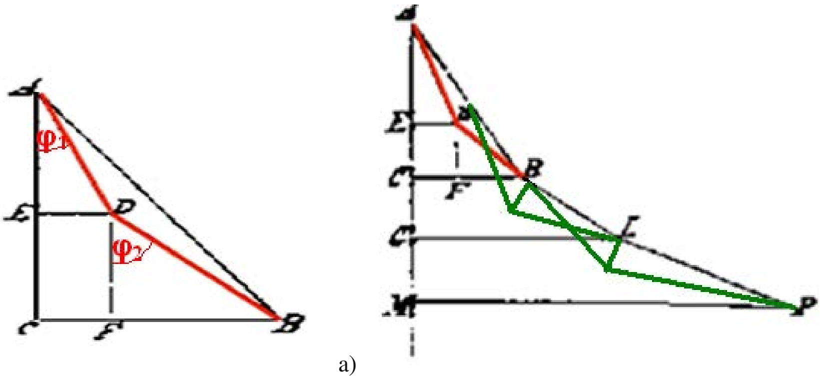
Leibniz did not present an algorithm for the discrete problem in Fig. 10b, although he was asked for this by Johann Bernoulli in a letter.

Using only two equidistant "finite elements", Fig. 11, the discrete solution is shown by the author in the sequal.

From the continuous minimum problem

$$T_{AB} = \int_A^B \mathrm{d}t = \int_{y_A}^{y_B} \frac{\mathrm{d}s(y)}{v(y)} = \int_{y_A}^{y_B} \frac{\sqrt{1+(x'(y))^2}}{\sqrt{2gy}}\mathrm{d}y = \min_{x(y)} \tag{26}$$

the discrete approximation with one discrete unknown $x_1(y_1)$ reads



a) b)

Fig. 10 a) Discrete stationarity condition: the gliding times t_{AB} and t_{ADB} must be equal, similar to Jacob Bernoullis derivation with discrete supports b) Leibniz' draft of a discrete variational method with equidistant support points E, C, C' and triangular (local) test and trial functions between these points

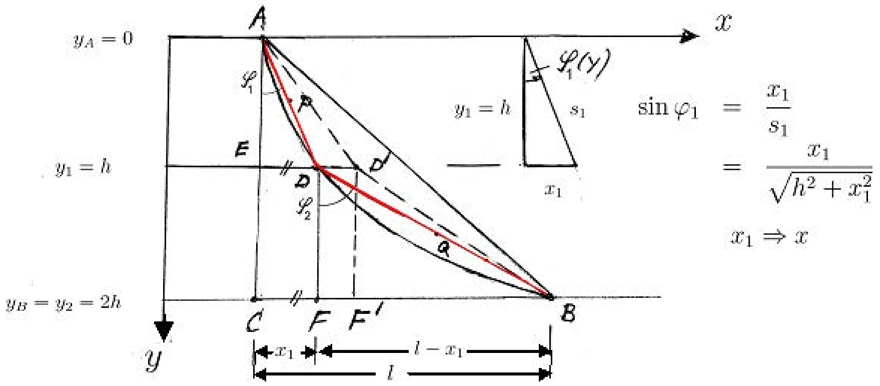


Fig. 11 Discrete variational algorithm for the brachystochrone problem with one unknown nodal position x_1, i.e. two equidistant "elements" with length h

$$T_{AB} = \frac{\overline{AD}}{v_{AD}} + \frac{\overline{DB}}{v_{DB}} = \frac{s_1}{v(h)} + \frac{s_2}{v(2h)} = \frac{\sqrt{h^2+x^2}}{\sqrt{2gh}} + \frac{\sqrt{h^2(l-x)^2}}{\sqrt{2g\cdot 2h}} = \min_{x_h(y)}. \tag{27}$$

The discrete stationarity condition, not given by Leibniz, is

$$\frac{\partial T_{AB}}{\partial x} \overset{!}{=} 0 \rightsquigarrow \frac{x}{\sqrt{h^2+x^2}} \cdot \frac{1}{\sqrt{2gh}} - \frac{l-x}{\sqrt{h^2+(l-x)^2}} \cdot \frac{1}{\sqrt{2g2h}} \overset{!}{=} 0 \tag{28}$$

$$\rightsquigarrow \frac{\sin\varphi_1}{v_1} - \frac{\sin\varphi_2}{v_2} \overset{!}{=} 0, \tag{29}$$

i.e. the same condition as for the optical path length in least time.

This condition, equ. (28), yields the 4th order polynomial

$$f(x) = x^4 - 2lx^3 + x^2(l^2 + h^2) + 2h^2lx - h^2l^2 \stackrel{!}{=} 0 \tag{30}$$

or with

$$\xi := \frac{x}{l}\,;\ \left(\frac{h}{l}\right)^2 = \frac{1}{4} \quad \rightsquigarrow \quad f(\xi) = \xi^4 - 2\xi^3 + \frac{5}{4}\xi^2 + \frac{1}{8}\xi - \frac{1}{4} \stackrel{!}{=} 0. \tag{31}$$

The linear approximation is $x_{1,lin} = 0,5l$, and the exact solution results in

$$x \to x_1 = 0,69l \tag{32}$$

as the first approximation with only two elements.

4.9 Discrete Variational Formulations by Schellbach

In 1851, Schellbach presented 12 discrete solutions of variational problems in the sense of Leibniz' draft for the brachistochrone problem with various boundary conditions and for related problems in analytical form, also using equidistant support points, Schallbach (1851), [23], Fig.12.

In this paper, entitled "Probleme der Variationsrechnung", Schellbach points out in the introduction: "The reasons for Bernoulli's, Euler's, and Lagrange's methods [for the variational calculus] cannot be clearly understood yet", and: "the variational calculus is the most abstract and most sublime area of all mathematics".

This gives the information that variational calculus and moreover discrete variational calculus was not yet well-known in the mathematical community in the middle of the 19th century.

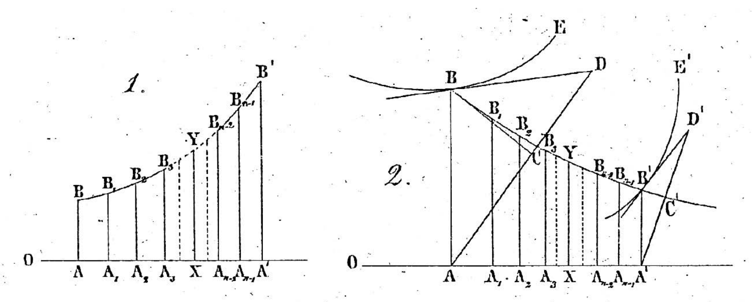


Fig. 12.1 & 12.2 Figures from K. H. Schellbach's discrete formulation and variational setting of the brachystochrone problem with coupled algebraic equations at equidistant points $A, A_1, A_2, \ldots, A'$ in analytical form

In this paper the following discrete problems are treated:

1. Minimum area of a polygon with fixed ends, given length, and extensions (Fig. 12.1)
2. Minimum area of a rotational surface with given meridian arc length and boundary conditions, with extended versions ((Fig. 12.1)
3. Brachistochrone problem with generalized boundary conditions in B and B (Fig. 12.2)
4. Brachistochrone problem in a resisting medium (Fig. 12.2)
5. Problem similar to 3. & 4., but with the condition of largest or smallest final velocity in B' (Fig. 12.2)
6. to 12. Further problems of this type

The numerical calculation of Schellbach's equations yields systems of non-linear algebraic equations for the treated problems. This can be conceived as a first special analytical version of the finite element method.

4.10 Finite Element Method for the Brachistochrone Problem in Today's Fashion

The variational problem of the brachistochrone reads, using the adequate coordinate representation $y = f(x)$, y the vertical coordinate in order to avoid a singularity at the starting point A (Fig. 13)

$$T = \frac{1}{\sqrt{2g}} \int_{y_0}^{y_1} \sqrt{\frac{1+(x')^2}{y}} \mathrm{d}y \to \min . \tag{33}$$

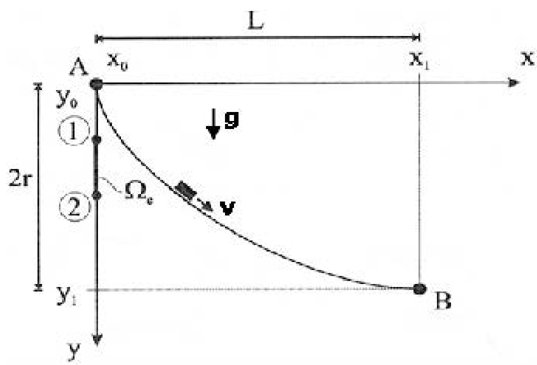


Fig. 13 Finite element analysis of the brachistochrone problem

The first variation (the stationarity condition) of the time functional follows as

$$\delta T = \frac{1}{\sqrt{2g}} \int_{y_0}^{y_1} \frac{x'}{\sqrt{y(1+(x')^2)}} \delta x' \mathrm{d}y \overset{!}{=} 0; \quad x' = \frac{\mathrm{d}x(y)}{\mathrm{d}y} \tag{34}$$

and the second variation (Hesse matrix, analytical tangent) reads

$$\delta^2 T = \frac{1}{\sqrt{2g}} \int_{y_0}^{y_1} \delta \bar{x}' \left[\frac{1}{\sqrt{y(1+(x')^2)}} - \frac{(x')^2}{\sqrt{y(1+(x')^2)^3}} \right] \delta x' \mathrm{d}y. \tag{35}$$

The discretization with linear finite elements for the dimensionless vertical coordinate $\eta = y/\ell_e$ as parametrized time variable realizes the original ideas of Jacob Bernoulli and Leibniz, Fig. 14.

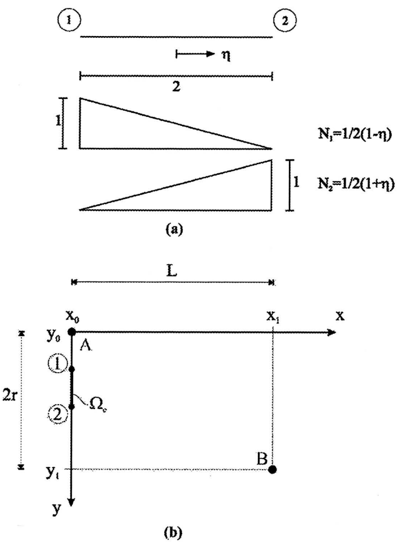


Fig. 14 Linear finite element test and trial shape functions for the discretization of the brachistochrone curve

The linear finite element Ansatz for the horizontal time-dependent coordinate $x(\eta)$, $\eta = f(t)$, reads, Fig. 14,

$$x_h = \sum_{I=1}^{2} N_I(\eta) x_I \quad \forall\, \Omega_e \subset \Omega \tag{36}$$

with the shape functions