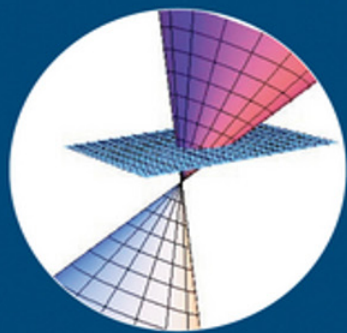
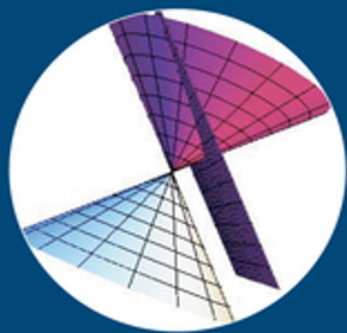
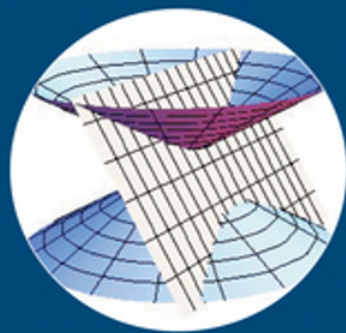


THIRD EDITION

# THE HISTORY OF MATHEMATICS

A BRIEF COURSE



ROGER L. COOKE

 WILEY



# **THE HISTORY OF MATHEMATICS**



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# THE HISTORY OF MATHEMATICS A BRIEF COURSE

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THIRD EDITION

Roger L. Cooke

Department of Mathematics and Statistics  
University of Vermont  
Burlington, VT



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## CONTENTS

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<b>PREFACE</b>	<b>xxiii</b>
Changes from the Second Edition	xxiii
Elementary Texts on the History of Mathematics	xxiv
<b>PART I. WHAT IS MATHEMATICS?</b>	
Contents of Part I	1
<b>1. Mathematics and its History</b>	<b>3</b>
1.1. Two Ways to Look at the History of Mathematics	3
1.1.1. History, but not Heritage	4
1.1.2. Our Mathematical Heritage	4
1.2. The Origin of Mathematics	5
1.2.1. Number	5
1.2.2. Space	5
1.2.3. Are Mathematical Ideas Innate?	7
1.2.4. Symbolic Notation	7
1.2.5. Logical Relations	7
1.2.6. The Components of Mathematics	8
1.3. The Philosophy of Mathematics	8
1.3.1. Mathematical Analysis of a Real-World Problem	9
1.4. Our Approach to the History of Mathematics	11
Questions for Reflection	12
<b>2. Proto-mathematics</b>	<b>14</b>
2.1. Number	14
2.1.1. Animals' Use of Numbers	14
2.1.2. Young Children's Use of Numbers	15
2.1.3. Archaeological Evidence of Counting	15
2.2. Shape	16
2.2.1. Perception of Shape by Animals	16
2.2.2. Children's Concepts of Space	16
2.2.3. Geometry in Arts and Crafts	17
2.3. Symbols	18

2.4.	Mathematical Reasoning	20
2.4.1.	Animal Reasoning	20
2.4.2.	Visual Reasoning	21
	Problems and Questions	22
	Mathematical Problems	22
	Questions for Reflection	24
<b>PART II. THE MIDDLE EAST, 2000–1500 BCE</b>		
	Contents of Part II	25
<b>3.</b>	<b>Overview of Mesopotamian Mathematics</b>	<b>27</b>
3.1.	A Sketch of Two Millennia of Mesopotamian History	27
3.2.	Mathematical Cuneiform Tablets	29
3.3.	Systems of Measuring and Counting	30
3.3.1.	Counting	31
3.4.	The Mesopotamian Numbering System	31
3.4.1.	Place-Value Systems	32
3.4.2.	The Sexagesimal Place-Value System	33
3.4.3.	Converting a Decimal Number to Sexagesimal	33
3.4.4.	Irrational Square Roots	36
	Problems and Questions	36
	Mathematical Problems	36
	Historical Questions	36
	Questions for Reflection	37
<b>4.</b>	<b>Computations in Ancient Mesopotamia</b>	<b>38</b>
4.1.	Arithmetic	38
4.1.1.	Square Roots	39
4.2.	Algebra	40
4.2.1.	Linear and Quadratic Problems	41
4.2.2.	Higher-Degree Problems	43
	Problems and Questions	44
	Mathematical Problems	44
	Historical Questions	44
	Questions for Reflection	44
<b>5.</b>	<b>Geometry in Mesopotamia</b>	<b>46</b>
5.1.	The Pythagorean Theorem	46
5.2.	Plane Figures	48
5.2.1.	Mesopotamian Astronomy	48
5.3.	Volumes	49
5.4.	Plimpton 322	49
5.4.1.	The Purpose of Plimpton 322: Some Conjectures	53



Problems and Questions	54
Mathematical Problems	54
Historical Questions	55
Questions for Reflection	55
<b>6. Egyptian Numerals and Arithmetic</b>	<b>56</b>
6.1. Sources	56
6.1.1. Mathematics in Hieroglyphics and Hieratic	57
6.2. The Rhind Papyrus	58
6.3. Egyptian Arithmetic	58
6.4. Computation	59
6.4.1. Multiplication and Division	61
6.4.2. “Parts”	62
Problems and Questions	65
Mathematical Problems	65
Historical Questions	65
Questions for Reflection	65
<b>7. Algebra and Geometry in Ancient Egypt</b>	<b>66</b>
7.1. Algebra Problems in the Rhind Papyrus	66
7.1.1. Applied Problems: The <i>Pesu</i>	67
7.2. Geometry	68
7.3. Areas	69
7.3.1. Rectangles, Triangles, and Trapezoids	69
7.3.2. Slopes	69
7.3.3. Circles	70
7.3.4. The Pythagorean Theorem	71
7.3.5. Spheres or Cylinders?	72
7.3.6. Volumes	73
Problems and Questions	76
Mathematical Problems	76
Historical Questions	76
Questions for Reflection	76
<b>PART III. GREEK MATHEMATICS FROM 500 BCE TO 500 CE</b>	
Contents of Part III	77
<b>8. An Overview of Ancient Greek Mathematics</b>	<b>79</b>
8.1. Sources	80
8.1.1. Loss and Recovery	81
8.2. General Features of Greek Mathematics	82
8.2.1. Pythagoras	83
8.2.2. Mathematical Aspects of Plato’s Philosophy	85

8.3.	Works and Authors	87
8.3.1.	Euclid	87
8.3.2.	Archimedes	87
8.3.3.	Apollonius	88
8.3.4.	Zenodorus	88
8.3.5.	Heron	88
8.3.6.	Ptolemy	89
8.3.7.	Diophantus	89
8.3.8.	Pappus	89
8.3.9.	Theon and Hypatia	89
	Questions	90
	Historical Questions	90
	Questions for Reflection	90
<b>9.</b>	<b>Greek Number Theory</b>	<b>91</b>
9.1.	The Euclidean Algorithm	92
9.2.	The <i>Arithmetica</i> of Nicomachus	93
9.2.1.	Factors vs. Parts. Perfect Numbers	94
9.2.2.	Figurate Numbers	95
9.3.	Euclid's Number Theory	97
9.4.	The <i>Arithmetica</i> of Diophantus	97
9.4.1.	Algebraic Symbolism	98
9.4.2.	Contents of the <i>Arithmetica</i>	99
9.4.3.	Fermat's Last Theorem	100
	Problems and Questions	101
	Mathematical Problems	101
	Historical Questions	102
	Questions for Reflection	102
<b>10.</b>	<b>Fifth-Century Greek Geometry</b>	<b>103</b>
10.1.	"Pythagorean" Geometry	103
10.1.1.	Transformation and Application of Areas	103
10.2.	Challenge No. 1: Unsolved Problems	106
10.3.	Challenge No. 2: The Paradoxes of Zeno of Elea	107
10.4.	Challenge No. 3: Irrational Numbers and Incommensurable Lines	108
10.4.1.	The Arithmetical Origin of Irrationals	110
10.4.2.	The Geometric Origin of Irrationals	110
10.4.3.	Consequences of the Discovery	111
	Problems and Questions	113
	Mathematical Problems	113
	Historical Questions	113
	Questions for Reflection	114

<b>11. Athenian Mathematics I: The Classical Problems</b>	<b>115</b>
11.1. Squaring the Circle	116
11.2. Doubling the Cube	117
11.3. Trisecting the Angle	122
11.3.1. A Mechanical Solution: The Conchoid	125
Problems and Questions	126
Mathematical Problems	126
Historical Questions	126
Questions for Reflection	127
<b>12. Athenian Mathematics II: Plato and Aristotle</b>	<b>128</b>
12.1. The Influence of Plato	128
12.2. Eudoxan Geometry	130
12.2.1. The Eudoxan Definition of Proportion	130
12.2.2. The Method of Exhaustion	131
12.2.3. Ratios in Greek Geometry	133
12.3. Aristotle	134
Problems and Questions	138
Mathematical Problems	138
Historical Questions	138
Questions for Reflection	139
<b>13. Euclid of Alexandria</b>	<b>140</b>
13.1. The <i>Elements</i>	140
13.1.1. Book 1	141
13.1.2. Book 2	141
13.1.3. Books 3 and 4	143
13.1.4. Books 5 and 6	143
13.1.5. Books 7–9	143
13.1.6. Book 10	144
13.1.7. Books 11–13	144
13.2. The <i>Data</i>	144
Problems and Questions	145
Mathematical Problems	145
Historical Questions	147
Questions for Reflection	147
<b>14. Archimedes of Syracuse</b>	<b>148</b>
14.1. The Works of Archimedes	149
14.2. The Surface of a Sphere	150
14.3. The Archimedes Palimpsest	153
14.3.1. The <i>Method</i>	154

14.4. Quadrature of the Parabola	155
14.4.1. The Mechanical Quadrature	155
14.4.2. The Rigorous Quadrature	156
Problems and Questions	158
Mathematical Problems	158
Historical Questions	158
Questions for Reflection	159
<b>15. Apollonius of Perga</b>	<b>160</b>
15.1. History of the <i>Conics</i>	161
15.2. Contents of the <i>Conics</i>	162
15.2.1. Properties of the Conic Sections	165
15.3. Foci and the Three- and Four-Line Locus	165
Problems and Questions	166
Mathematical Problems	166
Historical Questions	168
Questions for Reflection	168
<b>16. Hellenistic and Roman Geometry</b>	<b>169</b>
16.1. Zenodorus	169
16.2. The Parallel Postulate	171
16.3. Heron	172
16.4. Roman Civil Engineering	174
Problems and Questions	176
Mathematical Problems	176
Historical Questions	176
Questions for Reflection	176
<b>17. Ptolemy's Geography and Astronomy</b>	<b>177</b>
17.1. Geography	177
17.2. Astronomy	180
17.2.1. Epicycles and Eccentrics	181
17.2.2. The Motion of the Sun	182
17.3. The <i>Almagest</i>	184
17.3.1. Trigonometry	184
17.3.2. Ptolemy's Table of Chords	184
Problems and Questions	187
Mathematical Problems	187
Historical Questions	188
Questions for Reflection	188
<b>18. Pappus and the Later Commentators</b>	<b>190</b>
18.1. The <i>Collection</i> of Pappus	190
18.1.1. Generalization of the Pythagorean Theorem	191

18.1.2. The Isoperimetric Problem	191
18.1.3. Analysis, Locus Problems, and Pappus' Theorem	191
18.2. The Later Commentators: Theon and Hypatia	196
18.2.1. Theon of Alexandria	196
18.2.2. Hypatia of Alexandria	197
Problems and Questions	198
Mathematical Problems	198
Historical Questions	199
Questions for Reflection	199
<b>PART IV. INDIA, CHINA, AND JAPAN 500 BCE–1700 CE</b>	
Contents of Part IV	201
<b>19. Overview of Mathematics in India</b>	<b>203</b>
19.1. The <i>Sulva Sutras</i>	205
19.2. Buddhist and Jain Mathematics	206
19.3. The Bakshali Manuscript	206
19.4. The <i>Siddhantas</i>	206
19.5. Hindu–Arabic Numerals	206
19.6. Aryabhata I	207
19.7. Brahmagupta	208
19.8. Bhaskara II	209
19.9. Muslim India	210
19.10. Indian Mathematics in the Colonial Period and After	210
19.10.1. Srinivasa Ramanujan	210
Questions	211
Historical Questions	211
Questions for Reflection	211
<b>20. From the <i>Vedas</i> to Aryabhata I</b>	<b>213</b>
20.1. Problems from the <i>Sulva Sutras</i>	213
20.1.1. Arithmetic	213
20.1.2. Geometry	214
20.1.3. Square Roots	216
20.1.4. Jain Mathematics: The Infinite	217
20.1.5. Jain Mathematics: Combinatorics	217
20.1.6. The Bakshali Manuscript	218
20.2. Aryabhata I: Geometry and Trigonometry	219
20.2.1. Trigonometry	220
20.2.2. The <i>Kuttaka</i>	224
Problems and Questions	225
Mathematical Problems	225
Historical Questions	225
Questions for Reflection	225

<b>21. Brahmagupta, the <i>Kuttaka</i>, and Bhaskara II</b>	<b>227</b>
21.1. Brahmagupta's Plane and Solid Geometry	227
21.2. Brahmagupta's Number Theory and Algebra	228
21.2.1. Pythagorean Triples	229
21.2.2. Pell's Equation	229
21.3. The <i>Kuttaka</i>	230
21.4. Algebra in the Works of Bhaskara II	233
21.4.1. The <i>Vija Ganita (Algebra)</i>	233
21.4.2. Combinatorics	233
21.5. Geometry in the Works of Bhaskara II	235
Problems and Questions	237
Mathematical Problems	237
Historical Questions	238
Questions for Reflection	238
<b>22. Early Classics of Chinese Mathematics</b>	<b>239</b>
22.1. Works and Authors	240
22.1.1. The <i>Zhou Bi Suan Jing</i>	241
22.1.2. The <i>Jiu Zhang Suan Shu</i>	242
22.1.3. The <i>Sun Zi Suan Jing</i>	242
22.1.4. Liu Hui. The <i>Hai Dao Suan Jing</i>	242
22.1.5. Zu Chongzhi and Zu Geng	243
22.1.6. Yang Hui	243
22.1.7. Cheng Dawei	243
22.2. China's Encounter with Western Mathematics	243
22.3. The Chinese Number System	244
22.3.1. Fractions and Roots	245
22.4. Algebra	246
22.5. Contents of the <i>Jiu Zhang Suan Shu</i>	247
22.6. Early Chinese Geometry	249
22.6.1. The <i>Zhou Bi Suan Jing</i>	249
22.6.2. The <i>Jiu Zhang Suan Shu</i>	251
22.6.3. The <i>Sun Zi Suan Jing</i>	253
Problems and Questions	253
Mathematical Problems	253
Historical Questions	253
Questions for Reflection	253
<b>23. Later Chinese Algebra and Geometry</b>	<b>255</b>
23.1. Algebra	255
23.1.1. Systems of Linear Equations	256
23.1.2. Quadratic Equations	256
23.1.3. Cubic Equations	257
23.1.4. A Digression on the Numerical Solution of Equations	258

23.2. Later Chinese Geometry	262
23.2.1. Liu Hui	262
23.2.2. Zu Chongzhi	264
Problems and Questions	265
Mathematical Problems	265
Historical Questions	266
Questions for Reflection	266
<b>24. Traditional Japanese Mathematics</b>	<b>267</b>
24.1. Chinese Influence and Calculating Devices	267
24.2. Japanese Mathematicians and Their Works	268
24.2.1. Yoshida Koyu	269
24.2.2. Seki Kōwa and Takebe Kenkō	269
24.2.3. The Modern Era in Japan	270
24.3. Japanese Geometry and Algebra	270
24.3.1. Determinants	272
24.3.2. The Challenge Problems	273
24.3.3. Beginnings of the Calculus in Japan	274
24.4. <i>Sangaku</i>	277
24.4.1. Analysis	279
Problems and Questions	279
Mathematical Problems	279
Historical Questions	280
Questions for Reflection	280
<b>PART V. ISLAMIC MATHEMATICS, 800–1500</b>	
Contents of Part V	281
<b>25. Overview of Islamic Mathematics</b>	<b>283</b>
25.1. A Brief Sketch of the Islamic Civilization	283
25.1.1. The Umayyads	283
25.1.2. The Abbasids	284
25.1.3. The Turkish and Mongol Conquests	284
25.1.4. The Islamic Influence on Science	284
25.2. Islamic Science in General	285
25.2.1. Hindu and Hellenistic Influences	285
25.3. Some Muslim Mathematicians and Their Works	287
25.3.1. Muhammad ibn Musa al-Khwarizmi	287
25.3.2. Thabit ibn-Qurra	287
25.3.3. Abu Kamil	288
25.3.4. Al-Battani	288
25.3.5. Abu'l Wafa	288
25.3.6. Ibn al-Haytham	288
25.3.7. Al-Biruni	289

25.3.8. Omar Khayyam	289
25.3.9. Sharaf al-Tusi	289
25.3.10. Nasir al-Tusi	289
Questions	290
Historical Questions	290
Questions for Reflection	290
<b>26. Islamic Number Theory and Algebra</b>	<b>292</b>
26.1. Number Theory	292
26.2. Algebra	294
26.2.1. Al-Khwarizmi	295
26.2.2. Abu Kamil	297
26.2.3. Omar Khayyam	297
26.2.4. Sharaf al-Din al-Tusi	299
Problems and Questions	300
Mathematical Problems	300
Historical Questions	301
Questions for Reflection	301
<b>27. Islamic Geometry</b>	<b>302</b>
27.1. The Parallel Postulate	302
27.2. Thabit ibn-Qurra	302
27.3. Al-Biruni: Trigonometry	304
27.4. Al-Kuhi	305
27.5. Al-Haytham and Ibn-Sahl	305
27.6. Omar Khayyam	307
27.7. Nasir al-Din al-Tusi	308
Problems and Questions	309
Mathematical Problems	309
Historical Questions	309
Questions for Reflection	310
<b>PART VI. EUROPEAN MATHEMATICS, 500–1900</b>	
Contents of Part VI	311
<b>28. Medieval and Early Modern Europe</b>	<b>313</b>
28.1. From the Fall of Rome to the Year 1200	313
28.1.1. Boethius and the Quadrivium	313
28.1.2. Arithmetic and Geometry	314
28.1.3. Music and Astronomy	315
28.1.4. The Carolingian Empire	315



28.1.5. Gerbert	315
28.1.6. Early Medieval Geometry	317
28.1.7. The Translators	318
28.2. The High Middle Ages	318
28.2.1. Leonardo of Pisa	319
28.2.2. Jordanus Nemorarius	319
28.2.3. Nicole d'Oresme	319
28.2.4. Regiomontanus	320
28.2.5. Nicolas Chuquet	320
28.2.6. Luca Pacioli	320
28.2.7. Leon Battista Alberti	321
28.3. The Early Modern Period	321
28.3.1. Scipione del Ferro	321
28.3.2. Niccolò Tartaglia	321
28.3.3. Girolamo Cardano	321
28.3.4. Ludovico Ferrari	322
28.3.5. Rafael Bombelli	322
28.4. Northern European Advances	322
28.4.1. François Viète	322
28.4.2. John Napier	322
Questions	323
Historical Questions	323
Questions for Reflection	323
<b>29. European Mathematics: 1200–1500</b>	<b>324</b>
29.1. Leonardo of Pisa (Fibonacci)	324
29.1.1. The <i>Liber abaci</i>	324
29.1.2. The Fibonacci Sequence	325
29.1.3. The <i>Liber quadratorum</i>	326
29.1.4. The <i>Flos</i>	327
29.2. Hindu–Arabic Numerals	328
29.3. Jordanus Nemorarius	329
29.4. Nicole d'Oresme	330
29.5. Trigonometry: Regiomontanus and Pitiscus	331
29.5.1. Regiomontanus	331
29.5.2. Pitiscus	332
29.6. A Mathematical Skill: <i>Prosthaphæresis</i>	333
29.7. Algebra: Pacioli and Chuquet	335
29.7.1. Luca Pacioli	335
29.7.2. Chuquet	335
Problems and Questions	336
Mathematical Problems	336
Historical Questions	337
Questions for Reflection	337

<b>30. Sixteenth-Century Algebra</b>	<b>338</b>
30.1. Solution of Cubic and Quartic Equations	338
30.1.1. Ludovico Ferrari	339
30.2. Consolidation	340
30.2.1. François Viète	341
30.3. Logarithms	343
30.3.1. Arithmetical Implementation of the Geometric Model	344
30.4. Hardware: Slide Rules and Calculating Machines	345
30.4.1. The Slide Rule	345
30.4.2. Calculating Machines	345
Problems and Questions	346
Mathematical Problems	346
Historical Questions	346
Questions for Reflection	346
<b>31. Renaissance Art and Geometry</b>	<b>348</b>
31.1. The Greek Foundations	348
31.2. The Renaissance Artists and Geometers	349
31.3. Projective Properties	350
31.3.1. Girard Desargues	352
31.3.2. Blaise Pascal	355
Problems and Questions	356
Mathematical Problems	356
Historical Questions	357
Questions for Reflection	357
<b>32. The Calculus Before Newton and Leibniz</b>	<b>358</b>
32.1. Analytic Geometry	358
32.1.1. Pierre de Fermat	359
32.1.2. René Descartes	359
32.2. Components of the Calculus	363
32.2.1. Tangent and Maximum Problems	363
32.2.2. Lengths, Areas, and Volumes	365
32.2.3. Bonaventura Cavalieri	365
32.2.4. Gilles Personne de Roberval	366
32.2.5. Rectangular Approximations and the Method of Exhaustion	367
32.2.6. Blaise Pascal	368
32.2.7. The Relation Between Tangents and Areas	370
32.2.8. Infinite Series and Products	370
32.2.9. The Binomial Series	371
Problems and Questions	371
Mathematical Problems	371
Historical Questions	372
Questions for Reflection	372

<b>33. Newton and Leibniz</b>	<b>373</b>
33.1. Isaac Newton	373
33.1.1. Newton's First Version of the Calculus	373
33.1.2. Fluxions and Fluents	374
33.1.3. Later Exposition of the Calculus	374
33.1.4. Objections	375
33.2. Gottfried Wilhelm von Leibniz	375
33.2.1. Leibniz' Presentation of the Calculus	376
33.2.2. Later Reflections on the Calculus	378
33.3. The Disciples of Newton and Leibniz	379
33.4. Philosophical Issues	379
33.4.1. The Debate on the Continent	380
33.5. The Priority Dispute	381
33.6. Early Textbooks on Calculus	382
33.6.1. The State of the Calculus Around 1700	382
Problems and Questions	383
Mathematical Problems	383
Historical Questions	384
Questions for Reflection	384
<b>34. Consolidation of the Calculus</b>	<b>386</b>
34.1. Ordinary Differential Equations	387
34.1.1. A Digression on Time	389
34.2. Partial Differential Equations	390
34.3. Calculus of Variations	391
34.3.1. Euler	393
34.3.2. Lagrange	394
34.3.3. Second-Variation Tests for Maxima and Minima	394
34.3.4. Jacobi: Sufficiency Criteria	395
34.3.5. Weierstrass and his School	395
34.4. Foundations of the Calculus	397
34.4.1. Lagrange's Algebraic Analysis	398
34.4.2. Cauchy's Calculus	398
Problems and Questions	399
Mathematical Problems	399
Historical Questions	400
Questions for Reflection	400
<b>PART VII. SPECIAL TOPICS</b>	
Contents of Part VII	404
<b>35. Women Mathematicians</b>	<b>405</b>
35.1. Sof'ya Kovalevskaya	406
35.1.1. Resistance from Conservatives	408

35.2. Grace Chisholm Young	408
35.3. Emmy Noether	411
Questions	414
Historical Questions	414
Questions for Reflection	415
<b>36. Probability</b>	<b>417</b>
36.1. Cardano	418
36.2. Fermat and Pascal	419
36.3. Huygens	420
36.4. Leibniz	420
36.5. The <i>Ars Conjectandi</i> of James Bernoulli	421
36.5.1. The Law of Large Numbers	422
36.6. De Moivre	423
36.7. The Petersburg Paradox	424
36.8. Laplace	425
36.9. Legendre	426
36.10. Gauss	426
36.11. Philosophical Issues	427
36.12. Large Numbers and Limit Theorems	428
Problems and Questions	429
Mathematical Problems	429
Historical Questions	430
Questions for Reflection	431
<b>37. Algebra from 1600 to 1850</b>	<b>433</b>
37.1. Theory of Equations	433
37.1.1. Albert Girard	434
37.1.2. Tschirnhaus Transformations	434
37.1.3. Newton, Leibniz, and the Bernoullis	436
37.2. Euler, D'Alembert, and Lagrange	437
37.2.1. Euler	437
37.2.2. D'Alembert	438
37.2.3. Lagrange	438
37.3. The Fundamental Theorem of Algebra and Solution by Radicals	439
37.3.1. Ruffini	440
37.3.2. Cauchy	441
37.3.3. Abel	442
37.3.4. Galois	443
Problems and Questions	445
Mathematical Problems	445
Historical Questions	446
Questions for Reflection	446

<b>38. Projective and Algebraic Geometry and Topology</b>	<b>448</b>
38.1. Projective Geometry	448
38.1.1. Newton's Degree-Preserving Mapping	448
38.1.2. Brianchon	449
38.1.3. Monge and his School	450
38.1.4. Steiner	451
38.1.5. Möbius	452
38.2. Algebraic Geometry	453
38.2.1. Plücker	454
38.2.2. Cayley	455
38.3. Topology	456
38.3.1. Combinatorial Topology	456
38.3.2. Riemann	457
38.3.3. Möbius	458
38.3.4. Poincaré's <i>Analysis situs</i>	459
38.3.5. Point-Set Topology	461
Problems and Questions	462
Mathematical Problems	462
Historical Questions	463
Questions for Reflection	463
<b>39. Differential Geometry</b>	<b>464</b>
39.1. Plane Curves	464
39.1.1. Huygens	464
39.1.2. Newton	466
39.1.3. Leibniz	467
39.2. The Eighteenth Century: Surfaces	468
39.2.1. Euler	468
39.2.2. Lagrange	469
39.3. Space Curves: The French Geometers	469
39.4. Gauss: Geodesics and Developable Surfaces	469
39.4.1. Further Work by Gauss	472
39.5. The French and British Geometers	473
39.6. Grassmann and Riemann: Manifolds	473
39.6.1. Grassmann	474
39.6.2. Riemann	474
39.7. Differential Geometry and Physics	476
39.8. The Italian Geometers	477
39.8.1. Ricci's Absolute Differential Calculus	478
Problems and Questions	479
Mathematical Problems	479
Historical Questions	479
Questions for Reflection	479

<b>40. Non-Euclidean Geometry</b>	<b>481</b>
40.1. Saccheri	482
40.2. Lambert and Legendre	484
40.3. Gauss	485
40.4. The First Treatises	486
40.5. Lobachevskii's Geometry	487
40.6. János Bolyai	489
40.7. The Reception of Non-Euclidean Geometry	489
40.8. Foundations of Geometry	491
Problems and Questions	492
Mathematical Problems	492
Historical Questions	493
Questions for Reflection	493
<b>41. Complex Analysis</b>	<b>495</b>
41.1. Imaginary and Complex Numbers	495
41.1.1. Wallis	497
41.1.2. Wessel	498
41.1.3. Argand	499
41.2. Analytic Function Theory	500
41.2.1. Algebraic Integrals	500
41.2.2. Legendre, Jacobi, and Abel	502
41.2.3. Theta Functions	504
41.2.4. Cauchy	504
41.2.5. Riemann	506
41.2.6. Weierstrass	507
41.3. Comparison of the Three Approaches	508
Problems and Questions	508
Mathematical Problems	508
Historical Questions	509
Questions for Reflection	509
<b>42. Real Numbers, Series, and Integrals</b>	<b>511</b>
42.1. Fourier Series, Functions, and Integrals	512
42.1.1. The Definition of a Function	513
42.2. Fourier Series	514
42.2.1. Sturm–Liouville Problems	515
42.3. Fourier Integrals	516
42.4. General Trigonometric Series	518
Problems and Questions	519
Mathematical Problems	519
Historical Questions	519
Questions for Reflection	519

<b>43. Foundations of Real Analysis</b>	<b>521</b>
43.1. What is a Real Number?	521
43.1.1. The Arithmetization of the Real Numbers	523
43.2. Completeness of the Real Numbers	525
43.3. Uniform Convergence and Continuity	525
43.4. General Integrals and Discontinuous Functions	526
43.5. The Abstract and the Concrete	527
43.5.1. Absolute Continuity	528
43.5.2. Taming the Abstract	528
43.6. Discontinuity as a Positive Property	529
Problems and Questions	530
Mathematical Problems	530
Historical Questions	531
Questions for Reflection	531
<b>44. Set Theory</b>	<b>532</b>
44.1. Technical Background	532
44.2. Cantor's Work on Trigonometric Series	533
44.2.1. Ordinal Numbers	533
44.2.2. Cardinal Numbers	534
44.3. The Reception of Set Theory	536
44.3.1. Cantor and Kronecker	537
44.4. Existence and the Axiom of Choice	537
Problems and Questions	540
Mathematical Problems	540
Historical Questions	541
Questions for Reflection	541
<b>45. Logic</b>	<b>542</b>
45.1. From Algebra to Logic	542
45.2. Symbolic Calculus	545
45.3. Boole's <i>Mathematical Analysis of Logic</i>	546
45.3.1. Logic and Classes	546
45.4. Boole's <i>Laws of Thought</i>	547
45.5. Jevons	548
45.6. Philosophies of Mathematics	548
45.6.1. Paradoxes	549
45.6.2. Formalism	550
45.6.3. Intuitionism	551
45.6.4. Mathematical Practice	553

45.7. Doubts About Formalized Mathematics: Gödel's Theorems	554
Problems and Questions	555
Mathematical Problems	555
Historical Questions	555
Questions for Reflection	556
<b>Literature</b>	<b>559</b>
<b>Name Index</b>	<b>575</b>
<b>Subject Index</b>	<b>585</b>



## PREFACE

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Like its immediate predecessor, this third edition of *The History of Mathematics: A Brief Course* must begin with a few words of explanation to users of the earlier editions. The present volume, although it retains most of the material from the second edition, has been reorganized once again. In the first edition each chapter was devoted to a single culture or period within a single culture and subdivided by mathematical topics. In the second edition, after a general survey of mathematics and mathematical practice in Part I, the primary division was by subject matter: numbers, geometry, algebra, analysis, mathematical inference. After long consideration, I found this organization less desirable than a chronological ordering. As I said in the preface to the second edition,

For reasons that mathematics can illustrate very well, writing the history of mathematics is a nearly impossible task. To get a proper orientation for any particular event in mathematical history, it is necessary to take account of three independent “coordinates”: the time, the mathematical subject, and the culture. To thread a narrative that is to be read linearly through this three-dimensional array of events is like drawing one of Peano’s space-filling curves. Some points on the curve are infinitely distant from one another, and the curve must pass through some points many times. From the point of view of a reader whose time is valuable, these features constitute a glaring defect. The problem is an old one, well expressed eighty years ago by Felix Klein, in Chapter 6 of his *Lectures on the Development of Mathematics in the Nineteenth Century*:

I have now mentioned a large number of more or less famous names, all closely connected with Riemann. They can become more than a mere list only if we look into the literature associated with the names, or rather, with those who bear the names. One must learn how to grasp the main lines of the many connections in our science out of the enormous available mass of printed matter without getting lost in the time-consuming discussion of every detail, but also without falling into superficiality and dilettantism.

I have decided that in the lexicographic ordering of the three-dimensional coordinate system mentioned above, culture is the first coordinate, chronology the second, and mathematical content the third. That is the principle on which the first six parts of the present edition are organized. In the seventh and final part, which covers the period from 1800 on, the first coordinate becomes irrelevant, as mathematics acquires a worldwide scope. Because so much new mathematics was being invented, it also becomes impossible to give any coherent description of its whole over even a single decade, and so the chronological ordering has to become the second coordinate, as mathematical content becomes the first.

### Changes from the Second Edition

Besides the general reorganization of material mentioned above, I have also had a feeling that in the previous edition I succumbed in too many places to the mathematician’s impulse

to go into mathematical detail at the expense of the history of the subject and to discuss some questions of historical minutiae that are best omitted in a first course. I have therefore condensed the book somewhat. The main difference with earlier editions is that I have tried to adapt the text better to the needs of instructors. To that end, I have made the chapters more nearly uniform in length, usually ten to twelve pages each, putting into each chapter an amount of material that I consider reasonable for a typical 50-minute class. In addition, I have scrutinized the problems to be sure that they are reasonable as homework problems. They are of three types: (1) those that develop a mathematical technique, such as the Chinese method of solving polynomial equations numerically, the *kuttaka*, computation by the Egyptian method, *prosthaphæresis*, and the like; (2) those that ask the student to recall a specific set of historic facts (these generally have brief answers of a sentence or two and should be answerable directly from the narrative); and (3) those that ask the student to speculate and synthesize the history into a plausible narrative, including possible motives for certain investigations undertaken by mathematicians. In survey chapters at the beginning of some parts, only the last two types occur.

The book is divided into seven parts. The first six, comprising the first 34 chapters, contain as systematic a discussion as I can manage of the general history of mathematics up to the nineteenth century. Because it is aimed at a general audience, I have given extra attention to topics that continue to be in the school curriculum, while at the same time trying to discuss each topic within the context of its own time. At the end of each chapter are a few questions to provide a basis for classroom discussions. More such questions can be found in the accompanying teacher's manual. I believe that these 34 chapters, totaling about 400 pages, constitute a one-semester course and that any extra class meetings (I assume 42 such meetings) will be devoted to quizzes, midterms, and perhaps one or two of the specialized chapters in Part VII.

The seventh and last part of the book consists of more narrowly focused discussions. Except for Chapter 35, which discusses a small portion of the history of women in mathematics, these are updates, arranged by subject matter and carrying the history of the topics they treat into the twentieth century. Since this material involves modern mathematics, it is technically much more difficult than the first six parts of the book, and the mathematical homework problems reflect this greater difficulty, making much higher demands on the reader's mathematical preparation. Instructors will of course use their own judgment as to the mathematical level of their students. In some of these chapters, I have exceeded the self-imposed limit of 12 pages that I tried to adhere to in the first six parts of the book, assuming that instructors who wish to discuss one of these chapters will be willing to devote more than one class meeting to it.

### **Elementary Texts on the History of Mathematics**

A textbook on the history of mathematics aimed at a first course in the subject, whose audience consists of teachers, mathematicians, and interested students from other specialties, cannot be as complete or as focused as an encyclopedia of the subject. Connections with other areas of science deserve attention quite as much as historical issues of transmission and innovation. In addition, there are many mathematical skills that the reader cannot be presumed to have, and these need to be explained as simply as possible, even when the explanation does not faithfully reproduce the historical text in which the subject arose. Thus, I have hybridized and simplified certain mathematical techniques in order to provide a usable model of what was actually done while stripping away complications that make

the original texts obscure. This much sacrifice of historical accuracy is necessary, I believe, in order to get to the point within the confines of a single semester. At the same time, I think the exposition of these and other topics gives a reasonable approximation to the essence of the original texts.

This concept of a reasonable approximation to the original presents a problem that requires some judgment to solve: How “authentic” should we be when discussing works written long ago and far away, using concepts that have either disappeared or evolved into something very different? Historians have worked out ways of giving some idea of what original documents looked like. We can simply write numbers, for example, in our own notation. But when those numbers are part of a system with operational connections, it is necessary to invent something that is isomorphic to the original system, so that, for example, numbers written in sexagesimal notation still have a sexagesimal appearance, and computations done in the Egyptian manner are not simply run through a calculator and the output used. This problem is particularly acute in Euclidean geometry, which makes no reference to any units of length, area, or volume. The “Euclidean” geometry that is taught to students in high school nowadays freely introduces such units and makes use of algebraic notation to give formulas for the areas and volumes of circles, spheres, cones, and the like. This modernization conceals the essence of Euclid’s method, especially his theory of proportion. He did not speak of the area of a circle, for example, only of the ratio of one circle to another, proving that it was the same as the ratio of the squares having their diameters as sides (Book 12, Proposition 2). How much of that authentic Euclidean geometry, which I call *metric-free*, should the student be subjected to? Without it, many of the most important theorems proved by Euclid, Archimedes, and Apollonius look very different from their original forms. On the other hand, it *is* cumbersome to expound, and one is constantly tempted to capitulate and “modernize” the discussion. I have made the decision in this book to draw the line at conic sections, using symbolic notation to describe them, though I do so with a very bad conscience. But I would never dream of presenting, in an introductory text, the actual definition of the *latus rectum* given by Apollonius. I try to hold the use of symbolic algebra to a minimum, but compromises are necessary in the real world.

When it comes to algebra, symbolic notation is a very late arrival. Algorithms for solving cubic and quartic equations preceded it, and those algorithms are very cumbersome to explain without symbols. Once again, I surrender to necessity and try to present the essence of the method without getting bogged down in the technical details of the original works. There is a further difficulty that most students have learned algebra by rote and can carry out certain operations, but have no insight into the essence of the problems they have been taught to solve. They may know what American students call the FOIL method of solving quadratic equations with integer coefficients, and some of them may even remember the quadratic formula, but I have yet to encounter a student who has grasped the simple fact that solving a quadratic equation is a way of finding two numbers if one knows their sum and product. Nor have I found a student who has the more general insight that classical algebra is the search for ways of rendering explicit numbers that are determined only implicitly, even though this insight is crucial for recognizing algebra when it occurs in early treatises, where there is no symbolic notation.

Besides the enormous amount of mathematics that the human race has created, so enormous that no one can be really expert except over a tiny region of it, the historian has the additional handicap of trying to fit that mathematics into the context of a wide range of cultures, most of which will not be his or her area of expertise. I feel these limitations with

particular keenness when it comes to languages. Despite a lifetime spent trying to acquire new languages in what spare time I have had, I really feel comfortable (outside of English, of course) only when working in Russian, French, German, Latin, and ancient Greek. (I have acquired only a modest ability to read a bit of Japanese, which I constantly seek to expand.) Of course, having a language from each of the Romance, Germanic, and Slavic groups makes it feasible to attempt reading texts in perhaps two dozen languages, but one needs to be on guard and never rely on one's own translations in such cases. I am most sharply aware of my total dependence on translations of works written in Chinese, Sanskrit, and Arabic. Even though I report what others have said about certain features of these languages for the reader's information, let it be noted here and now that anything I say about any of these languages is pure hearsay.

Just to reiterate: One can really glimpse only a small portion of the history of mathematics in an introductory course. Some idea of how much is being omitted can be seen by a glance at the website at the University of St Andrews.

<http://www-history.mcs.st-and.ac.uk/>

That site provides biographies of thousands of mathematicians. Under the letter G alone there are 125 names, fewer than 40 of which appear in this book. While many of the "small fry" have made important contributions to mathematics, they do not loom large enough to appear on a map the size and scale of the present work. Thus, it needs to be kept in mind that the picture is being painted in very broad brush strokes, and many important details are simply not being shown. Every omission is regrettable, but omissions are necessary if the book is to be kept within 600 pages.

And, finally, a word about the cover. When I was asked what kind of design I wished, I thought of a collage of images encompassing the whole history of the subject: formulas and figures. In the end, I decided to keep it simple and let one part stand for the whole. The part I chose was the conic sections, because of the length and breadth of their influence on the history of the subject. Arising originally as tools to solve the problems of trisecting the angle and doubling the cube, they were the subject of one of the profoundest treatises of ancient times, that of Apollonius. Later, they turned out to be the key to solving cubic and quartic equations in the work of Omar Khayyam, and they became a laboratory for the pioneers of analytic geometry and calculus to use in illustrating their theories. Still later, they were a central topic in the study of projective geometry, and remained so in algebraic geometry far into the nineteenth century. It is no accident that non-Euclidean geometries are classified as elliptic and hyperbolic, or that linear partial differential equations are classified as elliptic, parabolic, and hyperbolic. The structure revealed by this trichotomy of cases for the intersection of a plane with a cone has been enormous. If any one part deserves to stand for the whole, it is the conic sections.

# WHAT IS MATHEMATICS?

This first part of our history is concerned with the “front end” of mathematics (to use an image from computer algebra)—its relation to the physical world and human society. It contains some general considerations about mathematics, what it consists of, and how it may have arisen. This material is intended as an orientation for the main part of the book, where we discuss how mathematics has developed in various cultures around the world. Because of the large number of cultures that exist, a considerable paring down of the available material is necessary. We are forced to choose a few sample cultures to represent the whole, and we choose those that have the best-recorded mathematical history. The general topics studied in this part involve philosophical and social questions, which are themselves specialized subjects of study, to which a large amount of scholarly literature has been devoted. Our approach here is the naive commonsense approach of an author who is not a specialist in either philosophy or sociology. Since present-day governments have to formulate *policies* relating to mathematics and science, it is important that such questions not be left to specialists. The rest of us, as citizens of a republic, should read as much as time permits of what the specialists have to say and make up our own minds when it comes time to judge the effects of a policy.

## Contents of Part I

1. Chapter 1 (Mathematics and Its History) considers the general nature of mathematics and gives an example of the way it can help to understand the physical world. We also outline a series of questions to be kept in mind as the rest of the book is studied, questions to help the reader flesh out the bare bones in the historical documents.
2. Chapter 2 (Proto-mathematics) studies the mathematical reasoning invented by people in the course of solving the immediate and relatively simple practical problems of administering a government or managing a construction site. In this area we are dependent on archaeologists and anthropologists for the historical information available.

