KINEMATIC
DIFFERENTIAL
GEOMETRY AND
SADDLE SYNTHESIS
OF LINKAGES
KINEMATIC DIFFERENTIAL GEOMETRY AND SADDLE SYNTHESIS OF LINKAGES

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Preface

This book introduces the kinematic geometry of linkages in both analysis and synthesis, and builds up a theoretical system from planar, spherical, to spatial. The presentation differs from traditional ones in the approaches of both the differential geometry for kinematic geometry and the saddle point program for kinematic synthesis of linkages. Kinematic geometry provides the theoretical basis for the kinematic synthesis, both precise and approximated, of linkages by invariants.

The kinematic geometry of a rigid body, logically the combination of the kinematics of a rigid body and the geometry of graphs, tries to study the local geometrical properties of loci from the point of view of continuous motion along trajectories, while this continuous motion can certainly be visualized as the differential of the Frenet frame of the trajectories with respect to its arc length. Therefore, differential geometry, of course, may be the first choice in research on the kinematic geometry of a rigid body. However, the current research situation is unfortunately quite different, and this is one of the reasons for the authors writing this book.

There are currently many methods to study the kinematic geometry of a rigid body, such as geometry, algebra, screws, matrices, complex numbers, vectors, etc., each with their own merit in different application cases. In fact, this originates from the geometry by Burmester, which converts the displacement (or movement) of a lamina at several finite separated planar positions into a geometrical graph by means of corresponding poles of rotation. The algebraic equations are then built up to analyze the properties of geometrical graphs, which expand the object of research to all graphs of the lamina. For modern mathematics with expressions of vector algebra and invariants of geometrical graphs, it is difficult to identify them as belonging to the traditional geometry or algebra. For example, the differential geometry of curves and surfaces, both persists in geometrical significance and avoids the effects of external factors on geometrical graphs. In particular, a moving Frenet frame with three mutually orthogonal axes, or the natural trihedron of a curve or ruled surface, moving along the curve or surface is introduced to examine the intrinsic geometrical properties in differential geometry, whose derivatives can be viewed as the motion conversion for a rigid body at infinitesimally separated positions, just like the poles in finite separated positions, which is believed to be a powerful tool of kinematic geometry for a rigid body, in both planar and spatial motion. The kinematic geometry of a rigid body with multiple degrees of freedom is studied in multi-dimensional space. Of course, there is a natural extension from two or three dimensions to multiple dimensions while the classical differential geometry is developed into modern differential geometry, such as differential
manifolds, Lie groups, and Lie algebras, although these are much more non-representational mathematical methods and the reader may have more difficulty understanding them.

The discrete kinematic geometry of a rigid body, naturally combining the discrete kinematics of a rigid body and the geometry of discrete graphs, studies the global geometrical properties of discrete trajectories, comprised of a series of discrete points or lines, which are globally compared with constraint curves or surfaces, while the differences between them or their errors have to be defined and estimated in terms of their invariants. Hence, the best uniform approximation in multi-dimensional space, or the saddle point programming approach, may be adopted first since it developed from one-dimensional space, or the interpolating approach of the Chebyshev polynomial originally, initially applied in the functional synthesis of linkages. The saddle point programming approach has been applied widely in geometrical error evaluations for manufacturing and measuring. However, the current objective function in the optimal synthesis of linkages for multiple positions, or the error evaluation method, is the least square structural error or the best square approximation, which intensively depends on the initial values and may be valid for the special cases but invalid for the general problems since the structural error is not uniformly defined and the design variables are redundant, other than the invariants in the approach of the saddle point program. This is another reason for authors to write the book.

The book has seven chapters and two appendices in the order of planar, spherical, and spatial kinematic geometry of a rigid body and synthesis of linkages, so it is easy for readers to gain familiarity with the differential geometry and gradually build up the theoretical system. Also, for the reader’s convenience, the required elemental knowledge of differential geometry is partly arranged in Chapter 1 for planar curves and Chapter 3 for space curves and surfaces. Chapters 1 and 2 describe the kinematic geometry and synthesis of planar linkages. Chapters 4 and 5 state the kinematic geometry and synthesis of spherical linkages, which is the bridge between the planar and spatial motion and a transition, even though it can be visualized as a special case of spatial motion. The kinematic geometry and synthesis of spatial linkages are respectively discussed in Chapters 6 and 7 in detail. In the appendices, the displacements of the spatial linkage RCCC are solved to provide the data for the numerical examples of kinematic geometry and synthesis of spatial linkages in the book.
Acknowledgments

Time goes so fast, and it has been over two decades since I first started research on the kinematic geometry of linkages as a PhD student under the guidance of Professor Dazhun Xiao and Professor Jian Liu (Dalian University of Technology). I can still remember the day when Professor Xiao told me honestly how arduous and challenging the study of the kinematic geometry of mechanisms was. I was deeply intrigued, and keen to discover the challenges and problems in the field of mechanisms. From that day on, I have been fully aware of the direction my research would take. Thanks to those unforgettable discussions with Professor Liu, I have gained many wonderful research ideas; he always supported and inspired me. I was also enlightened by many classical books, such as Kinematic Geometry of Mechanisms written by K.H. Hunt. This book would not have come into being if I had not been supported and encouraged by previous generations of scholars, both domestic and abroad, in the field of mechanisms, such as Professor Q.X. Zhang (Beihang University), Professor Y.L. Xiong (Huazhong University of Science and Technology), Professor H.M. Li (Harbin Institute of Technology), Professor S.X. Bai (Beijing University of Technology), Professor Z. Huang (Yanshan University), Professor T.L. Yang (Sinopec Jinling Petrochemical Co., Ltd), Professor H.J. Zou (Shanghai Jiao Tong University), Professor H.S. Yan (National Cheng Kung University), Professor C. Zhang (Tianjin University), Professor J.S. Dai (King’s College London), Professor J.M. McCarthy (University of California, Irvine), Professor Kwun-Lon Ting (Tennessee Tech University), Professor J.Q. Ge (Stony Brook University), and so on. I should also mention the new generation of mechanism scholars in China, such as Professor T. Huang (Tianjin University), Professor F. Gao (Shanghai Jiao Tong University), Professor Z.Q. Deng (Harbin Institute of Technology), Professor Y.Q. Yu (Beijing University of Technology), Professor J. Xie (Southwest Jiaotong University), Professor X.L. Ding (Beihang University), Professor Y.H. Yang (Tianjin University), Professor S. Lin (Tongji University), Professor S.J. Li (Northeastern University), and Professor W.Z. Guo (Shanghai Jiao Tong University).

Many thanks go to my students and colleagues, because the book contains not just my own PhD dissertation but also those of three of my students — Mr. Wei Wang (co-author), Dr. Tao Li, and Dr. Shufen Wang — and eight Master’s dissertations by Ms. Lihua Xiao, Mr. Jincang Zhou, Ms. Tianjian Li, Mr. Pengcheng Zheng, Mr. Baoying Zhang, Mr. Jianjun Zhang, Ms. Jie Chai, and Mr. Jinlei Li. Professor Huimin Dong, my classmate in graduate school, has been working with and helping me for 30 years. Professor Shudong Yu (Ryerson University), my classmate in college, Professor Yimin Tong, Professor Huili Wang, and Dr. Jin Qiu (Dalian University of Technology) have also given me enormous help in writing this book.
I want to express my gratitude to the National Natural Science Foundation of China (grants 59305033 and 59675003) for supporting my research into the kinematic geometry of mechanisms.

I would also like to thank the staff of John Wiley & Sons for their warm collaboration in presenting this book as accurately as possible in all its details.
1

Planar Kinematic Differential Geometry

Kinematics, a branch of dynamics, deals with displacements, velocities, accelerations, jerks, etc. of a system of bodies, without consideration of the forces that cause them, while kinematic geometry deals with displacements or changes in position of a particle, a lamina, or a rigid body without consideration of time and the way that the displacements are achieved. As a combination of kinematic geometry and differential geometry both in content and approach, kinematic differential geometry describes and studies the geometrical properties of displacements.

There are a number of articles and books on kinematic geometry. Pioneers such as Euler (1765), Savary (1830), Burmester (1876), Ball (1871), Bobillier (1880), and Müller (1892) established the theoretical foundation and developed the classical geometrical and algebraic approaches for studying kinematic geometry in two dimensions some hundred years ago. The classical geometric and algebraic approaches are still in use today. Differential geometry is favored by many researchers studying the geometrical properties of positions of a planar object, changes in its positions, and their relationships. Invariants, independent of coordinate systems, are introduced to describe the geometric properties concisely. Thanks to the moving Frenet frame for describing infinitesimally small variations of successive positions, the positional geometry can be naturally and conveniently connected to the time-independent differential movement of a planar object.

This chapter deals with the kinematic characteristics of a two-dimensional object (a point, a line) in a plane without consideration of time by means of differential geometry. Though abstract, the explanation is judiciously presented step by step for ease of understanding and will be a necessary foundation for studying the kinematic characteristics of a three-dimensional object by means of differential geometry in later chapters.
1.1 Plane Curves

1.1.1 Vector Curve

A plane curve \( \Gamma \) is represented in rectangular coordinates as

\[
\begin{align*}
    x &= x(t) \\
    y &= y(t)
\end{align*}
\]  

(1.1)

where \( t \) is a parameter. The above equation can be rewritten in the following way by eliminating the parameter \( t \):

\[ y = F(x) \]  

(1.2)

or in implicit form as

\[ F(x, y) = 0 \]  

(1.3)

In a fixed coordinate frame \( \{O; i, j\} \), the vector equation of curve \( \Gamma \) can be written as

\[ \Gamma : R = x(t)i + y(t)j \]  

(1.4)

or

\[ R = R(t) \]  

(1.5)

Obviously, both the magnitude and direction of \( R \) in equation (1.5) vary.

To describe a curve in the vector form, a real vector function, represented by a unit vector \( e_{I(\varphi)} \) with an azimuthal angle \( \varphi \) with respect to axis \( i \), measured counterclockwise, is defined as a vector function of a unit circle (see Fig. 1.1). A plane curve \( \Gamma \) can be denoted by the following vector function:

\[ R = r(\varphi)e_{I(\varphi)} \]  

(1.6)

In the above equation, the magnitude and direction of vector \( R \) depend on the scalar function \( r(\varphi) \) and the vector function of a unit circle \( e_{I(\varphi)} \).

Another vector function of a unit circle \( e_{II(\varphi)} = e_{I(\varphi+\pi/2)} \) can be obtained by rotating \( e_{I(\varphi)} \) counterclockwise about \( k \) by \( \pi/2 \) (in Chapters 1 and 2, \( k \) is the unit vector normal to the paper and directed toward the reader).

![Figure 1.1](image-url)  

**Figure 1.1** Vector function of a unit circle
The vector function of a unit circle has the following properties:

1. Expansion
   \[
   \begin{align*}
   e_{I(\phi)} &= \cos \phi \mathbf{i} + \sin \phi \mathbf{j} \\
   e_{II(\phi)} &= -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}
   \end{align*}
   \] (1.7)

2. Orthogonality
   For a unit orthogonal right-handed coordinate system \( \{O; e_{I(\phi)}, e_{II(\phi)}, k\} \) consisting of \( e_{I(\phi)}, e_{II(\phi)}, \) and \( k \), we have the following identities:
   \[
   e_{I(\phi)} \cdot e_{II(\phi)} = 0, \quad e_{I(\phi)} \times e_{II(\phi)} = k
   \] (1.8)

3. Transformation
   \[
   \begin{align*}
   e_{I(\theta+\phi)} &= \cos(\theta + \phi) \mathbf{i} + \sin(\theta + \phi) \mathbf{j} = \cos \theta e_{I(\phi)} + \sin \theta e_{II(\phi)} \\
   e_{II(\theta+\phi)} &= -\sin(\theta + \phi) \mathbf{i} + \cos(\theta + \phi) \mathbf{j} = -\sin \theta e_{I(\phi)} + \cos \theta e_{II(\phi)}
   \end{align*}
   \] (1.9)

4. Differentiation
   \[
   \frac{de_{I(\phi)}}{d\phi} = e_{II(\phi)}, \quad \frac{de_{II(\phi)}}{d\phi} = -e_{I(\phi)}
   \] (1.10)

The descriptive form of a curve depends on the chosen parameters and coordinates. A curve may have many descriptive forms, which differ in complexity if the parameters and reference coordinates are chosen differently. Below are three examples.

**Example 1.1** A circle with radius \( r \) and center point \( C \) is shown in Fig. 1.2. Write its equation in both vector and parameter forms.

**Solution**

The parameter equation of a circle in rectangular coordinates \( \{O; i, j\} \) can be written as

\[
\begin{align*}
   x &= x_C + r \cos \phi \\
   y &= y_C + r \sin \phi
   \end{align*}
\] (E1-1.1)

where \((x_C, y_C)\) are the coordinates of the center of the circle in the reference frame \( \{O; i, j\} \).
Alternatively, the same circle can be represented as a vector function of a unit circle:

\[ R = R_C + r e_{\text{I}(\varphi)} \]  

(E1-1.2)

**Example 1.2**  An involute is shown in Figs 1.3 and 1.4. Write its equation in both vector and parameter forms.

**Solution**

The equation of an involute can be written in three different forms using polar coordinates, rectangular coordinates, and a vector function of a unit circle, where \( r_b \) is the radius of the base circle.

1. Polar coordinates:

\[
\begin{align*}
  r & = \frac{r_b}{\cos \alpha} \\
  \theta & = \tan \alpha - \alpha
\end{align*}
\]

(E1-2.1)

![Figure 1.3](image3.png)  
**Figure 1.3**  An involute

![Figure 1.4](image4.png)  
**Figure 1.4**  An involute with a unit circle vector function
2. Rectangular coordinates:

\[
\begin{align*}
  x &= r_b \cos \varphi + r_b \varphi \sin \varphi \\
  y &= r_b \sin \varphi - r_b \varphi \cos \varphi
\end{align*}
\]  

(E1-2.2)

3. Vector function of a unit circle:

\[ R = r_b e_{I(\varphi)} - r_b \varphi e_{II(\varphi)} \]  

(E1-2.3)

**Example 1.3** A planar four-bar linkage is shown in Fig. 1.5. Write the equation of the coupler curve in both parameter and vector forms.

**Solution**

As shown in Fig. 1.5, links BC and AB, in a planar four-bar linkage ABCD with link lengths \( a_1, a_2, a_3, a_4 \), form an inclination angle \( \gamma \) and \( \varphi \) with respect to the fixed link. A moving rectangular coordinate system \( \{B; \vec{i}_m, \vec{j}_m\} \) attached to link BC and a fixed coordinate system \( \{A; \vec{i}_f, \vec{j}_f\} \) attached to the fixed link are established. Point \( P \) in the coupler with polar coordinates \( (r_P, \theta_P) \) can be represented in the coordinate system \( \{B; \vec{i}_m, \vec{j}_m\} \) as

\[
\begin{align*}
  x_m &= r_P \cos \theta_P \\
  y_m &= r_P \sin \theta_P
\end{align*}
\]  

(E1-3.1)

1. The parameter equation of coupler curves

A coupler curve traced by point \( P \) can also be expressed in the fixed frame \( \{A; \vec{i}_f, \vec{j}_f\} \) as

\[
\begin{align*}
  x &= r_P \cos (\theta_P + \gamma) + a_1 \cos \varphi \\
  y &= r_P \sin (\theta_P + \gamma) + a_1 \sin \varphi
\end{align*}
\]  

(E1-3.2)

*Figure 1.5* A planar four-bar linkage
A sextic algebraic equation can be deduced for a coupler curve if parameters $\gamma$ and $\varphi$ are replaced by function $\gamma = \gamma(\varphi)$ in the displacement solution of a four-bar linkage.

2. The vector equation of coupler curves

Link $AB$ rotates about joint $A$ of the fixed link $AD$, and link $BC$ rotates about joint $B$ of link $AB$. Since a circle can be expressed by a vector function of a unit circle, a coupler curve of a four-bar linkage can be written as

$$R_p = a_1e_{\varphi(\varphi)} + r_pe_{\theta(\varphi + \gamma)}$$  \hspace{1cm} (E1-3.3)

A point in link $AB$ traces a circle vector $a_1e_{\varphi(\varphi)}$ in the fixed frame $\{A; i_f, j_f\}$. A point in coupler link $BC$ produces a circle vector $r_pe_{\theta(\varphi + \gamma)}$ in the reference frame of link $AB$. The subscripts inside the brackets are independent variables. Here, we deal with the coupler point relative to the coordinate system by the vector function of a unit circle.

Based on the above three examples, we observe that the description of a plane curve in terms of a vector function of a unit circle is simpler than the traditional algebraic equation. Moreover, since a vector function of a unit circle has intrinsic properties, its successive derivatives with respect to the chosen parameters can be conveniently obtained.

Invariants of a curve, independent of the coordinate system used, can be used to simplify the equation of the curve, which is considered a general rule in differential geometry. The arc length of a curve, which is also termed a natural parameter, is an invariant. Other invariants will be introduced in the later of this chapter and other chapters of the book. For equation (1.4), $t$ can be replaced by $s$. The differential relationship between $s$ and $t$ can be written as

$$s = \int_{t_a}^{t_b} \left| \frac{dR}{ds} \right| dr, \quad ds = |dR| = \sqrt{\left( \frac{dx}{dr} \right)^2 + \left( \frac{dy}{dr} \right)^2 + \left( \frac{dz}{dr} \right)^2} \quad (1.11)$$

Then, the vector equation of curve $\Gamma$ is expressed in terms of $s$ as

$$\Gamma : R = R(s), \ s_a \leq s \leq s_b$$  \hspace{1cm} (1.12)

It is recognized that $|dR/ds| = 1$. Using the Taylor expansion, curve $\Gamma$ can be expressed in the neighborhood $\Delta s$ of point $s$ by

$$R(s + \Delta s) = R(s) + \frac{dR(s)}{ds} \Delta s + \frac{1}{2!} \frac{d^2R(s)}{ds^2} (\Delta s)^2 + \cdots + \frac{1}{n!} \frac{d^nR(s)}{ds^n} (\Delta s)^n + \epsilon_n(s, \Delta s)(\Delta s)^n$$  \hspace{1cm} (1.13)

where $\lim_{\Delta s \to 0} \epsilon_n(s, \Delta s) = 0$.

1.1.2 Frenet Frame

In a fixed frame, a curve is traced by a point of a moving body. There exists a connection between the point path and the moving body. A frame that moves along the curve can be employed to study the intrinsic geometrical properties of the curve.

Assume that the unit tangent vector of a plane curve $\alpha = dR(s)/ds$ is always in the direction of increasing arc length. Adopting the right-handed rule, as in the case of equation (1.8), the unit normal vector of a curve may be defined as $\beta = k \times \alpha$. A unit orthogonal right-handed coordinate system $\{\alpha, \beta, k\}$ may be uniquely established for each point $s$ on the curve.
This moving Cartesian reference frame is called the Frenet frame, or the moving frame of a plane curve (see Fig. 1.6). The Frenet frame for a plane curve may be defined as

\[
\begin{align*}
\frac{d\mathbf{R}}{ds} &= \mathbf{\alpha} \\
\frac{d\mathbf{\alpha}}{ds} &= k \mathbf{\beta} \\
\frac{d\mathbf{\beta}}{ds} &= -k \mathbf{\alpha}
\end{align*}
\] (1.14)

where \(k\), an invariant of the curve, is the curvature. Performing a dot product of both sides of the second equation in (1.14) with vector \(\mathbf{\beta}\), we obtain

\[
k = \frac{d\mathbf{\alpha}}{ds} \cdot \mathbf{\beta} = \left( \frac{d\mathbf{\alpha}}{ds}, k \times \mathbf{\alpha} \right) = \left( k, \frac{d\mathbf{R}}{ds}, \frac{d^2\mathbf{R}}{ds^2} \right)
\] (1.15)

If a vector equation with a general parameter \(t\) is given, as in equation (1.4) for a plane curve \(\Gamma\), the unit tangent vector \(\mathbf{\alpha}\) can be expressed as

\[
\mathbf{\alpha} = \frac{d\mathbf{R}}{ds} = \frac{d\mathbf{R}}{dt} \cdot \frac{dr}{ds} = \frac{dt}{ds} \left( \frac{dx}{dt} i + \frac{dy}{dt} j \right)
\] (1.16)

Utilizing the identity equation \(\mathbf{\beta} = k \times \mathbf{\alpha}\), the unit normal vector \(\mathbf{\beta}\) is obtained as

\[
\mathbf{\beta} = \frac{dt}{ds} \left( -\frac{dy}{dt} i + \frac{dx}{dt} j \right)
\] (1.17)

According to equation (1.11), the relationship between \(s\) and \(t\) is \(\frac{dt}{ds} = 1/\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\). Hence, the curvature \(k\) of a plane curve \(\Gamma\) can be written in terms of \(t\) as

\[
k = \frac{d\mathbf{\alpha}}{ds} \cdot \mathbf{\beta} = \frac{dt}{ds} \frac{d\mathbf{\alpha}}{dt} \cdot \mathbf{\beta} = \frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{dy}{dt} \frac{d^2x}{dt^2} \frac{\left[ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right]^{3/2}}
\] (1.18)
In order to explain the geometrical meaning of $k$, $\alpha$ and $\beta$ are projected onto each axis of the fixed coordinate system $\{O; i, j\}$. We have $\alpha = \{\cos \theta, \sin \theta\}$ and $\beta = k \times \alpha = \{-\sin \theta, \cos \theta\}$, where $\theta$ is the angle between $\alpha$ and axis $i$. The differentiation of $\alpha$ with respect to $s$ can be written as

$$\frac{d\alpha}{ds} = \frac{d\alpha}{d\theta} \frac{d\theta}{ds} = \frac{d\theta}{ds} \{-\sin \theta, \cos \theta\} = \frac{d\theta}{ds} \beta$$

(1.19)

From the second equation in (1.14), another expression for the curvature, $k = \frac{d\theta}{ds}$, can be obtained. The geometrical meaning of $k$ is the rate of change in angle $\theta$ with respect to $s$.

The curvature here may be positive or negative, depending on its direction around the curve, while the current curvature is always positive (used in the book for readers to identify the difference between them). As shown in Fig. 1.7, if $\alpha$ points in the direction of increasing arc length, then $\beta$ always points toward the left-hand side of the curve. Hence, if $\alpha$ and $\beta$ lie on opposite sides of the curve at a point, $k$ is positive, otherwise, $k$ is negative. In a special case, if $k$ at a point is zero, $\alpha$ is coincidental with the curve at that point. The point is then called an inflection point.

With the definition of the curvature, both local and global properties of a plane curve can be determined. We have the following theorem:

**Theorem 1.1**

Given a continuously differentiable function $k(s)$ in interval $(s_a, s_b)$ and an initial point $R_a$ along with a unit tangent vector $\alpha_a$, there exists only one regular plane curve having the given curvature $k(s)$.

Although a plane curve may be expressed in different forms for different coordinate systems and parameters, the curvature determines the curve uniquely according to Theorem 1.1 because it is an invariant of the curve and independent of the coordinate system used. The curvature function is usually termed the natural equation of a curve in two dimensions.

As special cases, a circle has constant curvature while a straight line has zero curvature. The local shape of a plane curve is close to a circle in the neighborhood of point $s$ if the curvature of the curve is constant at point $s$ or in its vicinity. To define closeness, a contact order between two curves is introduced. Two plane curves have two common points at two infinitesimally separated positions if they are tangent to each other. We define this case as “first-order contact.” Order-$n$ contact is defined as the contact between two curves having $n+1$ common points at
infinitesimally separated positions. According to this definition, a plane curve and a circle have first-order contact if the curve is tangent to the circle. Second-order contact implies that a curve and a circle have three common points at infinitesimally separated positions; the circle is then called the osculation circle (see Fig. 1.8), located by three infinitesimally separated points and whose radius is the radius of the curvature circle at the contact point. Third-order contact between a curve and a circle indicates that there are four common points at infinitesimally separated positions. In this case, the differential curvature of the curve with respect to the natural parameter at point $s$ must be zero, or $d\kappa/ds = 0$. Similarly, if the contact order between a curve and a circle is $n$, the successive derivatives up to order $n - 2$ of the curvature with respect to the natural parameter at position $s$ must be zero. The vector of center of the osculating circle, or the curvature center of a curve $\Gamma$ at the contact point $s$, can be written as

$$R_C = R + \frac{1}{k}\beta$$

(1.20)

Each point on curve $\Gamma$ has a corresponding curvature center. The loci of all centers of the curvature circles of a curve is another curve, called the evolute of curve $\Gamma$. In a special case, if curve $\Gamma$ is a circle, $k = \text{const.}$, the evolute degenerates to a fixed point.

We can also define contact orders between a curve and a straight line. A plane curve and a straight line have first-order contact, which means that the line is tangent to the curve at the contact point. Second-order contact implies that a curve and a straight line have three common points at infinitesimally separated positions; the curvature of the curve at the contact point, or the inflection point of the curve, must be zero. Third-order contact requires that a curve and a line have four common points at infinitesimally separated positions, in which the curvature of the curve at the contact point is zero and whose differential with respect to the natural parameter at position $s$ is zero, or $d\kappa/ds = 0$.

For convenience, in discussing the global geometrical properties of a curve in two dimensions, we introduce the following definitions.

- **Closed plane curve.** A plane curve $\Gamma$ is a closed curve during interval $[s_a, s_b]$ if $R(s_a) = R(s_b)$ is satisfied.
• **Simple closed plane curve.** A closed plane curve is a simple closed plane curve if it has no intersections with itself, or is without self-intersection points.

• **Simple convex closed plane curve.** A simple closed plane curve is a simple convex closed plane curve if the tangent vector of the curve at every point is always located on one side of the curve.

As examples, the curve in Fig. 1.9(a) is a simple closed and non-convex plane curve. The curve in Fig. 1.9(b) is a simple closed and convex plane curve, often called an *oval curve* because of its similarity to a goose egg shape. The curve in Fig. 1.9(c) is a non-simple closed plane curve.

From the definition of a simple convex closed plane curve and the geometrical meaning of curvature, we have:

**Theorem 1.2**  
A simple closed plane curve is a simple convex closed plane curve if and only if the curvature remains greater than or equal to zero \( k \geq 0 \) at every point for a properly chosen positive direction of increasing arc length.

In particular, a simple convex closed plane curve is an oval curve if the curvature is not equal to zero at any point of the curve. From Theorem 1.2, we have the following corollary:

**Corollary 1.1**  
A simple closed plane curve must be an oval curve if the curvature remains sign-invariant at every point of the curve for a properly chosen positive direction of increasing arc length.

### 1.1.3 Adjoint Approach

A point \( P \) moves along a plane curve \( \Gamma_p \) in the fixed coordinate system \( \{O; i, j\} \). Another point \( P^* \), which does not belong to curve \( \Gamma_p \), traces a different curve \( \Gamma_{p^*} \) in the same coordinate system \( \{O; i, j\} \). If each position of point \( P^* \) at \( \Gamma_{p^*} \) always corresponds to a position of point \( P \) at \( \Gamma_p \), point \( P^* \) is said to be adjoint to point \( P \), and curve \( \Gamma_{p^*} \) is said to be adjoint to curve \( \Gamma_p \). \( \Gamma_p \) is defined as the *original curve*, and \( \Gamma_{p^*} \) is called the *adjoint curve* of \( \Gamma_p \) (see Fig. 1.10).

![Figure 1.9](image-url)  
(a) A simple closed and non-convex plane curve  
(b) An oval curve  
(c) A non-simple closed plane curve
The Frenet frame \(\{R_P; \alpha, \beta\}\) is set up at the original curve \(\Gamma_P\), and the vector equation of the adjoint curve \(\Gamma_P^*\) can be written as

\[
\Gamma_P^* : R_P^* = R_P + u_1 \alpha + u_2 \beta
\]  \hspace{1cm} (1.21)

where \((u_1, u_2)\) are the coordinates of point \(P^*\) in the Frenet frame \(\{R_P; \alpha, \beta\}\) of the original curve \(\Gamma_P\). Based on the Frenet formulas (1.14), the first derivative of the above equation with respect to the arc length \(s\) of the original curve \(\Gamma_P\) (not the arc length \(s^*\) of the adjoint curve \(\Gamma_P^*\)) is given by

\[
\begin{align*}
\frac{dR_P^*}{ds} &= A_1 \alpha + A_2 \beta \\
A_1 &= 1 + \frac{du_1}{ds} - ku_2 \\
A_2 &= ku_1 + \frac{du_2}{ds}
\end{align*}
\]  \hspace{1cm} (1.22)

where \(dR_P^*/ds\) is the tangent vector of the plane curve \(\Gamma_P^*\). The absolute motion of point \(P^*\) is examined in \(\{O; i, j\}\) and expressed by the moving Frenet frame \(\{R_P; \alpha, \beta\}\) of the original curve \(\Gamma_P\). Hence, \(\left(\frac{du_1}{ds}, \frac{du_2}{ds}\right)\) are the rates of change of the coordinates in the Frenet frame \(\{R_P; \alpha, \beta\}\); here, \((A_1, A_2)\) are the rates of change of the absolute motion of point \(P^*\) in the fixed frame and expressed by the Frenet frame \(\{R_P; \alpha, \beta\}\). In particular, \(P^*\) is a fixed point in \(\{O; i, j\}\); the absolute coordinates of point \(P^*\) do not change with \(s\) of the original curve \(\Gamma_P\), or \((A_1, A_2)\) are zero (i.e., \(dR_P^*/ds = 0\)). The last two expressions of (1.22) can be written as

\[
\begin{align*}
A_1 &= 1 + \frac{du_1}{ds} - ku_2 = 0 \\
A_2 &= ku_1 + \frac{du_2}{ds} = 0
\end{align*}
\]  \hspace{1cm} (1.23)

The above equations are called the fixed point conditions of an adjoint plane curve, or Cesaro’s fixed point conditions. That is, point \(P^*\) in the Frenet frame \(\{R_P; \alpha, \beta\}\) at that instant has to meet the same conditions as if it remained absolutely still in the fixed coordinate system.
The differential equation (1.23), in fact, implies the relationship between the motion of the Frenet frame \( \{ R_p; \alpha, \beta \} \) with respect to the original curve \( \Gamma_p \) and the motion of point \( P^* \) relative to the fixed frame \( \{ O; i, j \} \).

Cesaro once used a metaphor for this: the original curve \( \Gamma_p \) is like a winding river, whereas the Frenet frame \( \{ R_p; \alpha, \beta \} \) is like a boat flowing and going with the water in the river (see Fig. 1.11). The tangent vector \( \alpha \) goes ahead downstream with the second axis, or the normal vector \( \beta \), perpendicular to the boat and toward the left. The boatman can see the breathtaking scenery of banks and mountains all from the viewpoint of the boat as the coordinate system, and hence he knows everything about the curves of the river and detailed information about the banks.

A point \( P \) moves along a plane curve \( \Gamma_p \) in \( \{ O; i, j \} \). A straight line \( L \) passing through point \( P^* \), which does not belong to curve \( \Gamma_p \), traces a set of lines \( \Gamma^*_l \) in \( \{ O; i, j \} \), but each position of line \( L \) with point \( P^* \) always corresponds to a position of point \( P \) at \( \Gamma_p \), or line \( L \) with point \( P^* \) is adjoint to point \( P \). Hence, \( \Gamma^*_l \) is adjoint to curve \( \Gamma_p \). We designate \( \Gamma_p \) as an original curve and \( \Gamma^*_l \) as the set of adjoint lines of \( \Gamma_p \) (see Fig. 1.12).

The Frenet frame \( \{ R_p; \alpha, \beta \} \) is set up at the original curve \( \Gamma_p \), and the vector equation of the set of adjoint lines \( \Gamma^*_l \) can be written as

\[
\Gamma^*_l : R^*_l = R^*_p + \lambda l = R_p + u_1 \alpha + u_2 \beta + \lambda (l_1 \alpha + l_2 \beta), l_1^2 + l_2^2 = 1
\]  

(1.24)

where \( \lambda \) is a parameter for the straight line, \( l \) is a unit direction vector of the straight line described in the Frenet frame \( \{ R_p; \alpha, \beta \} \), which is a function of the arc length \( s \) of the original curve.
curve \( \Gamma_p \). Based on the Frenet formulas in equation (1.14), the first derivative of the above equation with respect to \( s \) of \( \Gamma_p \) is given by

\[
\begin{align*}
\frac{dR_P^*}{ds} &= A_1 \alpha + A_2 \beta + \lambda (B_1 \alpha + B_2 \beta) \\
A_1 &= 1 + \frac{du_1}{ds} - ku_2, \quad A_2 = ku_1 + \frac{du_2}{ds} \\
B_1 &= \frac{dl_1}{ds} - kl_2, \quad B_2 = kl_1 + \frac{dl_2}{ds}
\end{align*}
\]  

(1.25)

If all points on the straight line \( L \) are fixed points in \( \{O; i, j\} \) and do not change with \( s \) of \( \Gamma_p \), the line is a fixed line. We define it as an **absolute fixed line**. Hence, the last four expressions of equation (1.25) are equal to zero in this case, which leads to the conditions of an absolute fixed line in the fixed frame \( \{O; i, j\} \):

\[
\begin{align*}
A_1 &= 1 + \frac{du_1}{ds} - ku_2 = 0, \quad A_2 = ku_1 + \frac{du_2}{ds} = 0 \\
B_1 &= \frac{dl_1}{ds} - kl_2 = 0, \quad B_2 = kl_1 + \frac{dl_2}{ds} = 0
\end{align*}
\]  

(1.26)

If line \( L \) is always collinear with a fixed line in \( \{O; i, j\} \), but slides along the fixed line, the line \( L \) satisfies the following conditions:

\[
\frac{dl}{ds} = 0, \quad \frac{dR_P^*}{ds} \times l = 0
\]  

(1.27)

Substituting equation (1.24) into the above equations, we introduce the **quasi-fixed line conditions**

\[
B_1 = 0, \quad B_2 = 0, \quad l_2A_1 = l_1A_2
\]  

(1.28)

**Example 1.4** Represent a coupler curve by an adjoint curve for a planar linkage.

**Solution**

As shown in Example 1.3, a coupler curve of a four-bar linkage is concisely expressed in terms of a vector function of a unit circle. In fact, the coupler point is always adjacent to joint \( B \) when it traces a coupler curve. Path \( \Gamma_B \) of joint \( B \) in the fixed frame \( \{O; i, j\} \) is a circle, and is consequently taken as the original curve, while a coupler curve is viewed as an adjoint curve of \( \Gamma_B \). Hence, the Frenet frame \( \{R_B; \alpha, \beta\} \) of the original curve \( \Gamma_B \) is established as in Fig. 1.13. A coupler curve \( \Gamma_p \), or the adjoint curve of \( \Gamma_B \), can be expressed by the Frenet frame \( \{R_B; \alpha, \beta\} \) as

\[
R_p = R_B + u_1\alpha + u_2\beta = u_1\alpha + (u_2 - a_1)\beta
\]  

(E1-4.1)

where \((u_1, u_2)\), the coordinates of a coupler point \( P \) in the Frenet frame \( \{R_B; \alpha, \beta\} \), are

\[
\begin{align*}
u_1 &= r_p \sin(\theta_p - \varphi + \gamma) \\
u_2 &= -r_p \cos(\theta_p - \varphi + \gamma)
\end{align*}
\]  

(E1-4.2)

Here, angles \( \varphi, \gamma \) are identical to those in Example 1.3. Both angles are measured in the counterclockwise sense.
1.2 Planar Differential Kinematics

1.2.1 Displacement

1.2.1.1 A General Description of Plane Displacement

To describe the displacement of a moving body $\Sigma^*$ relative to a fixed body $\Sigma$ from one position to another, different reference frames are established. A moving Cartesian reference frame $\{O_m;i_m,j_m\}$ is set up and attached to the moving body $\Sigma^*$ and a fixed Cartesian reference frame $\{O_f;i_f,j_f\}$ is established in the fixed body $\Sigma$ (see Fig. 1.14). For planar motion, a body has three-degrees-of-freedom motion, moving along both $i_f$ and $j_f$ and rotating about $k$, or two linear displacements and one angular displacement. A point $O_m(x_{Omf}, y_{Omf})$ of $\Sigma^*$ is taken as a reference point, whose linear displacements $(x_{Omf}, y_{Omf})$ present a given movement of $\Sigma^*$ in $\{O_f;i_f,j_f\}$, and the angular $\gamma$ of $\Sigma^*$ denotes the angular displacement around $k$, completely describing a planar motion of $\Sigma^*$ relative to $\Sigma$.

An arbitrary point $P_m(x_{Pm}, y_{Pm})$ in $\Sigma^*$ corresponds to a position $P(x_{Pf}, y_{Pf})$ or a displacement in $\{O_f;i_f,j_f\}$, which are related by

$$\begin{bmatrix}
 x_{Pf} \\
 y_{Pf} \\
 1
\end{bmatrix} =
\begin{bmatrix}
 \cos \gamma & -\sin \gamma & x_{Omf} \\
 \sin \gamma & \cos \gamma & y_{Omf} \\
 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 x_{Pm} \\
 y_{Pm} \\
 1
\end{bmatrix}$$

(1.29)

We can analyze and determine the displacements of a point on $\Sigma^*$ if any two linear displacements $(x_{Omf}, y_{Omf})$ and an angular displacement $\gamma$ of $\Sigma^*$ relative to $\Sigma$ are given. In equation (1.29), the given movement of $\Sigma^*$ may be continuous, or discrete. For the former, its content belongs to the continuous kinematic geometry, or the kinematic geometry at infinitesimally separated positions, but the authors prefer the kinematic differential geometry since the continuous movement is achieved by the differential of the moving frame. The latter belongs to the discrete kinematic geometry (the authors appreciate this term, although its contents are not mature so far) or the kinematic geometry at finite separated positions corresponding to the classical curvature theory or the classical Burmester theory. All of these form a theoretical basis for kinematic synthesis of linkage. The kinematic geometry, either with continuous or discrete movement, has similar geometrical properties.
1.2.1.2 Descriptions of Plane Displacement by an Adjoint Approach

The point $P$ of $\Sigma^*$ is located by the Cartesian coordinates $(x_{pm}, y_{pm})$ or the polar coordinates $(r_{pm}, \theta_{pm})$ in $\{O_m; i_m, j_m\}$ of $\Sigma^*$, whose vector equation is

$$R_{pm} = x_{pm}i_m + y_{pm}j_m = r_{pm}e_1(\theta_{pm}) \quad (1.30)$$

The plane displacements of a moving body $\Sigma^*$ are examined in a fixed Cartesian reference frame $\{O_f; i_f, j_f\}$ of $\Sigma$. The relationship between the coordinate axes of $\{O_m; i_m, j_m\}$ and those of $\{O_f; i_f, j_f\}$ is

$$\begin{align*}
i_m &= \cos \gamma i_f + \sin \gamma j_f \\
j_m &= -\sin \gamma i_f + \cos \gamma j_f \quad (1.31)
\end{align*}$$

where $\gamma$ is the angle between $i_m$ and $i_f$. Hence, the plane displacements of $\Sigma^*$ can be represented by two linear displacements $(x_{omf}, y_{omf})$ of point $O_m$ and the rotational displacement $\gamma$ about $k$. Point $O_m$ moves along curve $\Gamma_{om}$, which is given as a regular plane curve with higher-order continuity.

A given curve $\Gamma_{om}$ in the fixed frame $\{O_f; i_f, j_f\}$ is designated as the curve traced by the origin point $O_m$ of the moving frame $\{O_m; i_m, j_m\}$ of $\Sigma^*$ (see Fig. 1.15); its vector equation can be written as

$$\Gamma_{om} : R_{om} = x_{omf}i_f + y_{omf}j_f \quad (1.32)$$

Point $P(x_{pm}, y_{pm})$ of $\Sigma^*$ traces a path $\Gamma_p$ in the fixed frame $\{O_f; i_f, j_f\}$; its vector equation is

$$\Gamma_p : R_p = R_{om} + R_{pm} = R_{om} + x_{pm}i_m + y_{pm}j_m \quad (1.33)$$

Taking the derivative of equation (1.33) with respect to time $t$, the absolute velocity of point $P(x_{pm}, y_{pm})$ can be written as

$$V_p = \frac{dR_p}{dt} = \frac{dR_{om}}{dt} + x_{pm} \frac{di_m}{dt} + y_{pm} \frac{dj_m}{dt} + \frac{dx_{pm}}{dt}i_m + \frac{dy_{pm}}{dt}j_m \quad (1.34)$$
The absolute velocity of the point \( P \) can be viewed as the superposition of two velocities, the following velocity of the moving frame \( \{ O_m; i_m, j_m \} \) and the relative velocity of point \( P \) with reference to the moving frame \( \{ O_m; i_m, j_m \} \). The position of the origin \( O_m \) and the orientation of \( \{ O_m; i_m, j_m \} \) in \( \Sigma^* \) not only affect the property and complexity of the following velocity in equation (1.34), \( \frac{dR_{Om}}{dt} + x_{Pm} \frac{di_m}{dt} + y_{Pm} \frac{dj_m}{dt} \), but also change the magnitude of the relative velocity with reference to the moving frame.

In general, point \( P \) is a fixed point in the moving body \( \Sigma^* \), and the moving frame \( \{ O_m; i_m, j_m \} \) is also fixed in \( \Sigma^* \). There is no relative motion between \( P \) and the moving frame \( \{ O_m; i_m, j_m \} \), or the components of the relative velocity in equation (1.34), \( \frac{dx_{Pm}}{dt} \) and \( \frac{dy_{Pm}}{dt} \), are zero. The following velocity of \( \{ O_m; i_m, j_m \} \) can be determined by equations (1.31) and (1.32).

As mentioned above, both the given curve \( \Gamma_{Om} \) and the relative angular displacement \( \gamma \) of \( \Sigma^* \) completely define the kinematic properties of a moving body \( \Sigma^* \). A point \( P \) of \( \Sigma^* \) traces a path \( \Gamma_P \) in \( \{ O_f; i_f, j_f \} \), while the origin point \( O_m \) of \( \Sigma^* \) moves along the given curve \( \Gamma_{Om} \) in \( \{ O_f; i_f, j_f \} \) simultaneously. Each position of \( P \) at \( \Gamma_P \) always corresponds to a position of \( O_m \) at \( \Gamma_{Om} \). In other words, \( P \) is adjoint to \( O_m \). Hence, any point of \( \Sigma^* \) traces a path, which can be viewed as an adjoint curve of the given curve \( \Gamma_{Om} \), or the displacements of \( \Sigma^* \) in plane motion can be expressed by the adjoint approach of curve \( \Gamma_{Om} \) if the given curve \( \Gamma_{Om} \) is a regular curve.

Utilizing the analytic properties of a given curve \( \Gamma_{Om} \), the natural arc length \( s \) is chosen as the parameter of the given curve \( \Gamma_{Om} \). We can set up the Frenet frame \( \{ R_{Om}; \alpha, \beta \} \) of the given curve \( \Gamma_{Om} \) as

\[
\alpha = \frac{dR_{Om}}{ds}, \quad \beta = k \times \alpha
\]

or

\[
\left\{ \begin{array}{l}
\alpha = \left( \frac{dx_{Om}}{ds} i_f + \frac{dy_{Om}}{ds} j_f \right) / \left[ \left( \frac{dx_{Om}}{ds} \right)^2 + \left( \frac{dy_{Om}}{ds} \right)^2 \right]^{1/2} \\
\beta = \left( -\frac{dy_{Om}}{ds} i_f + \frac{dx_{Om}}{ds} j_f \right) / \left[ \left( \frac{dx_{Om}}{ds} \right)^2 + \left( \frac{dy_{Om}}{ds} \right)^2 \right]^{1/2}
\end{array} \right.
\]

(1.35b)