OPTICAL DESIGN USING EXCEL®
Practical Calculations for Laser Optical System
HIROSHI NAKAJIMA
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About the Author

Hiroshi Nakajima graduated from Doshisha University (Kyoto, Japan) with a degree in electronics. From 1967 to 2002, he worked for Hitachi Electronics Engineering Co., Ltd, as an engineer. He was responsible for developing automated manufacturing systems for a variety of electrical products, including the following: automatic adjustment systems for color picture tubes (CPTs), optical inspection systems for the silicon wafer surfaces of integrated circuits (ICs), and optical inspection systems for hard disk (HD) surfaces. From 2003 to 2012, he worked for Hitachi Electronics Engineering Co. and Hitachi High-Technologies Corporation as a technical consultant. During this time, he was responsible for developing optical inspection systems for HD surfaces. The author’s specialty is developing machines which are capable of detecting microscopic surface defects (less than 0.1 μm in diameter) using laser optical systems. This book is largely based on his practical experience of working with laser optical systems.
Preface

Allow me to introduce myself. My name is Hiroshi Nakajima, and I am an electronic and optical engineer in Japan. During my job training, I had to study optics. It was tougher than I had anticipated. It took me a great deal of effort to acquire an understanding of optics and the various technologies that it employs. While I was studying optics, I always wondered if there was an easier way to master the subject. That is what motivated me to write this book.

There are lots of excellent books about theoretical and applied optics. However, there are very few books on the market that explain the practical calculation methods used for designing optical systems, with the aid of fully worked-out examples. This book is intended for the non-specialist reader, whose grasp of the mathematical complexities of optical theory is somewhat limited. For ease of comprehension, the technical explanations have been made as clear and simple as possible. The book provides a detailed explanation of the design of optical systems, starting from the basic theoretical principles and then moving on to practical, real-life examples, with supporting calculations included. By the time the reader finishes this book on optics, he/she should have a fairly complete understanding of the subject.

One of the distinguishing features of this book is that it contains a detailed treatment of optical laser systems, which generate a laser-focused beam and then use it to irradiate an object, thereby obtaining images through processes such as reflection, refraction, and diffraction. From a theoretical standpoint, light emitted from a laser is the simplest kind of light. Starting with lasers is an ideal way of acquiring an understanding of general optics, with a minimum of effort.

This book employs various calculation methods to describe the focusing of a laser beam on a spot. After exploring the basic principles of geometrical optics and simple optical systems, the reader will be provided with multiple worked examples of two- and three-dimensional ray tracing techniques. The more complex phenomena of interference and diffraction require a mathematical grasp of diffraction theory, which treats light as a wave. Next, we shall derive the equations for Gaussian beam focusing, a very simple and practical method for performing laser optical calculations. Finally, the reader will be introduced to some basic techniques that make use of diffraction calculations in order to solve applied problems relating to the wave motion of light.

Some of the calculations in this book (ray trace calculations and Gaussian beam focusing calculations) are performed directly on Excel spreadsheets, which makes them easy to understand, while others (ray trace calculations and diffraction calculations) employ VBA programming, where users can perform complex calculations by inputting simple settings. Readers can either choose to perform the ray trace calculations directly on Excel spreadsheets, or use VBA programs instead. Additionally, all of the calculations have been recorded on the Excel files available on the book’s companion web site. I sincerely hope that my book will help readers obtain an understanding of what is needed to design optical systems – especially laser systems.
Lastly, I would like to thank Dr Vincent J. Torley for patiently proofreading my English.

Hiroshi Nakajima

OUTLINE OF CONTENTS

Chapter 1 – Geometrical Optics

In this chapter, the reader is introduced to the basic principles of geometrical optics. The fundamental characteristics of light are discussed, including Fermat’s principle. Starting from these basic principles, we can derive the principles of paraxial theory, the five Seidel aberrations and the sine condition, which together form the basis of geometrical optics.

Chapter 2 – Examples of Simple Optical Design Using Paraxial Theory

The basic principles for designing optical systems are discussed in this chapter, for various types of lenses. Examples of simple optical systems are also provided. Additionally, various technical considerations relating to the design of laser optical systems are discussed.

Chapter 3 – Ray Tracing Applications of Paraxial Theory

In this brief chapter, we examine a method for performing ray tracings, using paraxial theory. Problems with practical applications of this method are also included.

Chapter 4 – Two-Dimensional Ray Tracing

Here, we discuss a method for performing ray tracing calculations in two dimensions, using the laws of refraction. We show how ray tracing calculations can be performed for rays traveling from the object to the image, for a variety of optical systems. We can calculate the ray aberration occurring on the image plane, as well as the wave-front aberration of an optical system. As this book is intended to give the reader an understanding of the practical applications of optical systems, this chapter includes fully worked solutions to real-life problems relating to several different types of lenses, including biconvex lenses, plano-convex (concave) lenses, cylindrical lenses, meniscus lenses, achromatic lenses, and aspheric lenses. We can also use aberration-free lenses in ray tracing, as a substitute for lenses whose specifications we do not know in detail.

Chapter 5 – Three-Dimensional Ray Tracing

Using the method of ray tracing discussed in Chapter 4, we can expand the number of dimensions from two to three, and thereby obtain a method for performing three-dimensional ray tracing calculations. Using this method, we can then generate spot diagrams on the image plane. Two- and three-dimensional ray tracing calculations can be performed on Excel cells directly, or on VBA programs. Support is provided for the reader, whether he/she chooses Excel or VBA.
Chapter 6 – Mathematical Formulae for Describing Wave Motion

Chapters 1–5 relate to geometrical optics. However, geometrical optics alone cannot provide us with a complete analysis of the behavior of light in all situations. In particular, it does not tell us how to design optical systems relating to the phenomena of interference and diffraction. In these cases, we need a mathematical theory which enables us to deal with light as a wave. In this chapter, we study the fundamental characteristics of light, the wave equations, and the mathematical expressions used to describe the amplitude and phase of waves, including Argand diagrams. (For the benefit of those readers who may be interested in further study, a detailed derivation of the equations for plane waves, spherical waves, and Gaussian beams is contained in the Appendices.)

Chapter 7 – Calculations for Focusing Gaussian Beams

By combining the lens formula in Chapter 1 and the Gaussian beam equations in this chapter, we can derive focusing equations for a Gaussian beam. In this chapter, we discuss the characteristics of Gaussian beams and derive the focusing equations for a Gaussian beam. The chapter also contains a treatment of the M-squared factor, which can be used to calculate the focused spot size of an actual laser beam whose beam quality is a little worse than the ideal Gaussian beam because of being superimposed with higher order lateral modes.

Chapter 8 – Diffraction: Theory and Calculations

In this chapter, we examine the theory and calculation methods of diffraction for various apertures, including a slit, a rectangular aperture, and a circular aperture.

Chapter 9 – Calculations for Gaussian Beam Diffraction

Here, we discuss methods for calculating diffraction irradiance, especially for Gaussian beams, as they are the laser beams which are generally used in real-life situations. In this book, the diffraction calculations are performed by numerical integration methods using the VBA program. The VBA program includes one-dimensional diffraction calculations (for the case of diffraction from a slit), two-dimensional R-θ diffraction calculations (diffraction from a circular aperture) and two-dimensional x-y diffraction calculations (diffraction from an aperture with an arbitrary shape). The numerous examples at the end of the chapter provide the reader with solutions to practical problems with real-life applications, which the reader is likely to encounter in an everyday context.

Appendices

During the preparation of this book, some of the more detailed theoretical explanations were edited from the text, for the sake of ease of comprehension. However, some of these detailed explanations have been included in the Appendices for the benefit of those seeking a deeper understanding of optics, and for anyone who may be interested in undertaking further studies in the future.
1. Geometrical Optics

1.1 Characteristics of Lasers

This book is about optical calculation methods and the principles for applying these methods to actual optical devices. Most of these devices use lasers, so we will begin by briefly examining the characteristics of lasers. The following exposition will be especially beneficial for readers whose understanding of lasers is rather limited.

The term LASER is an acronym for “Light Amplification by Stimulated Emission of Radiation.” Lasers have special characteristics that distinguish them from most light-emitting devices: a narrow, low-divergence beam (sharp directivity), and a very narrow wavelength spectrum (monochromaticity). These features of lasers make them ideally suited for the generation of high intensity beams.

The history of lasers goes back about half a century to 1960, when the first working laser was demonstrated. Lasers are now widely used in a variety of fields, including optical storage (e.g., CD drives and DVD drives), fiber-optic communication, manufacturing (especially for cutting, bending, welding, and marking materials), scientific measurement, and medicine. This diversity of application is due to the following four properties of lasers, which give them a commercial and scientific edge over other light-emitting devices.

1. **Monochromaticity**: Radiation that has a very narrow frequency band (or wavelength band) is said to possess the property of monochromaticity. Because the band is so narrow, the radiation can be regarded as having a single frequency (or alternatively, a single wavelength). Laser light typically has a very narrow frequency band (or wavelength band), which makes it an ideal example of the property of monochromaticity. Sunlight, by contrast, has a very broad band spectrum, with multiple frequencies and wavelengths.

2. **Beam directivity**: The term directivity, when used in relation to lasers, refers to the directional properties of the electromagnetic radiation they emit. A laser beam exhibits a very sharp (narrow) directivity, allowing it to propagate in a straight line with almost no expansion. By contrast, normal light sources such as flashlights and car headlights have broader directivity than lasers, so their beams cannot travel as far.

3. **Coherence**: If split laser beams which were emitted from the same source are superimposed, a fringe pattern will appear. This fringe pattern is never observed in an isolated beam. We refer to this phenomenon as interference caused by the wave characteristics of light. If these two beams traveling onto the same plane are superimposed while they are in phase (with their crests and troughs lined up), the resultant beam will appear brighter, but if these waves are superimposed with a 180° phase difference
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(i.e., if crests and troughs are superimposed), the beam will appear dark and the resultant amplitude will be zero. A laser can easily generate interference patterns because of its coherence (uniformity of phase). Sunlight cannot readily generate interference patterns, due to its incoherence: because its coherence time and coherence length are very short, and the phases of the superimposed waves will not come into phase very easily. Sunlight will only exhibit coherence over a very short interval both of time and space.

4. High concentration of energy (high intensity): A sheet of paper can be burnt simply by focusing sunlight on it, using a convex lens. A laser is much more concentrated: it can even weld two pieces of steel together. This is not merely due to the high power of the laser, but also because of the extremely high intensity of the laser beam, where the light energy is narrowly focused. It is relatively easy to concentrate a laser beam on a small target. A high intensity beam can be generated very easily using a laser.

All of these characteristics of lasers can best be summed up by the word “coherent.” A laser is both temporally coherent and spatially coherent. What this means is that a laser has a uniform phase over time at an arbitrary point in space, and it also has a uniform phase in space at an arbitrary point in time. Thus a laser has a uniform phase both in time and in space. A laser radiates a single-wavelength (more precisely, a very narrow wavelength spectrum) beam with a constant phase and it can propagate in a specific direction. An electric light, by contrast, radiates a multitude of different wavelengths at various phases and in all directions. Thus it is both spatially and temporally incoherent.

1.2 The Three Fundamental Characteristics of Light Which Form the Basis of Geometrical Optics

To understand geometrical optics, which is the most basic form of optics, we need to first study the fundamental characteristics of light – including light emitted by lasers. The following properties of light are confirmed by everyday experience:

1. Light rays travel in straight lines in a uniform medium.
2. Light rays are independent of one another.
3. Light rays can be reflected and refracted: they change their direction at a boundary between different media, in accordance with the laws of reflection and refraction.

The whole science of geometrical optics can be derived from these three characteristics of light.

1.2.1 Light Rays Travel in Straight Lines

Many common optical phenomena attest to the fact that light rays travel in straight lines. For instance, when the sun is shining outdoors, a tree casts a shadow whose shape is identical with its own. Without using any lenses, we can construct a pinhole camera that can capture the image of an object, simply by making a pinhole in one of the walls of a black box as shown in Figure 1.1.

1.2.2 Light Rays Act Independently of One Another

Figure 1.2 illustrates the independent action of light rays using the example of three spotlights whose light is of different colors: red, blue, and green. When we irradiate the same area on a white sheet of paper with
these three spotlights, we perceive the light as “white.” However, if we replace the paper with a mirror, and then project the reflected light onto another sheet of paper, the three separate beams of light reappear in their original colors of red, blue, and green. The fact that these three beams reappear in their original colors demonstrates that light rays coming from different sources act independently of one another after being reflected by the mirror. Likewise, the fact that the light from the three spotlights appears white when they are all focused on the same area of paper can be explained in terms of the constituent light wavelengths (red, blue, and green) reaching our eyes and acting on our retinas independently. It is the superposition of waves which causes us to perceive them as white.

1.2.3 Reflection of Light Rays

As shown in Figure 1.3a, when a light ray is reflected by a mirror at the point of incidence O, we can define the normal as an imaginary line through point O perpendicular to the mirror. The reflected ray will lie in the same plane as the incident ray and the normal to the mirror surface, and the angle of reflection will be the same as the angle of incidence. We are all familiar with this phenomenon from everyday experience. It can be described by the following equation:

\[ \theta_i = \theta_r \]  

(1.1)

1.2.4 Refraction of Light Rays

An object lying in a tub of water appears to be at a shallower depth than it actually is. This phenomenon can be explained by the refraction of light at the interface between the water and the air. As shown in Figure 1.3b, when a ray is refracted at a boundary plane between different media, the relationship between the angle of incidence and angle of refraction can be described by the following equation (Snell’s law):

\[ n_1 \sin \theta_i = n_2 \sin \theta_r \]  

(1.2)
Figure 1.3 (a) Reflection of a light ray and (b) refraction of a light ray

where

\[ n_1 = \text{Refractive index in medium 1} \]
\[ n_2 = \text{Refractive index in medium 2} \]
\[ \theta_i = \text{Angle of incidence} \]
\[ \theta_r = \text{Angle of refraction} \]

1.3 Fermat’s Principle

The law of rectilinear propagation and the law of reflection and refraction of light rays can both be derived from Fermat’s principle, as explained below [1].

(i) The velocity of light is inversely proportional to the refractive index of the medium in which light propagates. (In other words, light travels more slowly in a medium having a higher refractive index.)

The velocity of light in a vacuum is a constant, \( c \):

\[ c = 2.99792458 \times 10^8 \text{ m/s} \]  

The velocity of light \( v \) in a medium having refractive index \( n \) is:

\[ v = \frac{c}{n} \]  

In optical calculations, the optical path length \( L \) is defined as follows:

\[ L = nL_0 \]  

where \( L_0 = \text{physical length traversed by light in the medium} \). \( L_0 \) is defined as the optical path length in a vacuum, and the refractive index in a vacuum, \( n_0 = 1 \).

(ii) The path taken by a ray of light between any two points in a system is always the path that can be traversed in the least time (or the path that has the minimum optical path length) (Fermat’s Principle).

A more accurate and complete statement of Fermat’s principle is that a ray of light traveling from one point to another point must traverse an optical path length which corresponds to a stationary point, which means that it can either be a minimum, a maximum, or a point of inflection.
1.3.1 Rectilinear Propagation

Fermat’s principle entails that a ray of light must travel along a rectilinear path in a uniform medium.

As shown in Figure 1.4a, a ray of light traveling from A to B in a uniform medium takes the path that can be traversed in the least time, that is, the straight line AB.

1.3.2 Reflection

When a ray of light is reflected, it will be reflected at an angle that minimizes its optical path length.

Figure 1.4b illustrates a ray of light which proceeds from point A, is reflected by the mirror at P, and reaches B. Now let us assume that the starting point is A’ instead of A. (A’ is the “reflection” of A in the mirror.) Clearly, the length of A’PB is equal to APB. It is also obvious that the optical path length A’PB is minimized when A’PB is rectilinear. When this is the case, the angle of reflection \( \theta_r \) will equal the angle of incidence \( \theta_i \).

\[
\text{Angle of incidence } \theta_i = \text{angle of reflection } \theta_r \quad (1.1)
\]

1.3.3 Refraction

A ray of light traveling from one medium to another will be refracted at an angle that minimizes the optical path length.

Suppose that a ray of light proceeds from point A(0,a) inside a medium with a refractive index of \( n_1 \), then passes through the boundary plane at point B(x,0), and finally reaches point C(d,c), inside a medium with a refractive index \( n_2 \), as shown in Figure 1.5a.

The optical path length ABC will be:

\[
L = n_1 AB + n_2 BC \\
= n_1 \sqrt{x^2 + a^2} + n_2 \sqrt{(d - x)^2 + c^2} \quad (1.6)
\]

The derivative of \( L \) with respect to \( x \) will be:

\[
\frac{dL}{dx} = \frac{n_1 x}{\sqrt{x^2 + a^2}} - \frac{n_2 (d - x)}{\sqrt{(d - x)^2 + c^2}} \quad (1.7)
\]
When $dL/dx = 0$, the optical path length $L$ is minimized. That is,

$$
\frac{dL}{dx} = \frac{n_1 x}{\sqrt{x^2 + a^2}} - \frac{n_2 (d - x)}{\sqrt{(d - x)^2 + c^2}} = 0
$$

(1.8)

From Figure 1.5a, the following equations can be derived:

$$
\sin \theta_1 = \frac{x}{\sqrt{x^2 + a^2}}
$$

(1.9)

$$
\sin \theta_2 = \frac{d - x}{\sqrt{(d - x)^2 + c^2}}
$$

(1.10)

Combining Equation (1.8), Equation (1.9) and Equation (1.10) yields:

$$
n_1 \sin \theta_1 = n_2 \sin \theta_2
$$

(1.2)

Equation (1.2) is the conditional equation of refraction.

**Path Swum by a Lifesaver in An Emergency Rescue**

A lifesaver at a beach will swim in a path that minimizes the time to reach the victim in an emergency rescue (Figure 1.5b).

This is identical to the behavior of a ray of light which is described by Fermat’s principle.

**1.3.4 An Ideal Imaging System Using Lenses**

Let us imagine an ideal imaging system using multiple lenses. A ray of light proceeds from the point object $A$, passes through the lens system, and reaches imaging point $B$, as in Figure 1.6. If the system is an ideal imaging system, these three routes must have the same optical path length (Fermat’s principle).

In an ideal imaging system, the optical path length from the object to the image will have a constant value of $\Sigma L$:

$$
\sum L = n_1 L_1 + n_2 L_2 + n_3 L_3 + n_4 L_4 + n_5 L_5 = \text{constant}
$$

(1.11)
Note
The refractive index of a medium is not always the same: it varies slightly, depending on the wave-
length of the light traveling through it. Thus lenses will exhibit some degree of chromatic aberration.
Spherical lenses also exhibit various monochromatic aberrations, which will be discussed in Section 1.6.
In actual imaging systems, multiple lens systems are generally used in order to minimize or eliminate
these aberrations.

1.4 Principle of Reversibility
Since Fermat’s principle only relates to the path and not the direction traveled, all of the foregoing
equations describing light rays apply equally to light traveling along the same path in a backwards
direction.

If there is a ray of light which starts from point A, passes through an optical system, and reaches point
B, it follows that the ray starting at B and reaching A must travel along exactly the same path. This
insight, which has immense practical value, is referred to as the “Principle of reversibility.”

1.5 Paraxial Theory Using Thin Lenses
We will now examine several methods of imaging using lenses [1]. First, using the concept of a minimal
optical path length, we will discuss how rays can create an image of an object.

1.5.1 Equation of a Spherical Lens Surface
The equation of a spherical lens surface S whose center of curvature lies on the z-axis and whose surface
touches the origin as in Figure 1.7, can be expressed by:
\[ x^2 + y^2 + (z - r)^2 = r^2 \]  (1.12)
where \( r \) is the curvature radius of the lens surface.

Equation (1.12) can be rewritten as:
\[ z = \frac{1}{2r} (x^2 + y^2 + z^2) \]  (1.13)

When we use a thin lens, we can neglect \( z^2 \) because \( z^2 \ll x^2 + y^2 \) in thin lenses. Hence
\[ z = \frac{1}{2c} (x^2 + y^2) \]  (1.14)
\[ z = \frac{1}{2} c (x^2 + y^2) \]  (1.15a)
\[ c \equiv \frac{1}{r} \] (curvature)  (1.15b)

In Equation (1.15a), since the surface of the lens bulges toward the incident rays coming from the
light source, it is convex and has a positive curvature. In a two-dimensional space, Equation (1.15a)
Figure 1.7  Spherical lens surface

Figure 1.8  Refraction of a light ray by a convex lens

simplifies to:

\[ z = \frac{1}{2} cy^2 \]  \hspace{1cm} (1.16)

Equation (1.16) describes a convex lens with a curvature of \( c \), whose thickness decreases as the vertical height of the lens increases.

1.5.2 Wave Front Radii of Incident Rays versus Rays Refracted by a Convex Lens

A collimated light beam (whose rays are parallel) is incident onto a convex lens having a refractive index of \( n \), a thickness of \( d \) in the center, and front and back surfaces whose curvatures are \( c_1 (\equiv 1/r_1) \) and \( c_2 (\equiv 1/r_2) \), respectively. Suppose that the front surface of the lens touches the origin, as shown in Figure 1.8.

The wave front surface (defined as a set of points whose optical path lengths from a point light source all have the same value) of the incident rays will lie on a plane \( \Sigma \), as the light beam is collimated. Using the concept of optical path length, we can illustrate how the shape of the wave front changes, after passing through the lens.

Suppose the ray at a height \( Y \) travels a distance \( Z \) parallel to the \( z \)-axis, while the ray going along the \( z \)-axis travels through the center of a lens whose thickness is \( d \). After passing through the lens, the wave front will be a surface consisting of points having the same optical path length from the light source.
From Equation (1.16), the thickness $t$ of the lens at height $Y$ will be:

$$t = d - \frac{1}{2}(c_1 - c_2) Y^2$$

(1.17)

The optical path length of the ray $L_0$ which travels through the center of the lens will be:

$$L_0 = nd$$

(1.18)

The optical path length of the ray $L_Y$ which travels through the lens at a vertical height of $Y$ will be:

$$L_Y = nt + Z - t$$

(1.19)

The condition for both rays having the same optical path length is that $L_0 = L_Y$. Thus

$$nd = n\left\{d - \frac{1}{2}(c_1 - c_2) Y^2\right\} + Z - d + \frac{1}{2}(c_1 - c_2) Y^2$$

(1.20)

In other words,

$$Z = d + \frac{1}{2}(n - 1)(c_1 - c_2) Y^2$$

(1.21)

Equation (1.21) represents a sphere with a curvature of $(n - 1)(c_1 - c_2)$, which includes the rear surface of the lens on the axis. From the preceding discussion, it follows that the surface of the wave front, after passing through the lens, will be transformed into a sphere whose equation is represented by Equation (1.21).

**Note**

The foregoing equations all presuppose that we can apply the thin lens approximation: light rays maintain the same vertical height while passing through a lens.

### 1.5.3 The Refractive Power of a Lens and the Thin Lens Equations

Let us now apply Equation (1.21) to incident rays which are not parallel but diverging. From Equation (1.21), the ray of light at a vertical height of $Y$ will travel a distance of $\frac{1}{2}(n - 1)(c_1 - c_2) Y^2$ further than the ray along the $z$-axis, which travels through the lens at the point where it is thickest (Figure 1.9). Putting it another way, after passing through a convex lens, the wave front curvature will increase by a factor of $(n - 1)(c_1 - c_2)$. We refer to this value as the refractive power of the lens, $K$.

$$K = (n - 1)(c_1 - c_2)$$

(1.22)

As a result, a convex lens will refract the incident rays inwards. Figure 1.9 shows that a convex lens will refract a diverging beam of incident light (which has a negative wave front curvature) inwards, generating a converging beam of light (with a positive wave front curvature). The refractive power $K$ will be the difference in wave front curvatures between the light rays behind the lens and those in front of the lens.

![Figure 1.9 Refraction by a lens](image-url)
We can define $K$ as the (refractive) power of a lens. The point light source $P$ situated at a distance $s(-)$ in front of the lens generates a wave front $\Sigma$ (a sphere with its center at $P$) at the lens front. After passing through the lens, the wave front $\Sigma'$ is transformed into a sphere centered at $P'$ which is situated at a distance $s'(+)\text{ to the right of the lens along the z-axis.}$ The wave front curvatures of $\Sigma$ and $\Sigma'$ will be $1/s$ and $1/s'$. The curvature after passing through the lens $1/s'$ should be the value added by the (refractive) power $K$ to the initial curvature of the wave front before passing through the lens $1/s$. That is,

\[
\frac{1}{s'} = \frac{1}{s} + K \tag{1.23}
\]

More generally, we can replace $K$ with the inverse of the focal length $1/f$. This gives us:

\[
\frac{1}{s'} = \frac{1}{s} + \frac{1}{f} \tag{1.24}
\]

where $s$ is the distance of the object from the lens, and $s'$ is the distance of the image from the lens. These values will be positive when they are situated to the right of the lens and negative when they are situated to the left of the lens. $f$ is the focal length of the lens.

Equation (1.24) is the famous Gaussian lens formula. Using this formula, we can easily calculate the distance of the image. If $s > 0$, the object is situated behind the lens, so the object will be a virtual object. If $s' < 0$, the image is situated in front of the lens, so the image will be a virtual image.

From Equation (1.22), the focal length of a thin lens can be expressed as follows:

\[
\frac{1}{f} = (n-1) (c_1 - c_2) = (n-1) \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \tag{1.25}
\]

If the lens is plano-convex (or plano-concave), $c_2 = 0 (r_2 = \infty)$, and we can then write:

\[
\frac{1}{f} = \frac{(n-1)}{r} \tag{1.26}
\]

1.5.4 Imaging Equations for a Lens

1.5.4.1 A Method of Drawing Rays Before and After Passing through the Lens, in Order to Obtain Their Image Point

Figure 1.10 illustrates the rays of light before and after going through the lens, where

- $F$ = First focal point
- $F'$ = Second focal point
- $H$ = Primary principal point ($H_1 H H_2 = \text{primary principal plane}$)
- $H'$ = Secondary principal point ($H'_1 H'H'_2 = \text{secondary principal plane}$)
- $f$ = First focal length $(-)$
- $f'$ = Second focal length $($)$
- $z$ = Distance of object from the first focal point $(-)$
- $s$ = Distance of object from the primary principal point $(-)$
- $z'$ = Distance of image from the second focal point $($)$
- $s'$ = Distance of image from the secondary principal point $($)$
- $A$ = Object
- $A B$ = Normal to the axis (B is situated on the axis)
- $A'$ = Image
- $A'B'$ = Normal to the axis (B' is situated on the axis).