Frege
A Critical Introduction

Harold W. Noonan
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Preface

In this book I present a study of the most important themes in the work of the great German philosopher and logician Gottlob Frege. The exposition follows the order in which these themes appear in Frege’s work. Thus, after an introductory chapter outlining the background to Frege’s thought and setting him in the context of his time, the second chapter explains his fundamental advances in logic, as found in his first book, Conceptual Notation. The third chapter is devoted to his discussion of number in his masterpiece, The Foundations of Arithmetic, and its successor The Basic Laws of Arithmetic, including an account of the inconsistency discovered in his system by Bertrand Russell, ‘Russell’s Paradox’. The remaining two chapters are concerned with the most significant and influential of his writings on philosophical logic and meaning: ‘Function and Concept’, ‘On Concept and Object’, ‘On Sense and Reference’ and ‘Thoughts’.

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H. W. N.
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Introduction:
Frege’s Life and Work

Biography

Friedrich Ludwig Gottlob Frege was the founder of modern mathematical logic, which he created in his first book, *Conceptual Notation, a Formula Language of Pure Thought Modelled upon the Formula Language of Arithmetic* (Begriffsschrift, eine der arithmetischen nachgebildete Formalsprache des reinen Denkens (1879), translated in Frege 1972). This describes a system of symbolic logic which goes far beyond the two thousand year old Aristotelian logic on which, hitherto, there had been little decisive advance. Frege was also one of the main formative influences, together with Bertrand Russell, Ludwig Wittgenstein and G. E. Moore, on the analytical school of philosophy which now dominates the English-speaking philosophical world. Apart from his definitive contribution to logic, his writings on the philosophy of mathematics, philosophical logic and the theory of meaning are such that no philosopher working in any of these areas today could hope to make a contribution without a thorough familiarity with Frege’s philosophy. Yet in his lifetime the significance of Frege’s work was little acknowledged. Even his work on logic met with general incomprehension and his work in philosophy was mostly unread and unappreciated. He was, however, studied by Edmund Husserl, Bertrand Russell, Ludwig Wittgenstein and Rudolf Carnap and via these great figures he has eventually achieved general recognition.

Frege’s life was not a personally fulfilled one (for more detailed accounts of the following see Bynum’s introduction to Frege 1972
and Beaney's introduction to Frege 1997). His wife died twenty years before his own death in 1925 (he was survived by an adopted son, Alfred) and, ironically, his life's work in the philosophy of mathematics, to which he regarded all the rest of his efforts as subordinate, that is, his attempted demonstration that arithmetic was a branch of logic (the 'logicist thesis' as it is now called), was dealt a fatal blow by Bertrand Russell, one of his greatest admirers, who showed that it entailed the inconsistency that now bears his name ('Russell's Paradox'). Nevertheless, Frege perhaps gained some comfort from the respect accorded to him by Russell and by Wittgenstein, who met Frege several times and revered him above all other philosophers. In retrospect, indeed, many would perhaps say that in philosophy generally, as distinct from the narrower branches of logic and the philosophy of mathematics, Frege's greatest contribution was the advance in the philosophy of logic and language which made Wittgenstein's work possible.

Little is known of Frege's personality and life outside philosophy. Apparently his politics and social views, as recorded in his diaries, reveal him to have been, in his later years, extremely right-wing, strongly opposed to democracy and to civil rights for Catholics and Jews. Frege's greatest commentator, Michael Dummett, expresses great shock and disappointment (1973: xii) that someone he had revered as an absolutely rational man could have been imbued with such prejudices. But a more generous view is the one expressed by another great Frege scholar, Peter Geach. Geach writes that while Frege was indeed imbued with typical German conservative prejudices, 'to borrow an epigram from Quine, it doesn't matter what you believe so long as you're not sincere. Nobody can really imagine Frege as an active politico devoted to some course like Hitler's' (1976c: 437).

We have, however, a presentation of the more attractive side of Frege in an account Wittgenstein gives of his encounters with him:

I was shown into Frege's study. Frege was a small, neat man with a pointed beard who bounced around the room as he talked. He absolutely wiped the floor with me, and I felt very depressed; but at the end he said 'You must come again,' so I cheered up. I had several discussions with him after that. Frege would never talk about anything but logic and mathematics, if I started on some other subject, he would say something polite and then plunge back into logic and mathematics. He once showed me an obituary on a colleague, who,
it was said, never used a word without knowing what it meant; he
expressed astonishment that a man should be praised for this! The
last time I saw Frege, as we were waiting at the station for my train,
I said to him 'Don't you ever find any difficulty in your theory that
numbers are objects?' He replied: 'Sometimes I seem to see a difficulty
but then again I don't see it.' (Included in Anscombe and Geach 1961)

Rudolf Carnap, who attended Frege's lectures in 1914, also presents
a vivid image:

Frege looked old beyond his years. He was of small stature, rather
shy, extremely introverted. He seldom looked at his audience. Ordinarily we saw only his back, while he drew the strange diagrams of
his symbolism on the blackboard and explained them. Never did a
student ask a question or make a remark, whether during the lecture
or afterwards. The possibility of a discussion seemed to be out of the
question. (Carnap 1963: 5)

Frege was born in 1848 in Wismar on the German Baltic coast.
He attended the Gymnasium in Wismar for five years (1864–9),
passed his Abitur in the spring of 1869 and then entered Jena
University.

There Frege spent two years studying chemistry, mathematics
and philosophy. He then transferred to the University of Göttingen
(perhaps influenced by one of his mathematics professors, Ernst
Abbe), where he studied philosophy, physics and mathematics.

In 1873 Frege presented his doctoral dissertation, On a Geometri-
cal Representation of Imaginary Figures in a Plane (in Frege 1984: 1–55),
which extended the work of Gauss on complex numbers, and was
granted the degree of Doctor of Philosophy in Göttingen in Decem-
ber of that year.

Frege then applied for the position of Privatdozent (unsalaried
lecturer), at the University of Jena. Among the documents he sup-
plied in support of his application was his Habilitationsschrift
(postdoctoral dissertation required for appointment to a university
teaching post), 'Methods of Calculation Based upon an Amplifica-
tion of the Concept of Magnitude' (in Frege 1984: 56–92). In this
piece there first emerges Frege's interest in the concept of a function
which, as we shall see, was to play an absolutely central role
throughout his philosophy.

Frege's work was judged acceptable by the Jena mathematics
faculty, and in a prescient report Ernst Abbe speculated that it con-
tained the seeds of a viewpoint which would achieve a durable
advance in mathematical analysis. Frege was therefore allowed to proceed to an oral examination, which he passed, though he was judged to be neither quick-witted nor fluent. After a public disputation and trial lecture in May 1874 he was appointed Privatdozent at Jena, where he remained for the rest of his career.

Initially Frege had a heavy teaching load and he only published four short articles (see Frege 1984: 93–100), three of them reviews and one an article on geometry, before 1879, when Conceptual Notation was published. Nevertheless, these were probably the happiest years of his life. He was young, ambitious, with a plan of his life's work (as we see from the Preface to Conceptual Notation) already formed. He was, moreover, well thought of by the faculty and by the best mathematics students at Jena. The description of his 'student friendly' lecturing style quoted from Carnap earlier fits with Abbe's evaluation of Frege for the university officials in 1879. Abbe reported that Frege's courses were little suited to please the mediocre student 'for whom a lecture is just an exercise for the ears'. But 'Dr Frege, by virtue of the great clarity and precision of his expression and by virtue of the thoughtfulness of his lectures is particularly fit to introduce aspiring listeners to the difficult material of mathematical studies -- I myself have repeatedly had the opportunity to hear lectures by him which appeared to me to be absolutely perfect on every fundamental point' (quoted in Frege 1972: 8).

Absolute perfection on every fundamental point was indeed the aim -- and the achievement -- of Frege's Conceptual Notation, which he conceived as the necessary starting point of his logicist programme. It appeared in 1879 and partly as a result Frege was promoted to the salaried post of special (ausserordentlicher) Professor. The promotion was granted on the strength of a recommendation by Frege's mentor Ernst Abbe, who wrote with appreciation of Conceptual Notation. His remarks were again prescient. He thought that mathematics 'will be affected, perhaps very considerably, but immediately only very little, by the inclination of the author and the content of the book'. He continued by noting that some mathematicians 'will find little that is appealing in so subtle investigations into the formal interrelationships of knowledge', and 'scarcely anyone will be able, offhand, to take a position on the very original cluster of ideas in this book . . . it will probably be understood and appreciated by only a few' (quotations from Frege 1972: 16).

Abbe's pessimism about the immediate reception of Frege's work was wholly justified. It received at least six reviews, but none
showed an appreciation of the book's significance, even though some of the reviewers were eminent logicians. The reviews by Schröder in Germany and Venn in England must have been particularly bitter disappointments. Frege's work was judged inferior to the Boolean logic of his leading contemporaries and his 'conceptual notation' dismissed as 'cumbrous and inconvenient' (by Venn) and 'a monstrous waste of space' which 'indulges in the Japanese custom of writing vertically' (by Schröder).

It was an unfortunate outcome but neither without precedent, nor, in retrospect, surprising. The extent of Frege's achievement was something that could not possibly have been expected by a reviewer asked to give an initial assessment of his work. One is reminded of the similar reception of David Hume's *Treatise of Human Nature* which, likewise, as Hume famously put it, 'fell dead-born from the press'. And, as also in the case of Hume, the poor reception of Frege's work was partly his own fault - arising from the 'manner rather than the matter' of presentation, to use Hume's words - and something that could have been anticipated. Frege did not explain clearly and thoroughly the purpose of *Conceptual Notation* and did not justify and illustrate the advantages of his bizarre-looking two-dimensional notation and its superiority to those available at the time. One can thus sympathize with the first reviewers. As a recent commentator has put it: 'The odds that Frege's work was the production of a genius rather than a crackpot may have seemed long indeed to his colleagues and contemporaries' (Boolos 1998: 144).

As a result of the poor reception of *Conceptual Notation*, Frege postponed his plan, announced in its preface, to proceed immediately to the analysis of the concept of number. Instead he attempted to answer his critics. He wrote two papers comparing his logical symbolism with that of Boole. The first, 'Boole's Logic Calculus and the Concept-Script' (now published in Frege 1979: 9-46) was rejected by three journals. The second, a much shorter version of the first, 'Boole's Logical Formula Language and my Conceptual Notation' (now in Frege 1979: 47-52) was also rejected. Finally Frege managed to get published a more general justification of his conceptual notation, 'On the Scientific Justification of a Conceptual Notation' (now in Frege 1972: 83-9), and was able to deliver a lecture, also subsequently published, at a meeting of the Jenaischüe Gesellschaft für Medicin und Naturwissenschaft, in which he compared his symbolism with Boole's ('On the Aim of the Conceptual Notation', now in Frege 1972: 90-100).
Introduction: Frege’s Life and Work

The disappointing reviews of Conceptual Notation thus sidetracked Frege into a frustrating episode of self-justification. But they also had the effect of making him more aware of how he must present his work if it was to be appreciated. Instead of proceeding straight from Conceptual Notation to a formal demonstration, in his symbolic notation, of the derivability of arithmetic from logic, as anticipated in the Preface to Conceptual Notation, Frege decided to produce an informal sketch of his derivation in ordinary German, set out against the background of a critique of traditional (including Kantian and empiricist) views of number. The result was his masterpiece, The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number (Die Grundlagen der Arithmetik: eine logische mathematische Untersuchung über den Begriff der Zahl) published in 1884 (Frege 1968).

Once again, as in the case of Conceptual Notation, Frege viewed this only as a preliminary stage in his logicist project. He thought that he had made the ‘analytic character of arithmetical propositions’ (i.e. their derivability from logical laws by definition) ‘probable’, but to prove his thesis he needed to produce ‘a chain of deductions with no link missing’ using principles of inference all of which could be recognized as purely logical (Frege 1968: 102).

Frege could have hoped that after Foundations his achievement of this project would have been eagerly awaited by scholars. For Foundations is indeed, as Frege intended, a brilliantly written exposition of his views, both negative and positive. In fact, it received only three reviews, all of them hostile (one, by Cantor, criticizing Frege on the basis of the misunderstanding that he took numbers to be sets of physical objects), and remained largely unread and unnoted for nearly twenty years. A partial explanation of this situation is perhaps the poor reception of Conceptual Notation, which could not have added to Frege’s reputation or predisposed mathematicians and philosophers to think his subsequent work worthy of the effort needed to understand it. But whatever the case, the result was that Frege had no choice but to persevere with his logicist project unacknowledged and unsupported by any encouragement from his peers.

The next stage in this project appeared as volume 1 of The Basic Laws of Arithmetic (Grundgesetze der Arithmetik) in 1893 (see Frege 1962, translated in part in Frege 1964). However, in the intervening nine years Frege’s views on the underlying philosophy of language and logic of Foundations developed rapidly, necessitating a complete rewriting of a large preliminary manuscript for Basic Laws. It was
in this period that he published, in the early 1890s, his three best known papers ‘Function and Concept’ (Funktion und Begriff), ‘On Sense and Reference’ (Über Sinn und Bedeutung) and ‘On Concept and Object’ (Über Begriff und Gegenstand) (all in Frege 1969). All three of these are now regarded as classic works in the philosophy of language, and the second, in particular, must be read by anyone who wishes to understand twentieth-century analytic philosophy at all, but their importance for Frege was that they set out the changes in his views from the time of Foundations and prepared their readers for Basic Laws.

In this period, also, notice began to be taken notice of Frege’s works when the Italian logician Peano cited them in print and Husserl began to correspond with Frege.

With volume 1 of Basic Laws written, Frege should now have been able to look forward to its publication and the recognition his work had for so long gone without. However, so poorly had his previous work been received that no publisher would print the lengthy manuscript as a whole. Frege eventually got an agreement from Hermann Pohle of Jena, who had published ‘Function and Concept’, to publish it in two volumes, with the publication of the second volume being conditional on the success of the first. In this way volume 1 was eventually printed in 1893.

Frege evidently anticipated that his work was likely, once more, to fail to gain the recognition it deserved. He acknowledged that:

An expression cropping up here or there, as one leafs through these pages, may easily appear strange and create prejudice... Even the first impression must frighten people off: unfamiliar signs, pages of nothing but alien looking formulas... I must relinquish as readers all those mathematicians who, if they bump into logical expressions such as 'concept', 'relation', 'judgement', think: metaphysica sunt, non leguntur, and likewise those philosophers who at the sight of a formula cry: mathematica sunt, non leguntur; and the number of such persons is surely not small. Perhaps the number of mathematicians who trouble themselves over the foundations of their science is not great, and even those frequently seem to be in a great hurry until they have got the fundamental principles behind them. And I scarcely dare hope that my reasons for painstaking rigour and its inevitable lengthiness will persuade many of them. (Frege 1964: xi-xii)

For this reason Frege made great efforts to make his work more accessible to his readers. He gave hints in the Preface as to how to
read the book to achieve a speedy understanding and in the text he
prefaced his proofs with rough outlines to bring out their signifi-
cance. He also attempted to provoke other scholars to respond to
his work by attacking rival theories.

It was all to no avail. Volume 1 of Basic Laws received just two
reviews, both unfavourable, one of only three sentences, and was
otherwise ignored. As a result the publisher refused to publish the
remainder of Frege's work and volume 2 eventually had to be pub-
lished a decade later by Frege at his own expense.

Nevertheless, publication of volume 1 at least led to an improve-
ment in Frege's material circumstances, with his promotion in 1896
to the rank of Honorary Ordinary Professor. This was unsalaried
but without administrative duties. Frege was able to accept this post
because he was offered a stipend from the Carl Zeiss Stiftung,
founded and sustained by his mentor Ernst Abbe. Consequently
Frege now had more time for his research, and in the decade pre-
ceding the publication of volume 2 of Basic Laws engaged in corre-
spondence with a variety of scholars, and published a number of
articles and reviews of other authors as well as carrying forward his
work on the Basic Laws.

One of the scholars Frege corresponded with in this period was
Peano, who had written the longer of the two reviews of volume 1
of Basic Laws The review started an exchange of views and led
Peano to make modifications in his logical symbolism (Frege's cor-
respondence with Peano is published in Frege 1980: 108–29; his two
pieces explaining the superiority of his logical notation to that of
Peano are published in Frege 1980: 112–18). Another fateful result
of Frege's coming to the notice of Peano was that Russell, who
adopted Peano's notation, learned of his work. As Russell himself
tells the story (Russell 1959: 65):

I did not read [the Begriffsschrift] . . . until I had independently
worked out a great deal of what it contained . . . I read it in 1901 . . . .
What first attracted me to Frege was a review of a later book of
his by Peano [Peano's review of volume 1 of the Grundgesetze] accus-
ing him of unnecessary subtlety. As Peano was the most subtle
logician I had at that time come across, I felt that Frege must be
remarkable.

Apart from Peano, another scholar on whom Frege had some
influence during this period prior to the publication of the second
volume of Basic Laws was Husserl, the founder of the continental
phenomenological school. Husserl began as a disciple of Brentano and an advocate of psychologism (the attempt to base logic and arithmetic on psychology). In 1891 he published the first volume of his Philosophy of Arithmetic. This contained criticisms of Frege and Frege responded in 1894 with a scathing review. Husserl was converted from psychologism and became henceforth its strong opponent, developing the notion of the noema of an act of thought, which corresponds to, but is intended to generalize, Frege’s notion of sense.

Thus, although his own work was still neglected, Frege could at least take comfort from the fact that he was now known and respected by some of the most eminent scholars of the day, and look forward to a better reception for volume 2 of Basic Laws. Despite the neglect of his work, he himself never doubted its achievement. In the final paragraph to the Preface of volume 1 of Basic Laws he raises the question of the possibility of someone deriving a contradiction in his system, but dismisses it with total confidence:

It is prima facie improbable that such a structure could be erected on a base that was uncertain or defective. . . . As a refutation in this I can only recognize someone’s actually demonstrating either that a better, more durable edifice can be erected upon other fundamental convictions, or else that my principles lead to manifestly false conclusions. But no one will be able to do that. (1964: xxvi)

Disaster struck in the form of a modestly expressed letter from Russell which arrived in June 1902, as the second volume of Basic Laws was in press. Russell’s letter pointed out that the contradiction now known as ‘Russell’s Paradox’ was derivable in Frege’s logical system. After expressing his admiration for Frege’s work and his substantial agreement, Russell writes:

I have encountered a difficulty only on one point. You assert (p.17) that a function could also constitute the indefinite element. This is what I used to believe, but this view now seems to me dubious because of the following contradiction: let w be the predicate of being a predicate which cannot be predicated of itself. Can w be predicated of itself? From either answer follows its contradictory. We must therefore conclude that w is not a predicate. Likewise, there is no class (as a whole) of those classes which, as wholes, are not members of themselves. From this I conclude that under certain circumstances a definable set does not form a whole. (Frege 1969: 130-1)
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Frege recognized at once the seriousness of the difficulty Russell had explained and identified Basic Law (V) as its origin. He wrote back to Russell:

Your discovery of the contradiction has surprised me beyond words and, I should almost like to say, left me thunderstruck because it has rocked the ground on which I meant to build arithmetic. It seems accordingly that the transformation of the generality of an equality [Gleichheit] into an equality of value ranges is not always permissible, that my law (V) is false, and that my explanations do not suffice to secure a reference [Bedeutung] for my combination of signs in all cases. I must give some further thought to the matter. It is all the more serious as the collapse of my law (V) seems to undermine not only the foundations of my arithmetic but the only possible foundations of arithmetic as such. (Frege 1969: 132–3)

Frege attempted to develop a response to the paradox and published an amendment in an appendix to volume 2 of Basic Laws. However, the amended system can also be proved to be inconsistent (see Quine 1955; Geach 1956) and although it is unclear when Frege finally accepted that his work had been fatally undermined, the third volume of Basic Laws was never published and at the end of his life Frege admitted that his logicist programme had been a failure, and attempted in his last years to found arithmetic on geometry.

After the discovery of the paradox Frege was now to suffer personal tragedy. His wife died, leaving him to bring up his adopted son, Alfred, on his own. He published little following this, apart from several articles on the foundations of geometry which arose from his correspondence with Hilbert before the disclosure of Russell’s Paradox, and three articles against ‘formalist’ arithmetic in response to an attack by his Jena colleague Johannes Thomae. However, Frege did engage in extensive correspondence during this period and continued his lectures at Jena. And this was the time that he met Wittgenstein, who wrote to him after reading an account of his views in Russell’s Principles of Mathematics. The correspondence led to a meeting and as well as discussing his own views with Wittgenstein Frege also made the suggestion that Wittgenstein should go to Cambridge to study with Russell.

It was also during this period that Rudolf Carnap attended Frege’s lectures. Like Wittgenstein, Carnap greatly admired Frege’s work and developed and disseminated his ideas when he subsequently became influential.
Frege retired from lecturing in 1918 and moved from Jena to Bad Kleinen, near his home town of Wismar. He did not cease working and appears to have gained renewed vigour in this later period. At any rate he wrote a series of papers, of which the first, 'Thoughts' ('Der Gedanke'), has had more influence and attracted more discussion than any of Frege's papers apart from 'On Sense and Reference'. During this time, too, Frege came to believe that arithmetic must have a geometrical foundation. In a piece entitled 'Numbers and Arithmetic' written in the last year of his life he wrote:

The more I have thought the matter over, the more convinced I have become that arithmetic and geometry have developed on the same basis — a geometrical one in fact — so that mathematics in its entirety is really geometry. Only on this view does mathematics present itself as completely homogeneous in nature. Counting, which arose psychologically out of the demands of practical life, has led the learned astray. (Frege 1979: 275–7)

Thus Frege at last abandoned the view he had held ever since his first publication, that arithmetic, unlike geometry, was a source of a priori knowledge requiring no foundation in intuition.

However, Frege did not have the time left to pursue his new ideas and he died in 1925, aged seventy-seven, before he was able to know of the widespread influence his work was to have.

He bequeathed his unpublished writings to his son Alfred with the following note attached (now printed in Frege 1979: ix):

Dear Alfred,

Do not despise the pieces I have written. Even if all is not gold, there is gold in them. I believe there are things which will one day be priced much more highly than they are now. Take care that nothing gets lost.

Your loving father.

It is a large part of myself that I bequeath to you herewith.

Alfred handed over Frege's papers to Heinrich Scholz of the University of Münster in 1935. Unfortunately the originals were destroyed by Allied bombing during the war. However, copies had been made of most of the important pieces and eventually, after a long delay, due to Scholz's own illness and death, they were published in German in 1969 and in English in 1979. Meanwhile Frege's correspondence was edited and published in German in 1976 and in English (in an abridged edition) in 1980.
The Origin and Development of Frege's Philosophy

It is clear that from the start of his career Frege was interested in seeking a foundation for arithmetic. How important he took it to be for a mathematician to be clear about the fundamentals of his subject is made very obvious in a harsh review, his first publication after his appointment as Privatdozent at Jena, of a book on The Elements of Arithmetic by one H. Seager (in Frege 1984). Frege writes:

After some particularly unfortunate explanations of the calculating operations and their symbols, some propositions are presented in the second and third chapters under the title of 'the fundamental theorems and most essential transformation formulas'. These propositions, which actually form the foundation of the whole of arithmetic, are lumped together without proof; while, later, theorems of a much more limited importance are distinguished with particular names and proved in detail . . . The amplification of concepts which is so highly important for arithmetic, and is often the source of great confusion for the student, leaves much to be desired . . . The result of all these deficiencies will be that the student will merely memorize the laws of arithmetic and become accustomed to being satisfied with words he does not understand.

Whether it was reading Seager's book that stimulated in Frege the ambition to set arithmetic on pure logical foundations we do not know. But we will understand this project better if we place it in the philosophical and mathematical context of his time. In particular, it will be illuminating to look briefly at the links between Frege's project and the Kantian doctrine of the synthetic *a priori* and the associated notion of pure intuition; the development of non-Euclidean geometry; and the arithmetization of analysis.

For Kant mathematics was an epistemological puzzle, combining two apparently irreconcilable features: necessity and substantiality. Mathematical propositions seem to state truths that could not be otherwise. But at the same time they appear to represent genuine extensions of our knowledge. In this respect, Kant thought, they were like the maxim of universal causation, that every event has a cause, whose problematic status Kant's predecessor Hume, the great British empiricist, had brought to his attention. Hume had operated with a dichotomy, between relations of ideas and matters of fact, which did not allow any place for a proposition of this character. Thus he claimed that the causal maxim was, in fact, a merely
contingently true statement of a matter of fact and that our ascribing to it the character of a necessary truth was a mistake whose psychological origin he took it to be one of his principal achievements to have explained. Kant would not accept this, however, but neither was he willing to accept that the causal maxim merely expressed a Humean relation of ideas, and so, like the proposition that every effect has a cause, was trivially true in virtue of its meaning. Both in this case and the case of mathematics, Kant thought, what we had to acknowledge was the existence of propositions which fell on neither side of Hume’s dichotomy.

Kant discusses this problem in *The Critique of Pure Reason* (1781) within the framework of a pair of distinctions: (i) between *a priori* and *a posteriori* knowledge and (ii) between analytic and synthetic judgements. He explains the first term of the first distinction as follows:

We shall understand by *a priori* knowledge, not knowledge independent of this or that experience, but knowledge absolutely independent of all experience. (Kant 1929 A2/B3: 43)

*A posteriori* knowledge, then, is knowledge that does require experience.

Kant’s second distinction he explains as follows:

Either the predicate B belongs to the subject A, as something which is (covertly) contained in this concept A, or B lies outside the concept A, although it does indeed stand in connection with it. In the one case I entitle the judgement analytic, in the other synthetic. (Kant 1929 A6/B10: 48)

In an analytic judgement, Kant says, in thinking the subject term one thinks the predicate term, so no new knowledge can be expressed in an analytic judgement. He illustrates this distinction with the following example:

Analytic judgement: All Bodies are Extended
Synthetic judgement: All Bodies are Heavy

Thus for Kant, there are four possible categories of judgement. The synthetic *a posteriori*, the synthetic *a priori*, the analytic *a posteriori* and the analytic *a priori*. The first category, illustrated by the judgement ‘All Bodies are Heavy’ is unproblematic, as is the fourth,
Illustrated by the judgement ‘All Bodies are Extended’, the third category is unproblematically empty. For Kant it is the second category, the synthetic a priori, which is of interest. It is in this category that he places the causal maxim and mathematical propositions, as extending our knowledge – since the concept of the predicate is not thought in thinking the concept of the subject – and at the same time as necessary and universally true and, therefore, knowable independently of the contingencies of particular features of our experience.

Kant’s argument for the synthetic a priori character of mathematics is clearly expressed in his Prolegomena (1783) (Kant 1959). First he argues:

Properly mathematical propositions are always judgements a priori, and not empirical, because they carry with them necessity, which cannot be taken from experience. (1959: 18–19)

Next he argues, illustrating his point with his favourite mathematical proposition, that \( 7 + 5 = 12 \) is synthetic because twelve can never be found in the analysis of the sum of seven and five:

The concept of twelve is in no way already thought merely by thinking this unification of seven and five, and though I analyse my concept of such a possible sum as long as I please, I shall never find the twelve in it. We have to go outside these concepts and with the help of the intuition which corresponds to one of them, our five fingers for instance, (or as Segner does in his Arithmetic) five points, add to the concept of seven, unit by unit, the five given in intuition. Thus we really amplify our concept by this proposition \( 7 + 5 = 12 \), and add to the first concept a new one which was not thought in it. That is to say, arithmetical propositions are always synthetic, of which we shall be the more clearly aware if we take rather larger numbers. For it is then obvious that however we might turn and twist our concept, we could never find the sum by means of mere analysis of our concepts without seeking the aid of intuition. (1959: 19–20)

Kant thinks that the same is true of geometrical truths, e.g. that a straight line is the shortest distance between two points:

That the straight line between two points is the shortest, is a synthetic proposition. For my concept of straight contains nothing of quantity,