Classic Texts in the Sciences

Gerolamo Saccheri

Euclid Vindicated from Every Blemish

Edited and Annotated
by Vincenzo De Risi

Translated by G. B. Halsted
and L. Allegri

Birkhäuser

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Foreword to the English Edition

This book is the English translation of my annotated edition of Saccheri's *Euclides vindicatus*, published in Italian for the 'Edizioni della Normale' in 2011. With respect to the Italian edition, I have corrected some minor mistakes and typos, added some bibliographical references and deleted those that may have no interest for the English-speaking reader.

I have not re-translated Saccheri into English (a difficult task that awaits a native English-speaking scholar), and have based the present edition on Halsted's classical translation which first appeared in volume form in 1920 and was subsequently reprinted several times. Halsted's translation received some favorable reviews by R.C. Archibald and A. Emch ("Amer. Math. Monthly", 1921; "Bull. Amer. Math. Soc.", 1922) and harsh criticisms by T. Heath ("Nature", 1922). It seems to me fairly good, and Halsted's geometrical skills allowed him to produce a text that contains no significant mathematical misunderstandings. Today his language may sound old-fashioned, but it mimics Saccheri's own baroque and cumbersome Latin very well, which is almost impossible to translate into modern, elegant prose. Nonetheless, I have taken the liberty of modifying and correcting Halsted's version at some places. Besides correcting a few minor mistakes noticed by Heath, and a pair of mathematical imprecisions, my changes mostly regard the choice of terminology, some of which I have justified in my commentary. The most obvious modification is in the title of Saccheri's work whose translation varies largely depending on the translator or interpreter in question. In the *Introduction*, I have tried to justify the preservation of Euclid 'vindicated' rather than the more common 'emended' or 'corrected' (or Halsted's 'freed'), which do not seem to reflect the author's intentions. I prefer the term 'blemish' to other more abstract terms (such as 'error' or 'flaw'), given that Savile's original text makes use of an explicit corporal metaphor ("two blemishes, two moles in the most beautiful body of geometry").

Halsted's translation, in any case, covered only the First Book of *Euclid Vidicated*. A complete English translation of the Second Book is to be found in the doctoral dissertation of Linda Allegri (Columbia University, 1960) which was intended to complement Halsted's work. I publish Allegri's text here which, however, required a larger number of corrections and revisions. To my knowledge, this is the first complete English edition of Saccheri's masterwork.

My *Introduction* and *Notes* have been translated by Marco Santi and Caterina Benincasa, whose painstaking efforts deserve my deepest gratitude. I also thank Rebecca Rothfeld and Chiara Fabbrizi for the editing of the texts.

I would also like to express my thanks to Massimo Mugnai, Mariano Giaquinta and Paolo Freguglia for their invaluable help with the original Italian edition; to Massimo Mugnai and Massimo Girondino for providing me with their edition of Saccheri's *Logica* when it was still in proofs; to Marvin J. Greenberg, Victor Pambuccian, and Roshdi Rashed for their important suggestions on how to improve the English edition; to the many scholars who commented on my talks on Saccheri in several workshops organized by Arianna Betti, Michael Detlefsen and Roshdi Rashed.

I am grateful to my colleagues and friends at the Max Planck Institute for the History of Science for their support and advice, to the Library of the Institute for providing me with a digitalization of the Latin text of Saccheri and to the Max Planck Society for financial support.

Finally, and most of all, I thank Jürgen Jost for making this edition possible and for proposing that I publish it in this Series. I also thank the 'Edizioni della Normale' for allowing this translation. The present volume will soon be complemented (in the same Series) by a similar English edition of Lambert's *Theorie der Parallellinien*.

Berlin, January 2014

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Au commencement de la géométrie, on dit: “On donne le nom de parallèles à deux lignes qui, prolongées à l’infini, ne se rencontrent jamais.” Et, dès le commencement de la *Statique*, cet insigne animal de Louis Monge a mis à peu près ceci: *Deux lignes parallèles peuvent être considérées comme se rencontrant, si on les prolonge à l’infini*. Je crus lire un catéchisme et encore un des plus maladroits. Ce fut en vain que je demandai des explications à M. Chabert. Le monstre, s’approchant de son tableau en toile cirée et traçant deux lignes parallèles et très voisines, me dit: “Vous voyez bien qu’à l’infini on peut dire qu’elles se rencontrent.” Je faillis tout quitter. Un confesseur, adroit et bon jésuite, aurait pu me convertir à ce moment en commentant cette maxime: “Vous voyez que tout est erreur, ou plutôt qu’il n’y a rien de faux, rien de vrai, tout est de convention.”

(Stendhal, *Vie de Henry Brulard*)

Introduction

1. Euclid and Saccheri

Euclid Vindicated from Every Blemish, by the Jesuit mathematician Gerolamo Saccheri,[1] appeared in print in Milan in 1733. The work enjoyed little success in the eighteenth century, was completely forgotten during the following century, and was rediscovered

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[1] Saccheri was born in San Remo (a city under the dominion of Genoa) in 1667. In his youth, he lectured on grammar, philosophy, and theology at the Jesuit colleges in Cremona, Milan, Turin and Pavia. In 1699, he was appointed to the chair of mathematics at the university of Pavia, which he held till his death in 1733. It is known that he displayed outstanding mathematical abilities from an early age, and that he enjoyed the highest esteem in the Italian scientific circles of his time. He had close personal ties to the mathematicians Guido Grandi and Tommaso and Giovanni Ceva; we also possess letters that he exchanged with Vincenzo Viviani. He did not authored many scientific texts, and all sources report that his uncommon laziness as a writer left for later ages a less vivid impression of his intelligence and thought than his acquaintances had of him. Before *Euclid Vindicated*, his major work, Saccheri had published certain *Quaesita Geometrica* in 1693, an important *Logica Demonstrativa*, which appeared anonymously in 1696–1697 and was reprinted several times, and a *Neo-Statica* in 1708; he had also published a few theological works. Many details of Saccheri's life may be found in the biography written (in Italian) by Count Francesco Gambarana between 1733 and 1739. This text, of which we have two slightly different versions, remained manuscript and was recently published as an Appendix to the Italian edition of *Logica Dimostrativa*, ed. by M. Mugnai and M. Girondino, Pisa, Edizioni della Normale 2012. A previous account of Saccheri's life is that of A. Pascal, *Girolamo Saccheri nella vita e nelle opere*, "Giornale di Matematiche di Battaglini", 52, 1914, pp. 229–51; cf. also H. Bosmans, *Le géomètre Jérome Saccheri, S.J. (1667–1733)*, "Revue des Questions Scientifiques", 7, 1925, pp. 401–30.

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and circulated only in the early nineteen hundreds. Today it is rather well-known, at least in outline, and is usually considered to be the birthplace of research on non-Euclidean geometry. The strategy of *Euclid Vindicated* is also widely regarded as one of the largest misunderstandings in the whole history of mathematics – and the most felicitous error in eighteenth-century geometry – as Saccheri's intention was in fact to demonstrate Euclid's Fifth Postulate, the parallel axiom, and thus to prove the impossibility of the very non-Euclidean geometries of which he is today regarded as the father. He undertook to prove the Fifth Postulate *per absurdum* and sought to spot a contradiction in the vast geometric theory that he constructed, for the first time in history, on the negation of the Euclidean axiom – a geometric theory that we nowadays identify without doubt as a genuine and well-structured system of hyperbolic geometry. Saccheri did not find the supposed contradiction, as it was nowhere to be found, but he was unable to convince himself that the new geometry he had erected might in fact be a reasonable alternative to Euclid's *Elements* rather than a green-eyed monster: consequently, he pointed to a contradiction of his own making, and thereby proved himself to be nothing more than a Jesuit.[2] This effort notwithstanding, the sacrilege, so to speak, had already been committed, and Saccheri's outstanding achievements towards the construction of hyperbolic geometry, while disowned by their author and relegated to a book printed in quite few copies, sneaked into European mathematical culture and poisoned the minds of certain more acute, unprejudiced, or simply more modern geometers. One century after the Jesuit's death, these scholars eagerly welcomed Saccheri's 'monster' in their writings, thus celebrating the triumph of non-Euclidean geometry. Following this widespread story, Saccheri unwittingly (yet brilliantly) anticipated one of the most momentous conceptual revolutions in the genesis of contemporary mathematics.

Instead of following this approach and treating Saccheri as an unwitting innovator, it will be worthwhile to try and place *Euclid Vindicated* in the history of the mathematics of its age. This is not to deny that its significance extends beyond its time: it is indisputable that Saccheri's results on the theory of parallelism are extraordinarily more advanced than any previous treatment of the subject, and that he was able to manufacture, in almost perfect isolation, a construction of exceptional amplitude and depth. Rather, it is for just this reason that we need to inquire how, where and why a work with intents so classical as to appear reactionary, and contents so novel as to start a revolution, could emerge at the beginning of the eighteenth century.

The structure and aim of Saccheri's book are indeed completely classical, and the work is firmly embedded in a long tradition of commentary and elaboration on Euclid's *Elements*

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[2] The expression comes from Valéry, who levels the same charge of priestly pusillanimity against Saccheri and Cavalieri, as the latter, with his theory of indivisibles, almost arrived at the genuine theory of infinitesimal calculus – only to fail in the end. Cf. P. VALÉRY, *Mélanges*, in *Oeuvres*, ed. by J. Hytier, Paris, Gallimard 1957, vol. 1, p. 473. At any rate, Cavalieri belonged to the Jesuates (a mendicant order suppressed in 1668), not the Jesuits.

dating back to the Renaissance. This tradition represents one of the most precious fruits of sixteenth- and seventeenth-century mathematics. The way for Saccheri had been paved by the great sixteenth-century commentaries on Euclid (most notably, that of Clavius), and the tradition had subsequently consolidated through the production of texts for the study and teaching of elementary geometry, especially in Italy. These works began as commentaries on the *Elements*, but tended to transform gradually into autonomous treatises on certain controversial issues in the classical text. Most of these commentaries endeavored to reconstruct Euclid's intentions, and in certain cases did so with the help of philological tools. Other commentators openly distanced themselves from the ancient mathematician's solutions and aimed at producing a clearer or more rigorous foundation for the first elements of geometry. Thus, before Saccheri's *Euclides vindicatus,* we find in Italy alone a *Euclides restitutus* by Borelli (1658), a *Euclides adauctus et methodicus* by Guarini (1671), a *Euclide restituto* by Giordano (1680) and a *Euclides reformatus* by Marchetti (1709), along with countless other works with different titles and similar contents.[3] In fact, Saccheri's *Euclid Vindicated* represents in many ways the highest point and the ripest fruit (almost rotten, to be sure) of that direction of studies: not only because its mathematical content is undoubtedly more relevant than that of the older treatises, but also because it has assimilated their results to such extent that its theory stands on the shoulders of the whole tradition of sixteenth- and seventeenth-century commentaries on Euclid. Even the structure of *Euclid Vindicated* is autonomous from the partitions of the Euclidean text: Saccheri's book presents itself as a monographic treatment of the unconnected problems that still plagued would-be interpreters of the *Elements*.

Saccheri groups such problems under two headings, that of parallel theory (for which *Euclid Vindicated* is still famed) and that of the theory of proportions. His 1733 work is consequently divided into two books addressing these two topics, respectively. This arrangement was not at all unusual, as it was not out of the ordinary for a seventeenth-century geometer to indicate exactly the Fifth Postulate and the definitions of 'ratio' and 'proportion' as the two parts of elementary mathematics most in need of clarification and foundation. In 1621, Henry Savile described these difficult subjects as "two blemishes, two moles" spoiling Euclid's "beautiful body". Indeed, Savile even went so far as to establish a mathematics chair

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[3] The works just mentioned, which certainly represent the closest precedents for the *title* of Saccheri's work, differ consistently as to their intrinsic values. Guarini's book, for instance, is a ponderous and verbose commentary on Euclid in the Scholastic style and contains very little mathematics of any import. The work of Borelli, and, building on this, those of Giordano and Marchetti, are of great mathematical significance, and we will discuss them in the next sections. The title of Borelli's book was itself probably derived from the term and concept of 'restitution' as it was employed in a previous edition of Euclid published by F.F. CANDALE, *Euclidis Megarensis Mathematici clarissimi Elementa Geometrica, libris XV. ad germanam Geometriae intelligentiam è diversis lapsibus temporis iniuria contractis restituta*, Paris, Royer 1566.

in Oxford designed to produce the research needed to wash the ugly blemishes away.[4] Half a century later, in 1663, John Wallis, the Savilian Professor of Geometry, wrote a short treatise on these two topics, which he took to present a solution to all the difficulties associated with them. Wallis concluded that he had accomplished the task assigned to him by his remote benefactor: in his words, he had finally "vindicated" Euclid.[5]

It is therefore with one eye towards the Italian tradition of commentaries on Euclid, and the other towards the English conceptions of the two major foundational problems of elementary geometry, that Saccheri baptizes his *Euclid Vindicated*, which is hence firmly situated in the mathematical research of its age. Let us now briefly outline the state of the theories of the parallels and of proportions in Saccheri's era.

2. The theory of parallels in the seventeenth century

The Fifth Postulate in Book I of Euclid's *Elements* affirms that if two straight lines that are both intersected by a third form with it two interior angles (on the same side) whose sum is less than π (two right angles), then the two lines, if extended, will end up meeting (on that side). We know of early attempts to prove this postulate in Classical Antiquity. In fact, these attempts probably preceded the composition of the *Elements*, suggesting that Euclid perhaps assumed his Fifth Postulate unwillingly, because he could not devise the proof for it that he sought. Nowadays, we ignore the methods, developments, and results of these ancient discussions almost completely; perhaps we even ignore their aim, because the mathemati-

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[4] Sir Henry Savile (1549–1622), English philologist and humanist most noted for his contribution to the *King James Version* of the Bible. The Oxford professorships for geometry and astronomy that he instituted in 1619 have yielded an incredible scientific progeny and are still active to this day. In his *Praelectiones tresdecim in principium elementorum Euclidis*, Oxford, Lichfield&Short 1621, he wrote: "In pulcherrimo Geometriae corpore duo sunt naevi, duae labes, nec, quod sciam, plures, in quibus cluendis & emaculandis, cùm veterum tùm recentiorum, ut postea ostendam, vigilavit industria. Prior est hoc postulatum [the Fifth Postulate], posterior pertinet ad compositionem rationum" (lect. VII, p. 140).

[5] John Wallis (1613–1703), great English mathematician, was Savilian Chair at Oxford from 1649 until his death. The work in question is *De Postulato Quinto; & Quinta Definitione Lib.* 6 *Euclidis; Disceptatio Geometrica*, which is the transcription of two lectures given by Wallis on February 1551 and then on July 11th, 1663. It was published in 1693 as an appendix to *De Algebra tractatus cum variis appendicibus*, and in J. WALLIS, *Opera mathematica*, Oxford, Theatrum Sheldonianum 1693–1699, vol. 2, pp. 665–78; repr. Olms 1972, ed. by C.J. Scriba (we will always refer to this edition). *De Postulato Quinto* ends with the words: "Atque haec, in Euclidis vindicias, sufficiant". A comprehensive introduction to Wallis' mathematical activity is L. MAIERÙ, *John Wallis. Una vita per un progetto,* Catanzaro, Rubbettino 2007.

cal epistemology underpinning them is quite obscure to us.[6] At any rate, these attempts to prove the postulate went on uninterrupted throughout the following centuries, originating a tradition of foundational studies in geometry that, to be sure, pursued different aims (as well as methods) in different ages. This tradition, which is partially known to us, begins with the immediate heirs of Euclid in the Hellenistic age (Posidonius, Geminus), continues through the Roman Imperial age (Ptolemy) to Late Antiquity (Proclus, Simplicius), and disappears completely in the Christian Middle Ages, along with almost the entirety of geometrical studies. The tradition, however, survives and prospers in the Islamic Middle Ages through scores of studies of a very high quality (Thābit ibn Qurra, An-Nayrīzī, Avicenna, Al-Haytham, Khayyām, Nasīr ad-Dīn), whence it returns to Europe with a handful of Late Medieval and Early Humanistic treatises (Witelo, Gersonides, Alfonso). It finally blooms in the Renaissance, with the mature sixteenth- and seventeenth-century research to which Saccheri's work is immediately tied. A detailed retelling of this story, which has been told very well many times, does not belong here,[7] but some of the results, difficulties, and per-

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[6] The level of sophistication of these Ancient studies might have indeed rivaled the elegance and rigor of the *Elements*: cf. I. Mueller, *Remarks on Euclid's Elements* I*, 32 and the Parallel Postulate*, "Science in Context", 16, 2003, pp. 287–97. The somewhat old but classic – and groundbreaking – study on this subject is M. Dehn, *Beziehungen zwischen der Philosophie und der Grundlegung der Mathematik im Altertum*, "Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik", B4, 1938, pp. 1–28. I find it problematic, however, to interpret the attempted proofs of the Fifth Postulate in terms of non-Euclidean developments in the geometry of Classical Antiquity, which is the famous thesis put forward in I. Tóth, *Das Parallelenproblem im Corpus Aristotelicum*, "Archive for History of Exact Sciences", 3, 1967, pp. 249–422; also see I. Tóth, *Fragmente und Spuren nicheuklidischer Geometrie bei Aristoteles*, Berlin, DeGruyter 2010; and I. Tóth, *Aristotele e i fondamenti assiomatici della geometria. Prolegomeni alla comprensione dei frammenti non-euclidei nel "Corpus Aristotelicum" nel loro contesto matematico e filosofico*, Milano, Vita e Pensiero 1997.

[7] Today, many very valuable texts on the history of non-Euclidean geometries are available. The older classic, somewhat aged but still an excellent read, is R. Bonola, *La geometria non-euclidea. Esposizione storico-critica del suo sviluppo*, Bologna, Zanichelli 1906; translated into English by H.S. Carslaw as *Non-Euclidean Geometry*, Chicago, Open Court 1912; and anticipated by R. Bonola, *Sulla teoria delle parallele e sulle geometrie non-euclidee*, in *Questioni riguardanti le matematiche elementari*, ed. by F. Enriques, Bologna, Zanichelli 1924–1927 (1900^1), Book I, vol. 2, pp. 309–427. The most extensive work on the subject is currently J.-C. Pont, *L'aventure des parallèles. Histoire de la géométrie non euclidienne: précurseurs et attardés*, Berne, Lang 1986. Equally notable is B.A. Rosenfeld, *A History on Non-Euclidean Geometry. Evolution of the Concept of a Geometric Space*, transl. from the Russian by A. Shenitzer, New York, Springer 1988, which presents much interesting material, especially in the part devoted to Arabic-Persian mathematics. On the latter topic, however, the indispensable survey is still K. Jaouiche, *La théorie des parallèles en pays d'Islam*, Paris, Vrin 1986. Many mathematical handbooks of hyperbolic geometry also contain similar, usually shorter, historical overviews of the research on parallels. Finally, I may refer to the relevant section in the well-known bibliography P. Riccardi, *Saggio di una Bibliografia Euclidea*, Hildesheim, Olms 1974 (1887–1892^1), which is now quite dated and very incomplete, but nonetheless provides a good starting point for studying the Early Modern tradition of foundational work in elementary geometry. See also the more specific D.M.Y. Sommerville, *Bibliography of Non-Euclidean Geometry including the Theory of Parallels, the Foundations of Geometry, and Space of n Dimensions*, London, Harrison 1911.

sonalities that contributed in a special way to the genesis of *Euclid Vindicated* are worth mentioning. First, we consider some more remote discussions, which precede and, as it were, prepare the Renaissance studies.

The oldest, most natural, and most important of the misunderstandings interspersed throughout the history of the efforts to prove the Fifth Postulate is directly tied to the definition of parallel straight lines. Euclid's definition characterizes them as straight lines lying on the same plane and not intersecting. The definition had long been a subject of discussion and at times of criticism, since, firstly, the concept of non-intersection, i. e., of two straight lines that do not meet even if in(de)finitely extended, seemed to involve an illegitimate reference to the obscure concept of infinity. Secondly, the definition did not seem to present a distinctive property of parallelism as such, since there are asymptotic curves, which never intersect and which nonetheless no one wanted to qualify as parallels. Already Posidonius (first century B.C.) had therefore defined parallels as equidistant straight lines,[8] which appeared as a more complete and intuitive characterization than non-intersection. It is likely that this was the accepted definition even before the composition of the *Elements*; at any rate, it enjoyed exceptional success in the Medieval and Modern times. The problem, however, is that the existence and possibility of such equidistant parallel straight lines requires a demonstration. Though Euclid had proved the existence of his non-intersecting parallels (in *Elements* I, 31, which is independent of the Fifth Postulate), thereby establishing the possibility of his definition, he had not established the existence of equidistant straight lines, which he never considers. As a matter of fact, the existence of such equidistant straight lines can only be proved by assuming the Fifth Postulate itself: in hyperbolic geometry, the line that is in any point equidistant from a given straight line is not itself straight (but rather a curve, a *hypercycle*). Nevertheless, the clear and intuitive grasp of parallelism as equidistance, and the confusion about Euclid's existential proof (which was expected to remain valid even though the definition had changed), tied the intricate knot. Thus, there were many mathematicians who did not challenge the intrinsic possibility of equidistant straight lines, but rather just defined parallels as equidistant, assumed their existence, and then undertook a (perhaps difficult, but nevertheless possible) demonstration of the Fifth Postulate.

This *petitio principii*, of course, was uncovered early on – though not universally acknowledged, to be sure – so that the first and original paralogism in the history of the demonstrations of the parallel axiom evolved, in the mindset of some geometers, into a proof strategy: to prove the Fifth Postulate, one first had to show that the line equidistant from a straight line is itself straight. This argumentative path was very difficult, anyway, as no good definition of straight lines was available – Euclid's definition, as is well known, is so ambiguous as to be almost unserviceable – nor was it easy to find one. Thus, the history of proofs of the Fifth Postulate went for a long time hand in hand with that of the definitions of straight lines.

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[8] See Proclus Diadochus, *In primum Euclidis*, 176 (ed. by G. Friedlein, Leipzig, Teubner 1873). Posidonius of Apamea (ca. 135–50 B.C.) is the eminent Stoic philosopher. Cf. also Hero, *Definitiones* 70 (note, however, that Hero is talking about equidistant *lines*, γραμμαί, not about equidistant *straight lines*).

Beginning in the Middle Ages, there was yet another hurdle: the incorrect but very tempting argument involving the employment of motion in geometry. Let the end point of a line segment slide continuously along a straight line, and let the segment remain perpendicular to the line while moving: then the other end point will trace out a line, undoubtedly equidistant from the given straight line; and it was evident to all that the line generated in this way could not be other than a straight line itself. To be sure, the evident certainty attained in this way is not very different from that of the original proposition, that the line equidistant from a straight line is straight; but for many centuries the genetic character of this procedure deceived countless mathematicians, who could not explain *how* such a simple and uniform movement might generate a curve – in fact, it would be hard for anyone lacking an account of curvature, that is, anyone before Gauss, to provide such explanation.[9]

So it happened that most of the attempts to prove the Fifth Postulate in Ancient times (first of all that of Proclus, much celebrated by the Renaissance),[10] in the Middle Ages (virtually all of the Arabic demonstrations), in the Renaissance, and even long after Saccheri, were doomed by their appeal to equidistant lines. This happened either because their authors did not notice that the possibility of such lines was in want of a demonstration, because they

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[9] Arguably, the creator of the proof from motion was the Arab physician, mathematician and astronomer Thābit ibn Qurra (836–901). The argument was probably the most difficult to refute in the history of the alleged demonstrations of the Fifth Postulate – unless one banishes the use of kinematic concepts in geometry altogether, indeed a rather common option. Saccheri himself, who was among the few to dispute this proof, ran into a similar difficulty in Scholium 2 to Proposition 37 of *Euclid Vindicated*: he claimed that the line drawn by the end point of the moving segment must be, if not straight, at least of the same length as the underlying line (which is again false in hyperbolic geometry). Thābit's two works on the theory of parallels are translated in French in Jaouiche's above-mentioned edition, as well as in R. Rashed, C. Houzel, *Thābit ibn Qurra et la théorie des parallèles*, "Arabic Science and Philosophy", 15, 2005, pp. 9–55, which also offers a long commentary. An English edition of the text is in A.I. Sabra, *Thābit Ibn Qurra on Euclid's Parallels Postulate*, "Journal of the Warburg and Courtauld Institutes", 31, 1968, pp. 12–32.

[10] Proclus of Lycia (412–485), the great Neoplatonist philosopher, wrote an important commentary on Book I of Euclid's *Elements* that is one of the primary sources of our knowledge of Ancient mathematics. In the Renaissance, the Greek text was edited by Simon Grynaeus, who published it in his *editio princeps* of Euclid (Basel, Herwagen 1533). A famous Latin translation was prepared by Francesco Barozzi (Padova, Percacino 1560), who also began writing a commentary on the text. The demonstration in question is found in Proclus, *In primum Euclidis*, 371–73. The argument is discussed by Saccheri in his Scholium 1 after Proposition 21 of *Euclid Vindicated*.

thought they could rely on some ill-devised definition of straight line, or, finally, because they were convinced of the validity of the rigid motion argument.[11]

The mathematicians of the Islamic Middle Ages formulated another fallacious argument that gained much currency in the Renaissance and beyond. This argument, which may have first appeared in the mathematical work of Nasīr ad-Dīn, consists in splitting the proof of the Fifth Postulate into two steps and claiming that (1) if two straight lines intersected by a perpendicular line form internal angles that sum to less than π, then they approach each other; and (2) if two straight lines approach each other, then they meet. As a matter of fact, however, *each* of these statements is equivalent to the Fifth Postulate.[12] Indeed, it is only locally that statement (1) enjoys absolute validity: in the neighborhood of their point of intersection with the transversal line, the two straight lines do, in fact, approach each other – but nothing prevents that they may subsequently move away from each other (as is the case with *ultraparallel lines* in hyperbolic space). Statement (2), in turn, is falsified in hyperbolic geometry by the existence of *asymptotic* parallels; we should note that the possibility of asymptotic straight lines was already apparent to the thinkers of Classical Antiquity and had provided one of the most important (and, ultimately, correct) reasons to doubt the truth of the Fifth Postulate.[13] The tangle presented by these two results posed great difficulties for

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[11] This tendency can also be shown in the terminology: sometimes Euclid's 'parallels' are translated into Latin as *rectae aequidistantes*, even when the definition is correctly stated as that of straight lines that do not meet. This is the case with Gerard of Cremona's translation of the *Elements* from an Arabic original, which became the principal Medieval translation of Euclid's text: see P.M. Tummers, *The Latin Translation of Anaritius' Commentary on Euclid's Elements of Geometry, Books I–IV*, Nijmegen, Ingenium 1994, pp. 23–5. Something similar is still to be found in the important and otherwise very accurate sixteenth-century edition of Euclid by Federico Commandino (1509–1575), the last edition before Clavius': F. Commandino, *Euclidis Elementorum libri* XV*, unà cum Scolijs antiquis*, Pesaro, Camillo Franceschini 1572.

[12] Nasīr ad-Dīn at-Tūsī (1201–1274), Persian scientist. He formulated, and attempted to prove, the two principles in his *Treatise to Cure Doubts Regarding Parallel Lines* (before 1251) and later restated them in his commentary on Euclid's *Elements* (the so-called *Shorter Version*). After his death, a disciple composed another commentary on Euclid that he attributed to his mentor (the so-called *Longer Version* of 1298). The two theorems also played a crucial role in this later work. The *Longer Version*, however, only provides a proof of statement (2) and assumes the more delicate claim (1) as an unproved lemma. Nasīr ad-Dīn's proof is the latest refinement of several attempts to prove the Fifth Postulate that were envisaged in the Islamic world; the first sketch of this kind of proof is probably to be found in an Arabic manuscript published in A.I. Sabra, *Simplicius's Proof of Euclid's Parallels Postulate*, "Journal of the Warburg and Courtauld Institutes", 32, 1969, pp. 1–24, who attributes it to Simplicius. Jaouiche offers a French translation of Nasīr ad-Dīn's works in *La théorie des parallèles*.

[13] The argument was put forward in Antiquity by Geminus and discussed in Proclus' commentary. The main studies on asymptotic lines from the Renaissance, which Saccheri certainly knew (if indirectly), are surveyed in the two articles by L. Maierù, *L'influsso del Narbonense sui commentatori euclidei del Seicento italiano circa il problema delle parallele*, in Atti del Convegno *La Storia delle matematiche in Italia*, Cagliari, Università di Cagliari 1982, pp. 341–49, and L. Maierù, *Il "meraviglioso problema" in Oronce Finé, Girolamo Cardano e Jacques Peletier*, "Bollettino di Storia delle Scienze Matematiche", 4, 1984, pp. 141–70.

Modern mathematicians, who often made one of two mistakes: they either assumed one of the contentions as obvious and set out to prove the other (a task that is eminently feasible, for as soon as one ascribes truth to either (1) or (2), one has already admitted the Fifth Postulate); or they committed a paralogism in the proof of either claim and then exultantly, and correctly, inferred the other. It was especially statement (1) that proved to be a particular source of error, as the difference between global and local properties was difficult to grasp within the Euclidean synthetic and constructive framework. This crucial difference would start to become clearer only from the second half of the seventeenth century onwards, with the first results of the Calculus. Only Saccheri, at any rate, provided a complete clarification of the matter in classical terms. In this case as well, however, well into the eighteenth (and even nineteenth!) century several latecomers kept contriving proofs of the Fifth Postulate based on some version of Nasīr ad-Dīn's demonstration.

Beside those two sources of error, the theory of equidistance and the theory of local approach, a positive accomplishment of Late Antique and Medieval research should be mentioned, as it plays an exceptionally important role in the history of Saccheri's work. This achievement can be described as the elaboration of a new strategy for proving the Fifth Postulate, a strategy that was not based on criteria governing the intersection of straight lines, but rather on the consideration of the sum of the internal angles of certain figures. Nowadays, after Legendre's results, the most famous of these paths to prove Euclidean geometry is the one that takes as its starting point the sum of internal angles of a triangle: a kind of inversion of *Elements* I, 32, where Euclid infers from the Fifth Postulate that the sum of a triangle's internal angles equals π. For a long time, however, the most relevant, and indeed the most obvious, object of study for this direction of research were quadrilaterals. Geometers attempted to prove that the internal angle sum of a quadrilateral is 2π without relying on the Fifth Postulate, and then proceeded to deduce the Postulate from the former result: a perfectly legitimate approach, provided that one handles with due attention the system of principles employed. Although the traces of such procedure are already present in certain late antique attempts, it seems that the Persian Omar Khayyām was the first to extensively explore this path. He was followed by many other Islamic scientists.[14]

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[14] In fact, some moves in this direction are already to be found in Thābit ibn Qurra, but the one who made the most extensive use of this strategy is the great Persian poet and scientist Umar Khayyām (circa 1048–1131), whose *Explanation of the Difficulties in the Postulates of Euclid* probably dates from 1077. The manuscript of Khayyām's work first appeared in print in 1936, but many of its results were discussed in the already quoted *Treatise to Cure Doubts* by Nasīr ad-Dīn. Khayyām's *Explanation* consists of three books, devoted to precisely the same three blemishes mentioned by Saccheri in the Preface of *Euclid Vindicated*: Book I discusses the theory of parallels, Book II the definitions of the equality of ratios, and Book III the composition of ratios. On the whole, Khayyām's work can perhaps be considered the principal Medieval contribution on the foundations of geometry. The section devoted to the parallels can nowadays be read in French in Jaouiche's aforementioned book, or in English (partially) in "Scripta mathematica", 24, 1959, pp. 275–303. The part on proportions is available, again in French, edited by A. Djebbar, in "Farhang. Quarterly Journal of Humanities and Cultural Studies", 14, 2002, pp. 83–136. An edition of the entire work is in R. Rashed, B. Vahabzadeh, *Al-Khayyâm mathématicien*, Paris, Blanchard 1999.

In the European Renaissance, this route was taken by all the major mathematicians who attempted to prove the Fifth Postulate. The path, of course, was not viable, and their expectations had to be disappointed, but it was nonetheless a fruitful path: it led naturally to the examination of a geometry with angular 'excess' (spherical geometry) and one with angular 'defect' (hyperbolic geometry) with regard to the expected value of 2π, hence to exact quantitative considerations of the phenomenon of parallelism. In other words, this strategy represented the dawn of the concept of curvature, at which no-one could have arrived by way of mere consideration of the intersection or non-intersection of straight lines.

Let us now take a closer look at the discussion occurring in the seventeenth century. The real center of all the research on the foundations of classical geometry – not only concerning the theory of parallels – is the extensive commentary on Euclid's *Elements* by Christoph Clavius, which appeared in several editions from 1574.[15] This work represents a genuine watershed in foundational studies in geometry, since it offers one of the first (mostly) accurate editions and translations of the *Elements*, a text that had undergone significant corruptions during the Middle Ages, and it supplements the Ancient text with a systematic collection of all scholia and comments that the tradition regarded as required additions, together with relevant considerations authored by Clavius himself. This edition immediately stood out as the reference text for any scientific study of Euclid's work, a position it held for two centuries. Moreover, Clavius was the first and most prominent among Jesuit scientists. It is therefore not hard to understand why Saccheri's work is consistently informed by Clavius' commentary on Euclid.

Clavius' confrontation with the Fifth Postulate begins in the second edition (1589) of his commentary, probably because it is around this time that he comes upon a manuscript

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[15] Christoph Clavius (1538–1612), born in Germany but a resident at the Roman College from 1560, was the most important and authoritative Jesuit mathematician of the Renaissance. Beside his monumental commentaries on Euclid and on Sacrobosco's astronomical treatise, he is remembered principally as the architect of the Gregorian calendar reform. His commentary on the *Elements* is C. Clavius, *Euclidis Elementorum Libri* XV, Roma, Vincenzo Accolto 1574: it was subsequently reissued in several editions with variants and additions, the most important being the second edition of 1589. The last edition was also reprinted in the first of five volumes of C. Clavius, *Opera mathematica*, Mainz, Reinhard Eltz (et al.) 1611–1612; see the reprint edited by E. Knobloch, Olms 1999, to which I will consistently refer. A succinct but very precise introduction to Clavius' vast body of work is E. Knobloch, *Sur la vie et l'oeuvre de Christophore Clavius (1538–1612)*, "Revue d'Histoire des Sciences", 41, 1988, pp. 331–356; more recently and extensively, cf. S. Rommevaux, *Clavius, une clé pour Euclide au XVI siècle*, Paris, Vrin 2005. On the intellectual context of Jesuit mathematics, see U. Baldini, *Legem impone subactis. Studi su filosofia e scienza dei gesuiti in Italia, 1540–1632*, Roma, Bulzoni 1992.

version of Nasīr ad-Dīn's demonstration.[16] Thanks to Nasīr ad-Dīn, he realizes that Proclus' proof, which he included in the first edition of his *Euclid*, is gravely inadequate, insofar as it takes the existence of equidistant straight lines for granted. To contrive his own proofs of the Postulate, Clavius follows two distinct routes, namely, the two erroneous strategies outlined above. The first hinges on the claim that the line equidistant from a straight line is itself straight, which, Clavius believes, is established both by his peculiar interpretation of Euclid's definition of a straight line and by the argument from the rigid motion of a line segment. The second route is the same as in Nasīr ad-Dīn's bipartite theorem. In both cases, Clavius does not directly conclude the Fifth Postulate, but rather goes through a demonstration about the sum of the internal angles of a quadrilateral. Thus, Clavius' work contains all of the principal devices – and the principal errors – that later discussions, in particular Saccheri's, were to take as their starting point.

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[16] In the second edition of his commentary, Clavius prefaces the new proof of the Fifth Postulate with this remark: "Id quod in Euclide quodam Arabico factum etiam esse accepi, sed nunquam facta mihi est copia demonstrationem illam legendi, etsi obnixe illud iterum atque iterum ab eo, qui eum Euclidem Arabicum poßidet, flagitavi" (*Euclidis*, p. 50). We know, in fact, that a copy of Nasīr ad-Dīn's three works on parallel lines (including the spurious *Longer Version*) was owned (among many other Arabic manuscripts, such as that of Books 5–7 of Apollonius' *Conics*) by the Medici family; their librarian and orientalist Giovan Battista Raimondi was appointed to publish most of these manuscripts, and to this effect he founded and directed the Typographia Medicea in Rome. The Oriental Press was active from 1584 to the death of Raimondi (1614), and published several works in Arabic to foster the evangelization of the Islamic world. The relations between Clavius and Raimondi were not excellent, and Clavius complained about the delay in the publication of the Arabic texts, trying to supersede Raimondi in the editorial work or, at least, to have access to the manuscripts. We have (for instance) a letter of Guidobaldo del Monte to Clavius (from 1590), asking the Jesuit some questions on the *Conics* of Apollonius, if he could have access to Raimondi's treasure; but in 1605 Raimondi was still protecting Apollonius' manuscript, claiming to be almost ready for an edition of it (see U. Baldini, P.D. Napolitani, *Christoph Clavius. Corrispondenza*, Pisa, Department of Mathematics (preprint) 1992, letters 65 and 256). Apollonius was only published by Borelli in 1661. The publication of the Arabic Euclid was more fortunate, however, and Raimondi begun his edition of Nasīr ad-Dīn's *Longer Version* in 1588, publishing it in 1594. In this case, clearly, Clavius was able to look at the text already in 1588, and made good use of it in his 1589 edition of Euclid. What is more, in the apocryphal *Longer Version* the proof for statement (1) mentioned above was omitted, and it is only to be found in the original (unpublished) works of Nasīr ad-Dīn. Since Clavius' *Euclidis* offers such a proof and it contains the same error as Nasīr ad-Dīn's original demonstration, that is, the passage from a local to a global property, it seems plausible to me that Clavius became acquainted, through Raimondi, with the *Treatise to Cure Doubts* and the *Shorter Version* (or had some glimpses thereof). On the history of the Medici's manuscript, see the special issue of the "Cahiers d'histoire des mathématiques de Toulouse", 9, 1986, edited by J. Cassinet. On Clavius' knowledge of Arabic sources, see E. Knobloch, *La connaissance des mathématiques arabes par Clavius*, "Arabic Science and Philosophy", 12, 2002, pp. 257–84. On Borelli's important edition of Apollonius, see L. Guerrini, *Matematica ed erudizione. Giovanni Alfonso Borelli e l'edizione fiorentina dei libri* V*, VI e VII delle Coniche di Apollonio di Perga*, "Nuncius", 14, 1999, pp. 505–68.

The later developments, for the most part, relied passively on Clavius' proofs. It must be stressed that Clavius' commentary represented a masterpiece of rigor for the study of the foundations of elementary geometry – the standard it set was seldom met in the following decades. Clavius' work became, for instance, the reference text for all of mathematical teaching in Jesuit colleges. Yet mere decades later, it had already become necessary to compose and compile new treatises in elementary geometry, shorter and more accessible to students than the original Clavian work. The works by the Order's great professors, such as André Tacquet and Milliet Dechales, were composed for this purpose.[17] These texts aimed chiefly at a simplification of the original Euclidean edifice and were less oriented towards scientific study than Clavius' commentary. The consequence, at least insofar as it concerns us, is that all these authors simply followed the old master: they mostly restated Clavius' proof appealing to equidistant straight lines, though in a less rigorous version; on the other hand, Nasīr ad-Dīn's bipartite proof must have appeared unnecessarily complicated to them. It is evident that Saccheri was well acquainted with these works, and he might have even made use of them as textbooks for his courses; it is equally evident that he could not hold them in any scientific esteem, and, in fact, *Euclid Vindicated* represents a lively protest against the turn taken by mathematical teaching in Jesuit colleges.[18]

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[17] Andreas Tacquet (1612–1660) was a Flemish mathematician and a disciple of the Jesuit geometer Grégoire de Saint-Vincent (who was, in turn, a student of Clavius'), and is nowadays mostly remembered for some studies that were instrumental in the formulation of the Fundamental Theorem of Calculus. In his age, he was also renowned for his edition of Euclid's *Elements*, which was translated into several languages and enjoyed a very wide circulation for educational purposes: A. TACQUET, *Elementa Geometriae planae ac solidae*, Antwerp, Jacob van Meurs 1654. Claude François Milliet Dechales (1621–1678) held a professorship for mathematics in Turin, which explains Saccheri's familiarity with his work, as Saccheri also spent many years in that city. In 1660, Dechales wrote a commentary on some of the books of the *Elements*, which was translated into various languages. First and foremost, however, he is the author of a monumental mathematics textbook in four volumes that treats all sorts of subjects, but whose parts on elementary geometry depend heavily on Tacquet's treatise: C.F.M. DECHALES, *Cursus seu Mundus Mathematicus*, Lyon, Posuel&Rigaud 1690 (1674^1). Honoré Fabri (1607–1688) was a mathematician, physicist and theologian who came under attack from Rome on charges of Cartesianism. It is very likely that Saccheri was acquainted with Fabri's works and was afraid of meeting the same fate (see the notes on this point in the aforementioned edition of the *Logica Demonstrativa* by Mugnai and Girondino). Fabri wrote a short handbook of geometry for students, which enjoyed great popularity: H. FABRI, *Synopsis Geometrica*, Lyon, Molin 1669.

[18] There is a vast body of literature on mathematical teaching in Jesuit colleges, and specifically on the genesis of the Society's first *ratio studiorum*, with which Clavius was involved. Cf. for instance G. COSENTINO, *L'insegnamento delle matematiche nei collegi gesuitici nell'Italia settentrionale*, "Physis", 13, 1971, pp. 205–217; D.C. SMOLARSKI, S.J., *The Jesuit Ratio Studiorum, Christopher Clavius, and the Study of Mathematical Sciences in Universities*, "Science in Context", 15, 2002, pp. 447–64; the most comprehensive treatment is probably A. ROMANO, *La contre-réforme mathématique. Constitution et diffusion d'une culture mathématique jésuite à la Renaissance (1540–1640)*, Roma, École Française 1999. More generally on the subject cf. the two volumes edited by Feingold: *The New Science and Jesuit Science: Seventeenth Century Perspectives*, ed. by M. Feingold, Dordrecht, Kluwer 2003; *Jesuit Science and the Republic of Letters*, ed. by M. Feingold, Cambridge, MIT 2003.

A comparable development also took place outside the Jesuit milieu. Jansenist and Cartesian circles, which were farthest removed from the Society, asserted the intuitive validity of the first principles. For this reason, they rebuked Euclid's – to say nothing of Clavius' – excessive rigor on the charge of obscuring with its subtleties what is manifest to common understanding. In this context, the issue of providing a demonstration of an axiom would not cause much concern. Indeed, the great Arnauld accepted one of the versions of the Fifth Postulate as self-evident, while composing his *New Elements of Geometry,* where he aimed to present elementary mathematics in a more natural and simple way than Euclid had. Nonetheless, he also restates Nasīr ad-Dīn's proof.[19]

The need for rigor in proofs was felt more urgently in the Italian school of geometry, which placed as great an emphasis on foundational work as it did on performing its role as an educational institution. Borelli's *Euclides restitutus* (1658)[20] does not contain impor-

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[19] A. Arnauld, *Nouveaux Elémens de géometrie*, Paris, Savreux 1667, with a second, extensively revised edition in 1683; both editions (now very rare) can be read in the recent *Géométries de Port-Royal*, ed. by D. Descotes, Paris, Champion 2009, to which I shall consistently refer. This geometrical work by Arnauld (1612–1694), which was probably conceived after certain discussions with Pascal in Port-Royal, enjoyed considerable success and also represents in many ways an important example of the Cartesian epistemology of mathematics, beside Arnauld's more famous *Logic* (A. Arnauld, P. Nicole, *La logique ou l'art de penser*, Paris, Desprez 1683 (1662^1); reprint ed. by P. Clair and F. Girbal, Paris, PUF 1965). If Arnauld often writes in the *Logic* that it is a mistake to try to prove what is evident in itself (cf. for instance *Logique*, iv, 9; pp. 326–27), the same principle is applied in the *Nouveaux Elémens* to the parallel axiom: "Sixième Axiome. Deux lignes droites qui étant prolongées vers un même côté s'approchent peu à peu, se couperont à la fin. *Euclide prend cette proposition pour un principe et avec raison: car elle a assez de clarté pour s'en contenter, et ce serait perdre le temps inutilement que de se rompre la tête pour le prouver par un long circuit*" (p. 361). Although this position could hardly be more distant from Saccheri's or Clavius', it was common in the whole Cartesian tradition, as well as in the 'modern' Jesuit circles. One can read, as a further example, Malebranche's laconic remarks on the two Euclidean blemishes discussed in Savile's *Praelectiones*: Savile was, in the eyes of the French philosopher, nothing but a pedantic Englishman who wasted his time on trifles (*Recherche de la Vérité*, II, ii, 6; in N. Malebranche, *Oeuvres complètes*, vol. 1, ed. by G. Rodis-Lewis, Paris, Vrin 1991^3, pp. 297–301).

[20] Giovanni Alfonso Borelli (1608–1679) was perhaps the most important Italian scientist around the half of the seventeenth century. He was born in Naples and was related to philosopher Tommaso Campanella. After joining the Galilean school, he taught in Pisa, where he was the mentor of Alessandro Marchetti and was involved in a lively polemic with Vincenzo Viviani, after which he moved to Messina. Eventually, he left the city and sought refuge in Rome, as he was implicated in an anti-Spanish conspiracy, and it was in Rome that he met and befriended Vitale Giordano. He is chiefly known for his studies of mechanics and physiology, but he also wrote an important commentary on Euclid, which is one of Saccheri's primary sources: G.A. Borelli, *Euclides restitutus, sive priscae Geometriae Elementa brevius et facilius contexta, in quibus praecipue proportionum theoriae nova firmiorique methodo proponantur*, Pisa, Francesco Onofri 1658. The book had an important third edition (Roma, Mascardi 1679), which is the text Saccheri read (and from which I shall quote), and an Italian translation as *Euclide Rinnovato*, transl. from the Latin by D. Magni and corrected by the Author, Bologna, Giovan Battista Ferroni 1663. On the figure of Borelli and his ideas in the epistemology of mathematics, so important for Saccheri, cf. C. Vasoli, *Fondamento e metodo logico della geometria nell'Euclides restitutus del Borelli*, "Physis", 11, 1969, pp. 571–98; then U. Baldini, *Giovanni Alfonso Borelli e la rivoluzione scientifica*, "Physis", 16, 1974, pp. 97–128.

tant innovations in parallel theory, but it ventures a novel definition of parallelism, which Saccheri will starkly criticize. Most importantly, Borelli seems quite aware that Clavius' demonstrations fail to achieve their goal. In the end, however, he simply accepts the claim that an equidistant line is straight as an axiom. Vitale Giordano's *Euclide restituto* (1680),[21] on the other hand, puts forward another demonstration of the Fifth Postulate that rejects Clavius' proofs as insufficiently rigorous, and explores an autonomous path. Giordano's central argument is itself vitiated by a rather subtle error, but its principal feature is that he treats the theory of the quadrilaterals' internal angles at great length, producing new results that precipitate the direction later followed by Saccheri.

Finally, we cannot avoid mentioning Wallis' 1663 proof, which bore in many ways a prime theoretical and historical importance.[22] First of all, it is noteworthy that Wallis, in the course of his efforts, published for the first time Nasīr ad-Dīn's text,[23] which was thus made available to scholars without Clavius' mediation. Equally noteworthy is Wallis' explicit rejection of Nasīr ad-Dīn's conclusions. But most important of all is Wallis' original attempt to prove the Fifth Postulate itself, which demonstrates that the postulate is equivalent to the possibility of constructing triangles similar, though not congruent, to a given triangle: in other words, (non-trivial) transformations by similarity are only possible in Euclidean geometry, as opposed to hyperbolic or spherical geometry. Wallis reasoned that this amounted to a proof of the postulate, since it was derived from a principle that he held to be indisputable, namely, that of the possibility of similarity. Wallis also sketches a metaphysical proof, claiming that the possibility of transformations by similarity depends on the very nature of quantity, that is, on the fact that the category of *quality* (under which falls the *shape* of a geometric figure, construed as invariant by similarity) differs from that of *quantity* (which determines that figure's *magnitude*). Indeed, quantity and quality represent two distinct highest genera of being, such that it is always possible to vary the properties of the former while preserving those of the latter, and *vice versa*.

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[21] Vitale Giordano da Bitonto (1633–1711) retired in Rome to teach mathematics after a very turbulent youth as a soldier, during which he faced a couple of charges of assault and murder. His Italian commentary on Euclid is V. Giordano, *Euclide restituto, ovvero gli antichi elementi geometrici ristaurati e facilitati*, Roma, Angelo Bernabò 1686 (1680^1); although Giordano criticizes Borelli in several passages, his text was certainly written as a homage to the Neapolitan mathematician, and it was published the year after Borelli's death.

[22] It is the aforementioned proof in *De Postulato Quinto*, first edited in 1693.

[23] Nasīr ad-Dīn's *Longer Version*, which had been published in Rome in 1594, in Arabic, was translated into Latin in 1651, on the occasion of Wallis' first lecture in Oxford on the Fifth Postulate. This Latin version, by the English orientalist Edward Pocock, was published as an appendix to Wallis' *De Algebra Tractatus* in 1693. We know that Pocock had access to the aforementioned Medicean manuscript, and translated in Latin both the original work of Nasīr ad-Dīn and the spurious essay that was published in Rome. Wallis' *Algebra* only contains the latter, but Wallis himself had access to Pocock's translation of the former, which he added in manuscript to his own copy of the *Opera mathematica*. Pocock's unpublished translation is still available in manuscript at the Bodleian Library, and is reprinted in the edition of Cassinet (see above).

Wallis' demonstration bore great historical significance: not only was it very elegant and deep mathematically speaking, but it also inaugurated the metaphysical discussions of the parallel postulate that were to bloom fully in later centuries. Let me stress that the intense philosophical discussion on the *significance* of non-Euclidean geometries that agitated the cultural and scientific scene for a good half of the nineteenth century and even continued, though in an evolved form, after the discovery of the general theory of relativity – and which constitutes one of the most interesting aspects of a study of the history of parallelism – was a unique product of the nineteenth century. In fact, it seems that, in the whole course of Antiquity and the Early Modern Age, despite the attention lavished on the Fifth Postulate, not a single philosopher detected in the problem of parallels any special metaphysical difficulty (which means, a difficulty concerning the nature of space) – at least, no one before Wallis. If general philosophical debate on the topic arose only much later, after Gauss' discoveries and Kant's philosophy had gained sway, there is no doubt that some metaphysical minds of the eighteenth century had already grappled with the issue. This is true, for instance, of Leibniz, Thomas Reid, and Johann Lambert: all of them tackled the problem from the viewpoint inaugurated by Wallis. These discussions remained quite isolated, however, as it is confirmed by the work of many other geometers of the same century, who were completely insensitive to any philosophical implications of the subject. This is also true of Saccheri: although he was rather gifted in philosophy, a subtle logician, and a professor of theology, he never seemed to discern in the mathematical problem that most strained his intellectual forces any philosophical theme, nor anything motivating a project broader than the systemization of Euclidean axiomatics. On the contrary, he overtly dismissed Wallis' demonstration as too metaphysical. While the rebuttal of Wallis' proof was certainly a gain for the progress of science, it also typifies the epoch's struggle with the idea of a structural or functional interpretation of physical space, as opposed to a merely 'substantial' one.

3. The theory of proportions in the seventeenth century

Early Modern discussions on the theory of proportions were much broader and livelier than those on parallel theory. The problem of parallels, in fact, was nothing more than a byproduct of Renaissance research on the foundations of geometry, inherited from antique discussions since lost. The scientific reach of this issue was limited to interpreting classical texts or to teaching elementary geometry, and its scientific efficacy gradually decreased as modern mathematics moved on towards territories untouched by Euclid. On the contrary, the theory of proportions, thanks to its greater generality and its immediately operational aspects, was the classical mathematical device most suited to facilitating a transition towards the new seventeenth-century discoveries. Indeed, Fermat's and Descartes' wide-ranging algebraic reform of geometry, which represents one of the chief landmarks of Modern mathematics, was not applicable to mechanics, a discipline whose methods have little in common with the elegant purity of Algebra. The mathematization of natural philosophy – its metamorphosis into a scientific, quantitative mechanics – could not make use of Descartes' algebraic equations. Moreover, even within pure mathematics, the new Calculus, like the various studies of the indivisibles before it, was rendered possible by a reformed theory of proportions and rational numbers. Therefore, Book V of Euclid's *Elements*, where the theory of ratios receives its most extensive treatment, was one of the few vestiges of Greek mathematics that still had an active role to play in the age of the Scientific Revolution.

To be sure, there had been numerous foundational discussions of the theory of proportions in older times, and these discussions were quite akin in spirit to those concerning the Fifth Postulate. In particular, Euclid's outstanding definition of the equality of ratios, which in some sense represents the peak of abstraction of Classical mathematics (and which is probably to be ascribed to Eudoxus), was gravely misunderstood in the Middle Ages, mainly due to an intricate tangle of textual problems.[24] Yet, nothing could save the whole edifice of Euclid's theory from collapse, if this definition (as well as a few others) underwent textual corruption. A few attempts were made to fix the problem: Arabic mathematics explored an alternative theory, the so-called anthyphairetic theory of ratios,[25] whereas the European tradition resorted to numeric interpretations of proportions through the concept of the denomination of a ratio.[26] The fundamental issue of the treatment of incommensurable magnitudes, however, resisted all solution. Only in the sixteenth century, eventually, new editions of the *Elements* were sufficiently cleansed of textual corruption to allow scholars to

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[24] Let me note that Greek mathematics, on the other hand, never seemed to find anything problematic about the theory of proportions after Eudoxus' systematization. On the Ancient theory see for instance the classic, and controversial, F. BECKMANN, *Neue Gesichtpunkte zum 5. Buch Euklids*, "Archive for History of Exact Sciences", 3, 1967, pp. 1–144.

[25] This theory consists basically in defining the equality of two ratios as follows: by first applying an algorithm of successive divisions to each of them and then ascertaining the identity of the successions of factors in the two procedures. In the case of irrational magnitudes, this amounted essentially to a definition of the latter with reference to what we today call continued fractions. It should be noted that the procedure could be executed geometrically, and it was not necessary to work with numbers instead of segments. Whether such a theory could have already been present in Classical Antiquity and in pre-Euclidean mathematics (a thesis first put forward by Zeuthen and Becker, then followed by many others) is a question that is much discussed among historians of science. In any event, anthyphairesis was certainly employed in the Islamic world, where it was often ascribed to Greek mathematicians. Umar Khayyām was one of its greatest theorists, in the aforementioned Book II of his *Explanation*; cf. B. VAHABZADEH, *Al-Khayyām's conception of ratio and proportionality*, "Arabic Science and Philosophy", 7, 1997, pp. 247–63; B. VITRAC, *Umar al Khayyam et l'anthyphérèse: Étude du deuxième Livre de son commentaire "Sur certaines prémisses problématiques du Livre d'Euclide"*, "Farhang. Quarterly Journal of Humanities and Cultural Studies", 14, 2002, pp. 137–92; and more in general E.B. PLOOIJ, *Euclid's Conception of Ratio and his Definition of Proportional Magnitudes as Criticised by Arabian Commentators*, Rotterdam, Hengel 1950.

[26] In theories of this kind, every ratio between magnitudes is expressed by a numeric fraction associated with it. Such approaches proceed to define (for instance) the equality of ratios through the identity of fractions, and apply this method to the other Euclidean definitions. It is to be noted, nevertheless, that this did not entail the identification of a ratio with its fraction, as would occur as part of the algebraization of mathematics in the Modern Age. The two approaches also differed in some other details.

make sense of Euclid's original strategy. The point of reference is once again Clavius' commentary, with tens of pages devoted to scholia that expound the correct Eudoxian theory and amend the misunderstandings that had plagued geometry for centuries.

Clavius' commentary also inaugurated a more critical reception of the Euclidean theory. For the first time, scholars began to focus not only on the strengths but also on some flaws and difficulties (or blemishes) in the classical theory of proportions. One of the major issues called into question concerned the so-called principle of the existence of the fourth proportional: in some theorems, Euclid simply and tacitly accepts that, given any three magnitudes A, B, and C, there always exists a fourth magnitude D such that A:B::C:D. It appeared, however, that this assumption should be considered as an additional axiom, and, as such, it was re-classified and counted among the Euclidean axioms in the work of Clavius, followed in this by many later interpreters. One of the problems with this strategy was that the axiom appeared to be existential and not constructive: it warranted the generic assumption of the fourth element, but gave no clue for producing the fourth magnitude concretely. In sum, the dispute was between mathematical constructivism and non-constructivism – one that is certainly not easy to settle. The birth of a new mechanics made it even bitterer. Although the non-constructive axiom is (perhaps) not indispensable to the *Elements*, and alternative demonstrations for the theorems that Euclid proves by tacit recourse to the non-constructive principle can be provided, Clavius' explicit axiomatization induced a series of mathematicians, first and foremost Galileo, to undertake a radical reform of the Euclidean theory of proportions that went so far as to be *based* on the existence of a fourth proportional. With this new system they hoped to sidestep some of the weakest points of the Greek theory – which had originated chiefly, or maybe solely, for geometric purposes – and to widen the scope of the doctrine of ratios to include the non-geometric magnitudes that were now employed in the new natural science. This new theory of proportions, whose final form Galileo had only outlined, was later developed in order to provide a justification for those 'functional' procedures that were required by, and had in fact already been employed in, the mathematization of the physical world.[27]

This reform of the theory of proportions, required by mechanics, demanded in turn the theoretical transvaluation of an almost certainly apocryphal Euclidean definition, namely, that of the composition of ratios. Today we could say that such composition consisted in nothing more than the product of two fractions, but such an operation would have been hard to justify within the classical Euclidean context, where ratios were not seen as numbers or quantities, which can be multiplied by one another, but rather as mere relations. This spurious definition of the composition of ratios, which may have been introduced by some Late Antique commentator as a means of clarifying the terms employed in a couple of theo-

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[27] The standard reference work on the theory of proportions in seventeenth-century Italy is E. Giusti, *Euclides reformatus. La teoria delle proporzioni nella scuola galileiana*, Torino, Bollati Boringhieri 1993. For a comparison between the Galilean reform and Clavius' theory, which Saccheri will adopt with little alteration, cf. P. Palmieri, *The Obscurity of the Equimultiples. Clavius' and Galileo's Foundational Studies of Euclid's Theory of Proportions*, "Archive for History of Exact Sciences", 55, 2001, pp. 555–97.

rems of the *Elements*, represented initially only a problem for the interpreters of the Greek text. However, it ended up becoming, in the hands of Galileo and his followers, an essential instrument enabling those operations and calculations that were indispensable to the new sciences. So it happened that a very peripheral locus of Euclid's vast work rapidly came to the fore of the scientific debate, even though the correct strategy to deal with it was unclear. Savile certainly had such developments in mind when he came to regard the definition of the composition of ratios as one of the ugly blemishes on Euclid's body.

The transformation that the theory of proportions in the *Elements* underwent in the Modern Age was therefore radical: ultimately, it also affected its very meaning – which, to be sure, had been a matter of debate since the early Renaissance. The controversy was mainly about the object of the theory of ratios – the enigmatic concept of 'magnitude' around which all the theorems in Book V of the *Elements* revolve and which Euclid does not define elsewhere. If by that concept one means the three different (that is, non-homogeneous, according to the definition) geometrical magnitudes, i. e., lengths, areas, and volumes, and maybe, in addition to these, angular width or numbers, then the theory of proportions applies perfectly, and solely, to the objects with which the other twelve books of the *Elements* are concerned. On the other hand, one could see in the concept of magnitude a generality that is absent from the rest of Euclid's work and allows us to apply proportions and ratios, under the same laws expressed in the *Elements*, to time, motion, speed, musical intervals, and potentially many other things – in a word, to anything that falls under the more general concept of 'magnitude' as quantity. In this second case, a way was opened towards a general mathematical theory of magnitudes as a *mathesis universalis*, and indeed towards the application of Classical mathematics to the wide world of natural science.[28] Yet this broader move required a discussion of the very foundational questions (the Euclidean blemishes) associated with the definition of the equality of ratios, the existence of a fourth proportional, and the exact characterization of the operation of the composition of ratios.

At stake here was the complete reworking of a Classical theory that was very well structured and that could be adapted to novel applications only with difficulty. This exceptional enterprise spanned for about the entire first half of the seventeenth century and engaged chiefly the Galilean school – and, of course, the commentators of Euclid. The highest points of this endeavor are, first, the composition of the *Fifth Day* of Galileo's *Two New Sciences*, second, the analogous Galilean treatises by Torricelli and Viviani, and, finally and most importantly, the most advanced work of the Italian school on this subject, Borelli's *Euclides*

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[28] Among the general reference works on this point, see the classic G. Crapulli, *Mathesis Universalis. Genesi di un'idea nel XVI secolo*, Roma, Edizioni dell'Ateneo 1969; and more recently D. Rabouin, *Mathesis Universalis. L'idée de "mathématique universelle" d'Aristote à Descartes*, Paris, PUF 2009.

restitutus.[29] These works represent a genuine hand-to-hand combat with the Euclidean text, and combine inseparably attentive reading, loose interpretation, emendation, reformation, and explicit rebuttal.

The discussion on the theory of proportions, however, was not exclusively Italian, although in Italy it developed in constant proximity with the original Euclidean text and bore the most important results of geometrical significance.

In France, on the other hand, the most productive research in mathematics was moving in a completely different direction, and mostly consisted in a development of Descartes' *Géométrie*. Consequently, few of the great French geometers of the time would be concerned with the Euclidean theory of proportions, which had little to do with the new algebra. Nonetheless, we can find significant exceptions to this general direction if we examine French mathematical textbooks, where Euclid's *Elements* still played an essential role. Thus it happens that the didactic works of the mathematicians mentioned above, be they Jesuits or Jansenists, devote plenty of space to the interpretation and reformation of the theory of proportions. The motives that prompted the first and most widely followed of these geometers, André Tacquet, to undertake a revision of the Euclidean theory are hard to determine. He may have had educational concerns: perhaps he deemed Eudoxus' complex system of definitions too obscure for his students, and intended to simplify the theory of proportions; other parts of his revision of the *Elements*, in fact, had been informed by this procedure – at times to the detriment of the work's rigor. Alternatively, he may have been motivated by worries similar to those of the Galilean school, since he certainly had a strong interest in mechanics and was very probably acquainted with the works of the Italian scientists.[30] Finally, it is also possible that he was motivated by foundational concerns of older origin

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[29] This *Fifth Day* of the *Discourses and Mathematical Demonstrations Concerning Two New Sciences* (published in 1638) is devoted wholly to the theory of proportions. It was conceived by Galileo (1564–1642) as a later addition to the work and dictated – it seems – to Torricelli in 1641. The *Fifth Day* was only published in 1674, in the treatise on proportions by V. Viviani, *Quinto Libro degli Elementi di Euclide, ovvero Scienza Universale delle Proporzioni spiegata colla dottrina del Galileo*, Firenze, Condotta 1674; nowadays, it is to be found in G. Galilei, *Opere*, Firenze, Barbera 1968, vol. 8, pp. 347–62. Evangelista Torricelli (1608–1647) also wrote a *De Proportionibus Liber* (1647), which, despite being first printed in the twentieth century, was distributed in universities in manuscript form. Indeed, Borelli certainly relies on it. It can be read in E. Torricelli, *Opere*, ed. by G. Loria and G. Vassura, Faenza, Montanari 1919–1944, vol. 1, pp. 293–327; and a critical edition is in appendix to Giusti, *Euclides reformatus*, pp. 299–340. Vincenzo Viviani (1622–1703), 'Galileo's last disciple', offered a probably mistaken or at least misleading, but very influential, reading of Galileo's work on proportions, in the text mentioned above, and added his own considerations on the subject. Viviani also treated the topic in an Italian edition of the *Elements* (Firenze, Carlieri 1690) that enjoyed a wide circulation and was certainly known to Saccheri.

[30] Tacquet's theory of proportions is probably autonomous and original, but on several points it undoubtedly displays similarities with those developed in Italy. Cf. F. Palladino, *Sulla teoria delle proporzioni nel Seicento. Due "macchinazioni" notevoli: Le sezioni dei razionali del galileiano G.A. Borelli; Le classi di misura del gesuita A. Tacquet*, "Nuncius", 6, 1991, pp. 38–81.

and was not persuaded that Clavius' edition, with its amended Greek text and its diligent scholia detailing how all the Medieval and Renaissance mathematicians were mistaken, had itself solved the interpretive problems associated with the *Elements* – as this work offered (he may have thought) a structurally confused and wrong theory that called for a rather deep revision. Nowadays, it is not easy to share Tacquet's concerns. Eudoxus' theory strikes us as quite masterful, whereas Tacquet's own reform may appear weakly conceived; in any event, it seems undeniable that these concerns must have prompted the Flemish Jesuit's work. The other mathematicians mentioned above, from Dechales to Arnauld, followed his approach all the way.[31]

In England, the situation was different still, and the most significant trend was probably a gradual movement away from the Euclidean methods. Instead of following Euclid's approach, English scholars attempted to completely re-ground the theory of proportions with arithmetical or algebraic methods. The point of departure was probably the Medieval anthyphairetic theory, which allowed the first moves towards the introduction of the concept of an irrational number; a theory that surfaced in the European Renaissance and probably informed some important works on arithmetic of the sixteenth century (such as Stevin's *Arithmétique* from 1585).[32] This approach had a great success in Britain, and the works of the two greatest mid-century mathematicians, Wallis and Barrow, both point in this direc-

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[31] The French Jesuit mathematicians often refer to Tacquet explicitly, even going so far as to copy his definitions and demonstrations. Dechales, for instance, engages in this practice. Arnauld's case is more complex, but the first edition of his *Nouveaux Elémens* (1667) includes a theory of proportions that is very close to Tacquet's. Since we know with a good deal of certainty that Arnauld began elaborating his reform and systematization of the Euclidean *Elements* in 1655, and Tacquet's *Elementa Geometriae*, which explicitly pursue the same goal, appeared in 1654, it is even possible that the motivation for Arnauld's work (besides Pascal's suggestion to this effect) was Tacquet's edition, although Arnauld, to be sure, never mentions him. On the other hand, by the second edition of the *Nouveaux Elémens* (1683) Arnauld has completely reworked his treatment of proportions. In this later text, he employs a positively 'modern' method, which rests on arithmetization rather than on Euclid's or Tacquet's synthetic geometry: it seems that Arnauld had Nonancourt's early work in mind, rather than English writings on the subject. Cf. F. de Nonancourt, *Euclides Logisticus sive de ratione euclidea*, Louvain, Bouvet 1652, also reprinted in the aforementioned *Géométries de Port-Royal*, pp. 801–20. The mathematician François de Nonancourt (1624–1686) was close to the Port-Royal school from 1669 and was in frequent personal contact with Arnauld in the years 1679–1680.

[32] The anthyphairetic theory, in fact, allowed one to take a continuous fraction as the expression of a ratio, and thus may represent the first introduction of irrational numbers; it arrived to the West through An-Nayrizi's commentary. Stevin's *Arithmétique* is to be found in his *Oeuvres mathematiques*, ed. A. Girard, Leyden 1634 (vol. 1). On Clavius' (and thus Saccheri's) connection with Stevin and this tradition of studies, see A. Malet, *Renaissance notions of number and magnitude*, "Historia Mathematica", 33, 2006, pp. 63–81.