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Alexander J. Zaslavski

Nonconvex Optimal Control and Variational Problems

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Alexander J. Zaslavski

Nonconvex Optimal Control and Variational Problems

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Alexander J. Zaslavski
Department of Mathematics
Technion - Israel Institute of Technology
Haifa, Israel

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Preface

This monograph is devoted to the study of nonconvex optimal control and variational problems. It contains a number of recent results obtained by the author in the last 15 years. The Tonelli classical existence theorem in the calculus of variations [81] is based on two fundamental hypotheses concerning the behavior of the integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex (or exhibit a more special convexity property in case of a multiple integral with vector-valued functions) with respect to the last variable. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals which are crucial in most of the existence proofs, although there are some interesting theorems without convexity (see Chap. 16 of [21] and [19, 20, 28, 61, 63]). Since in this book we do not use convexity assumptions on integrands the situation becomes more difficult. We overcome this difficulty using the so-called generic approach which is applied fruitfully in many areas of analysis (see, for example, [6, 67, 69, 71, 72, 99, 106] and the references mentioned there).

According to the generic approach we say that a property holds for a generic (typical) element of a complete metric space (or the property holds generically) if the set of all elements of the metric space possessing this property contains a G_δ everywhere dense subset of the metric space which is a countable intersection of open everywhere dense sets.

In [86, 88] it was shown that the convexity condition is not needed generically and not only for the existence but also for the uniqueness of a solution and even for well-posedness of the problem (with respect to some natural topology in the space of integrands). More precisely, in [86, 88] we considered a class of optimal control problems (with the same system of differential equations, the same functional constraints, and the same boundary conditions) which is identified with the corresponding complete metric space of cost functions (integrands), say F. We did not impose any convexity assumptions. These integrands are only assumed to satisfy the Cesari growth condition. The main result in [86, 88] establishes the existence of an everywhere dense G_δ-set $F' \subset F$ such that for each integrand in F' the corresponding optimal control problem has a unique solution. It should be mentioned that the author obtained this generic existence result in [86] for general Bolza and Lagrange optimal control problems. This result was published in [88].

The next step was done in a joint paper by Alexander Ioffe and the author (see [42]). There we introduced a general variational principle having its prototype in the variational principle of Deville, Godefroy, and Zizler [30]. A generic existence result in the calculus of variations without convexity assumptions was then obtained as a realization of this variational principle. It was also shown in [42] that some other generic well-posedness results in optimization theory known in the literature and their modifications are obtained as a realization of this variational principle.

The work [86, 88] became a starting point of the author's research on optimal control and variational problems without convexity assumptions which have been continued in the last 15 years. Many results obtained during this period of time are presented in the book. Among them generic existence results for different classes of optimal control problems are collected in Chaps. 2–5. Any of these classes of problems is identified with a functional space equipped with a natural complete metric and it is shown that there exists a G_δ everywhere dense subset of the functional space such that for any element of this subset the corresponding optimal control problem possesses a unique solution and that any minimizing sequence converges to this unique solution. These results are obtained as realizations of variational principles which are generalizations or concretization of the variational principle established in [42].

In Chaps. 6–9 we show nonoccurrence of the Lavrentiev phenomenon for many optimal control and variational problems without convexity assumptions.

We say that the Lavrentiev phenomenon occurs for an optimal control problem if its infimum on the full admissible class of trajectory-control pairs is less than its infimum on a subclass of trajectory-control pairs with bounded controls.

The Lavrentiev phenomenon in the calculus of variations was discovered in 1926 by M. Lavrentiev in [45]. There it was shown that it is possible for the variational integral of a two-point Lagrange problem, which is sequentially weakly lower semicontinuous on the admissible class of absolutely continuous functions, to possess an infimum on the dense subclass of C^1 admissible functions that is strictly greater than its minimum value on the admissible class. Since this seminal work, the Lavrentiev phenomenon is of great interest in the calculus of variations and optimal control [1, 8, 9, 21, 25, 26, 35, 49, 53, 60, 78–80]. Nonoccurrence of the Lavrentiev phenomenon was studied in [1, 25, 26, 35, 49, 79, 80]. It should be mentioned that Clarke and Vinter [25] showed that the Lavrentiev phenomenon cannot occur when a variational integrand $f(t, x, u)$ is independent of t.

In Chaps. 6–9 we consider large classes of optimal control problems identified with the corresponding complete metric spaces of integrands $f(t, x, u)$ depending on t. We establish that for most integrands (in the sense of Baire category) the infimum on the full admissible class of trajectory-control pairs is equal to the infimum on a subclass of trajectory-control pairs whose controls are bounded by a certain constant.

In Chaps. 10–12 we study turnpike properties of approximate solutions of variational and optimal control problems. To have this property means, roughly speaking, that the approximate solutions are determined mainly by the integrand (objective function) and are essentially independent of the choice of interval and end point conditions, except in regions close to the end points.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [77]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path).

We study the turnpike property of approximate solutions of the variational problems with integrands f which belong to a complete metric space of functions $\mathcal{M}$. We do not impose any convexity assumption on f. This class of variational problems was studied in Chap. 2 of [99] for integrands $f \in \mathcal{M}_{\rm co}$, where the space $\mathcal{M}_{\rm co}$ consists of all integrands $f \in \mathcal{M}$ which are convex with respect to the last variable (derivative). In Chap. 2 of [99] we showed that the turnpike property holds for a typical integrand $f \in \mathcal{M}_{\rm co}$. In this book we extend the turnpike result of [99] established for the space $\mathcal{M}_{\rm co}$ to the space of integrands $\mathcal{M}$. We also study turnpike properties for a class of discrete-time optimal control problems.

Haifa, Israel

Alexander J. Zaslavski

Contents

Chapter 1
Introduction

1.1 Generic Existence of Solutions of Optimal Control Problems

Let $-\infty < T_1 < T_2 < \infty$, $A \subset [T_1, T_2] \times R^n$ be a closed subset of the tx-space R^{n+1} and let $A(t)$ denote its sections, that is

$$A(t) = \{x \in R^n : (t, x) \in A\}, \quad t \in [T_1, T_2].$$

For every $(t, x) \in A$ let $U(t, x)$ be a given subset of the u-space R^m, $x = (x_1, \ldots x_n)$, $u = (u_1, \ldots u_m)$.

Let M denote the set of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$ and let $B_1, B_2 \subset R^n$ be closed. We assume that the set M is closed and $A(t) \neq \emptyset$ for every $t \in [T_1, T_2]$. Let $H(t, x, u) = (H_1(t, x, u), \ldots H_n(t, x, u))$ be a given continuous function defined on M.

We say that a pair $x : [T_1, T_2] \to R^n, u : [T_1, T_2] \to R^m$ is admissible if $x = (x_1 \ldots, x_n)$ is an absolutely continuous (a.c.) function, $u = (u_1, \ldots, u_m)$ is a measurable function and the following relations hold:

$$x(t) \in A(t),\ t \in [T_1, T_2], \quad x(T_i) \in B_i,\ i = 1, 2,$$
$$u(t) \in U(t, x(t)), \quad x'(t) = H(t, x(t), u(t)), \quad t \in [T_1, T_2] \text{ (a.e.)}.$$

Denote by Ω the set of all admissible pairs (x, u). We suppose that $\Omega \neq \emptyset$.

We are concerned with the existence of the minimum in Ω of the functional

$$\int_{T_1}^{T_2} f(t, x(t), u(t))dt + h(x(T_1), x(T_2)),$$

where $h : B_1 \times B_2 \to R^1$ is a lower semicontinuous bounded below function and f belongs to a space of functions described below.

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The existence of solutions of optimal control problems arising in various areas of research and the related lower semicontinuity of integral functionals were studied in [2,12,13,21,24,39,40,74,75] and others. Since the seminal work of Tonelli [81], it is well known that certain convexity assumptions are crucial to the existence of optimal solutions in problems of the calculus of variations and optimal control [21,23,59,65,74].

In this section we present the generic existence result obtained in [86, 88].

In [86, 88] we studied the existence of optimal solutions for a general class of optimal control problems. We considered optimal control problems with integrands $f(t,x,u)$ from a complete metric space of functions, which only satisfy a growth condition common in the literature, and established for a generic integrand f the existence result. More precisely, we showed that in the complete metric space of functions there exists a subset which is a countable intersection of open everywhere dense sets such that for each integrand belonging to this subset the corresponding optimal control problem has a unique solution, and moreover, this solution is stable under small perturbations of the integrand f.

Thus, instead of considering the existence problem for a single integrand, we investigate it for the space of all such integrands equipped with some natural metric and show that the existence result is valid for most of these integrands. This allows us to establish the existence result without convexity conditions and other restrictive assumptions.

Denote by $C_l(B_1 \times B_2)$ the set of all lower semicontinuous bounded below functions $h : B_1 \times B_2 \to R^1$ and denote by $C(B_1 \times B_2)$ the set of all continuous functions $h \in C_l(B_1 \times B_2)$. For the set $C_l(B_1 \times B_2)$ we consider the uniformity which is determined by the base

$$E_0(\epsilon) = \{(h_1,h_2) \in C_l(B_1 \times B_2) \times C_l(B_1 \times B_2) : \\ |h_1(z) - h_2(z)| \le \epsilon, z \in B_1 \times B_2\},$$

where $\epsilon > 0$. It is easy to verify that the uniform space $C_l(B_1 \times B_2)$ is metrizable and complete [44], and $C(B_1 \times B_2)$ is a closed subset of $C_l(B_1 \times B_2)$. We consider the topological space $C(B_1 \times B_2) \subset C_l(B_1 \times B_2)$ which has the relative topology.

Denote by $\mathfrak{M}_l$ the set of all lower semicontinuous functions $f : M \to R^1$ which satisfy the following growth condition.

(A) For each $\epsilon > 0$ there exists an integrable scalar function $\psi_\epsilon(t) \ge 0, t \in [T_1, T_2]$ such that $|H(t,x,u)| \le \psi_\epsilon(t) + \epsilon f(t,x,u)$ for each $(t,x,u) \in M$.

Growth condition (A) proposed by Cesari (see [21]) and its equivalents and modifications are rather common in the literature.

Denote by $\mathfrak{M}_c$ the set of all continuous functions $f \in \mathfrak{M}_l$. For $N, \epsilon > 0$ we set

$$E(N,\epsilon) = \{(f,g) \in \mathfrak{M}_l \times \mathfrak{M}_l : |f(t,x,u) - g(t,x,u)| \le \epsilon \\ ((t,x,u) \in M, |x|, |u| \le N), \quad |f(t,x,u) - g(t,x,u)| \\ \le \epsilon + \epsilon \sup\{|f(t,x,u)|, |g(t,x,u)|\} \quad (t,x,u) \in M\}.$$

We can show in a straightforward manner that for the set $\mathfrak{M}_l$ there exists the uniformity which is determined by the base $E(N, \epsilon)$, $N, \epsilon > 0$ [86, 88]. It is easy to verify that the uniform space $\mathfrak{M}_l$ is metrizable and complete. Clearly $\mathfrak{M}_c$ is a closed subset of $\mathfrak{M}_l$. We consider the topological space $\mathfrak{M}_c \subset \mathfrak{M}_l$ which has the relative topology and the spaces

$$\mathfrak{A}_l = \mathfrak{M}_l \times C_l(B_1 \times B_2), \quad \mathfrak{A}_c = \mathfrak{M}_c \times C(B_1 \times B_2),$$

which have the product topology.

We consider the functionals of the form

$$I^{(f,h)}(x,u) = \int_{T_1}^{T_2} f(t, x(t), u(t))dt + h(x(T_1), x(T_2)),$$

where $(x,u) \in \Omega$, $f \in \mathfrak{M}_l$, and $h \in C_l(B_1 \times B_2)$.

For each $f \in \mathfrak{M}_l$ and each $h \in C_l(B_1 \times B_2)$ we consider the problem of the absolute minimum

$$I^{(f,h)}(x,u) \to \min, \quad (x,u) \in \Omega,$$

and set

$$\mu(f,h) = \inf\{I^{(f,h)}(x,u) : (x,u) \in \Omega\}.$$

It is easy to see that

$$\mu(f,h) > -\infty \text{ for each } f \in \mathfrak{M}_l \text{ and each } h \in C_l(B_1 \times B_2).$$

Denote by $\mathrm{mes}(E)$ the Lebesgue measure of a measurable set $E \subset R^k$ and denote by $|\cdot|$ the Euclidean norm in R^k. Define

$$\mathfrak{A}_{l,\mathrm{reg}} = \{(f,h) \in \mathfrak{A}_l : \mu(f,h) < \infty\}, \quad \mathfrak{A}_{c,\mathrm{reg}} = \mathfrak{A}_{l,\mathrm{reg}} \cap \mathfrak{A}_c.$$

Denote by $\bar{\mathfrak{A}}_{l,\mathrm{reg}}$ the closure of $\mathfrak{A}_{l,\mathrm{reg}}$ in $\mathfrak{A}_l$ and by $\bar{\mathfrak{A}}_{c,\mathrm{reg}}$ the closure of $\mathfrak{A}_{c,\mathrm{reg}}$ in $\mathfrak{A}_c$. For each $h \in C_l(B_1 \times B_2)$ we define

$$\mathfrak{M}^h_{l,\mathrm{reg}} = \{f \in \mathfrak{M}_l : \mu(f,h) < \infty\}, \quad \mathfrak{M}^h_{c,\mathrm{reg}} = \{f \in \mathfrak{M}_c : \mu(f,h) < \infty\}.$$

Denote by $\bar{\mathfrak{M}}^h_{l,\mathrm{reg}}$ the closure of $\mathfrak{M}^h_{l,\mathrm{reg}}$ in $\mathfrak{M}_l$ and by $\bar{\mathfrak{M}}^h_{c,\mathrm{reg}}$ the closure of $\mathfrak{M}^h_{c,\mathrm{reg}}$ in $\mathfrak{M}_c$.

We showed in [86, 88] that $\mathfrak{A}_{l,\mathrm{reg}}$ is an open subset of $\mathfrak{A}_l$, $\mathfrak{A}_{c,\mathrm{reg}}$ is an open subset of $\mathfrak{A}_c$, and for each $h \in C_l(B_1 \times B_2)$, $\mathfrak{M}^h_{l,\mathrm{reg}}$ is an open subset of $\mathfrak{M}_l$, and $\mathfrak{M}^h_{c,\mathrm{reg}}$ is an open subset of $\mathfrak{M}_c$. We consider the topological subspaces $\bar{\mathfrak{A}}_{c,\mathrm{reg}} \subset \mathfrak{A}_c$, $\bar{\mathfrak{A}}_{l,\mathrm{reg}} \subset \mathfrak{A}_l$, $\bar{\mathfrak{M}}^h_{l,\mathrm{reg}} \subset \mathfrak{M}_l$, $\bar{\mathfrak{M}}^h_{c,\mathrm{reg}} \subset \mathfrak{M}_c$ $(h \in C_l(B_1 \times B_2))$ with the relative topology.

In [86, 88] we established the following results which show that generically the optimal control problem considered in this section has a unique solution.

Theorem 1.1. *There exist a set $\mathfrak{F}_l \subset \bar{\mathfrak{A}}_{l,\mathrm{reg}}$ which is a countable intersection of open everywhere dense subsets of $\bar{\mathfrak{A}}_{l,\mathrm{reg}}$ and a set $\mathfrak{F}_c \subset \bar{\mathfrak{A}}_{c,\mathrm{reg}} \cap \mathfrak{F}_l$ which is a* *countable intersection of open everywhere dense subsets of* $\bar{\mathfrak{A}}_{c,\mathrm{reg}}$*, such that for each* $(f,h) \in \mathfrak{F}_l$ *the following assertions hold:*

1. $\mu(f,h) < \infty$*, and there exists a unique* $(x^{(f,h)}, u^{(f,h)}) \in \Omega$ *for which*

$$I^{(f,h)}(x^{(f,h)}, u^{(f,h)}) = \mu(f,h).$$

2. *For each* $\epsilon > 0$ *there exist a neighborhood* U *of* (f,h) *in* $\mathfrak{A}_l$ *and a number* $\delta > 0$ *such that for each* $(g,\xi) \in U$ *and each* $(x,u) \in \Omega$ *satisfying* $I^{(g,\xi)}(x,u) \le \mu(g,\xi) + \delta$*, the following relation holds:*

$$mes\{t \in [T_1,T_2] : |x(t) - x^{(f,h)}(t)| + |u(t) - u^{(f,h)}(t)| \ge \epsilon\} \le \epsilon.$$

Note that by the Baire category theorem the set $\mathfrak{F}_l$ is nonempty and in fact everywhere dense in $\bar{\mathfrak{A}}_{l,\mathrm{reg}}$.

Theorem 1.2. *Let* $\eta \in C_l(B_1 \times B_2)$ *be fixed and let* $\mathfrak{F}_l, \mathfrak{F}_c$ *be as guaranteed in Theorem 1.1. Then there exist a set* $\mathfrak{F}_l^\eta \subset \bar{\mathfrak{M}}_{l,\mathrm{reg}}^\eta$ *which is a countable intersection of open everywhere dense subsets of* $\bar{\mathfrak{M}}_{l,\mathrm{reg}}^\eta$ *and a set* $\mathfrak{F}_c^\eta \subset \bar{\mathfrak{M}}_{c,\mathrm{reg}}^\eta \cap \mathfrak{F}_l^\eta$ *which is a countable intersection of open everywhere dense subsets of* $\bar{\mathfrak{M}}_{c,\mathrm{reg}}^\eta$*, such that*

$$\mathfrak{F}_l^\eta \times \{\eta\} \subset \mathfrak{F}_l.$$

It thus follows from Theorem 1.2 that for a fixed $\eta \in C_l(B_1 \times B_2)$ we have the properties of existence, uniqueness, and stability for all pairs (f,η) with f in $\mathfrak{F}_l^\eta$.

It should be mentioned that in [86,88] we established extensions of Theorems 1.1 and 1.2 for a class of optimal control problems with the Cinquini growth condition [22] and for a class of optimal control problems with multiple integrals.

In this book we present several generalizations and extensions of Theorems 1.1 and 1.2.

In Chap. 2 we prove generic existence results for classes of optimal control problems in which constraint maps are also subject to variations as well as the cost functions. These results were obtained in [87, 90]. More precisely, we establish generic existence results for classes of optimal control problems (with the same system of differential equations, the same boundary conditions, and without convexity assumptions) which are identified with the corresponding complete metric spaces of pairs (f,U) (where f is an integrand satisfying a certain growth condition and U is a constraint map) endowed with some natural topology. We will show that for a generic pair (f,U) the corresponding optimal control problem has a unique solution. In Sects. 2.1–2.9 we prove generic existence results for classes of optimal control problems with integrands satisfying the Cesari growth condition obtained in [87] while in Sects. 2.10–2.13 we prove generic existence results for classes of optimal control problems with integrands satisfying the Cinquini growth condition obtained in [90].

In [86, 88] we considered a class of optimal control problems which is identified with the corresponding complete metric space of integrands, say $\mathcal{F}$. We did not impose any convexity assumptions. The main result in [86, 88] establishes that for a generic integrand $f \in \mathcal{F}$ the corresponding optimal control problem is well posed. In Chap. 3 we study the set of all integrands $f \in \mathcal{F}$ for which the corresponding optimal control problem is well posed. We show that the complement of this set is not only of the first category but also of a σ-porous set. This result was obtained in [89].

In Chap. 4 we study variational problems in which the values at the end points are also subject to variations. Using the Baire category approach and the porosity notion we show that most variational problems are well posed. In Sects. 4.1–4.5 we prove generic results obtained in [92] while in Sects. 4.6–4.11 we prove porosity results obtained in [93].

In Chap. 5 we prove a generic existence and uniqueness result for a class of optimal control problems in which the right-hand side of differential equations is also subject to variations as well as the integrands. The results of Chap. 5 were obtained in [94].

In this book we usually consider topological spaces with two topologies where one is weaker than the other. (Note that they can coincide.) We refer to them as the weak and the strong topologies, respectively. If (X, d) is a metric space with a metric d and $Y \subset X$, then usually Y is also endowed with the metric d (unless another metric is introduced in Y). Assume that X_1 and X_2 are topological spaces and that each of them is endowed with a weak and a strong topology. Then for the product $X_1 \times X_2$ we also introduce a pair of topologies: a weak topology which is the product of the weak topologies of X_1 and X_2 and a strong topology which is the product of the strong topologies of X_1 and X_2. If $Y \subset X_1$, then we consider the topological subspace Y with the relative weak and strong topologies (unless other topologies are introduced). If (X_i, d_i), $i = 1, 2$ are metric spaces with the metrics d_1 and d_2, respectively, then the space $X_1 \times X_2$ is endowed with the metric d defined by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),\ (x_1, x_2),\ (y_1, y_2) \in X_1 \times X_2.$$

1.2 Lavrentiev Phenomenon

In Chaps. 6–9 we study nonoccurrence of the Lavrentiev phenomenon in optimal control and in the calculus of variations.

The Lavrentiev phenomenon in the calculus of variations was discovered in 1926 by M. Lavrentiev in [45]. There it was shown that it is possible for the variational integral of a two-point Lagrange problem, which is sequentially weakly lower semi-continuous on the admissible class of absolutely continuous functions, to possess an infimum on the dense subclass of C^1 admissible functions that is strictly greater than its minimum value on the admissible class. Since this seminal work the Lavrentiev phenomenon is of great interest. See, for instance, [1, 2, 8, 9, 21, 25, 26, 35, 49, 53, 60, 78–80] and the references mentioned there. Mania [53] simplified the original example of Lavrentiev. Ball and Mizel [8, 9] demonstrated that the Lavrentiev phenomenon can occur with fully regular integrands. Sarychev [78] constructed a broad class of integrands that exhibit the Lavrentiev phenomenon. Nonoccurrence of the Lavrentiev phenomenon was studied in [1, 2, 25, 26, 35, 49, 79, 80].

Clarke and Vinter [25] showed that the Lavrentiev phenomenon cannot occur when a variational integrand $f(t, x, u)$ is independent of t. Sychev and Mizel [80] considered a class of integrands $f(t, x, u)$ which are convex with respect to the last variable. For this class of integrands they established that the Lavrentiev phenomenon does not occur.

Sarychev and Torres [79] studied a class of optimal control problems with control-affine dynamics and with continuously differentiable integrands $f(t, x, u)$. For this class of problems they established Lipschitzian regularity of minimizers which implies nonoccurrence of the Lavrentiev phenomenon.

In [97] we studied nonoccurrence of Lavrentiev phenomenon for two classes of nonconvex nonautonomous variational problems with integrands $f(t, x, u)$. For the first class of integrands we proved the existence of a minimizing sequence of Lipschitzian functions while for the second class we showed that an infimum on the full admissible class is equal to the infimum on a set of Lipschitzian functions with the same Lipschitzian constant. Here we present these results.

Assume that $(X, ||\cdot||)$ is a Banach space. Let $-\infty < \tau_1 < \tau_2 < \infty$. Denote by $W^{1,1}(\tau_1, \tau_2; X)$ the set of all functions $x : [\tau_1, \tau_2] \to X$ for which there exists a Bochner integrable function $u : [\tau_1, \tau_2] \to X$ such that

$$x(t) = x(\tau_1) + \int_{\tau_1}^{t} u(s)ds,\ t \in (\tau_1, \tau_2]$$

(see, e.g., [16]). It is known that if $x \in W^{1,1}(\tau_1, \tau_2; X)$, then this equation defines a unique Bochner integrable function u which is called the derivative of x and is denoted by x'.

We denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue measurable set $\Omega \subset R^1$.

Let $a, b \in R^1$ satisfy $a < b$. Suppose that $f : [a, b] \times X \times X \to R^1$ is a continuous function such that the following assumptions hold:

(A1) $f(t, x, u) \geq \phi(||u||)$ for all $(t, x, u) \in [a, b] \times X \times X$, where $\phi : [0, \infty) \to [0, \infty)$ is an increasing function such that

$$\lim_{t\to\infty} \phi(t)/t = \infty.$$

(A2) For each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(t, x_1, u) - f(t, x_2, u)| \leq \epsilon \max\{f(t, x_1, u), f(t, x_2, u)\}$$

for each $t \in [a, b]$, each $u \in X$ satisfying $||u|| \geq \Gamma$ and each $x_1, x_2 \in X$ satisfying

$$||x_1 - x_2|| \leq \delta,\ ||x_1||, ||x_2|| \leq M.$$

(A3) For each $M, \epsilon > 0$ there exists $\delta > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le \epsilon$$

for each $t \in [a, b]$ and each $x_1, x_2, y_1, y_2 \in X$ satisfying

$$||x_i||, ||y_i|| \le M, \ i = 1, 2$$

and

$$||x_1 - x_2||, \ ||y_1 - y_2|| \le \delta.$$

Let $z_1, z_2 \in X$. Denote by $\mathcal{B}$ the set of all functions $v \in W^{1,1}(a, b; X)$ such that $v(a) = z_1$, $v(b) = z_2$. Denote by $\mathcal{B}_L$ the set of all $v \in \mathcal{B}$ for which there is $M_v > 0$ such that

$$||v'(t)|| \le M_v \text{ for almost every } t \in [a, b].$$

Clearly for each $v \in \mathcal{B}$ the function $f(t, v(t), v'(t))$, $t \in [a, b]$ is measurable. In [97] we considered the variational problem

$$I(v) := \int_a^b f(t, v(t), v'(t))dt \to \min, \ v \in \mathcal{B}$$

and established the following result.

Theorem 1.3. $\inf\{I(v) : \ v \in \mathcal{B}\} = \inf\{I(v) : \ v \in \mathcal{B}_L\}$.

It is not difficult to see that the following propositions hold.

Proposition 1.4. *Let* $\phi : [0, \infty) \to [0, \infty)$ *be an increasing function such that* $\lim_{t\to\infty} \phi(t)/t = \infty$*,* $g : [a, b] \times X \to R^1$ *be a continuous function such that*

$$g(t, u) \ge \phi(\|u\|) \textit{ for all } (t, u) \in [a, b] \times X,$$

and let $h : [a, b] \times X \to [0, \infty)$ *be a continuous function. Assume that for* $\xi = g, h$ *the following property holds:*

(A4) For each $M, \epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|\xi(t, x_1) - \xi(t, x_2)| \le \epsilon$$

for each $t \in [a, b]$ *and each* $x_1, x_2 \in X$ *satisfying*

$$||x_i|| \le M, \ i = 1, 2, \ ||x_1 - x_2|| \le \delta.$$

Then (A1)–(A3) hold with the function

$$f(t, x, u) = h(t, x) + g(t, u), \ (t, x, u) \in [a, b] \times X \times X.$$

Proposition 1.5. *Let* $\phi : [0, \infty) \to [0, \infty)$ *be an increasing function such that* $\lim_{t\to\infty} \phi(t)/t = \infty$*,* $g : [a, b] \times X \to R^1$ *be a continuous function such that*

$$g(t,u) \geq \phi(\|u\|) \textit{ for all } (t,u) \in [a,b] \times X,$$

and let $h : [a,b] \times X \to [0,\infty)$ *be a continuous function such that*

$$\inf\{h(t,x) : (t,x) \in [a,b] \times X\} > 0.$$

Assume that (A4) holds with $\xi = g, h$. *Then the function* $f(t,x,u) = g(t,u)h(t,x)$, $(t,x,u) \in [a,b] \times X \times X$ *satisfies (A1)–(A3).*

Corollary 1.6. *Let* $X = R^n$, $\phi : [0,\infty) \to [0,\infty)$ *be an increasing function such that*

$$\lim_{t \to \infty} \phi(t)/t = \infty,$$

$g : [a,b] \times X \to R^1$ *be a continuous function such that*

$$g(t,u) \geq \phi(\|u\|) \textit{ for all } (t,u) \in [a,b] \times X, \tag{1.1}$$

let $h : [a,b] \times X \to [0,\infty)$ *be a continuous function such that*

$$\inf\{h(t,x) : (t,x) \in [a,b] \times X\} > 0, \tag{1.2}$$

and let

$$f(t,x,u) = g(t,u)h(t,x), \ (t,x,u) \in [a,b] \times X \times X. \tag{1.3}$$

Then

$$\inf\{I(v) : v \in \mathcal{B}\} = \inf\{I(v) : v \in \mathcal{B}_L\}.$$

It should be mentioned that there are many examples of integrands of the form (1.3) for which the Lavrentiev phenomenon occurs. Corollary 1.6 shows that if such integrands satisfy inequalities (1.1) and (1.2), then the Lavrentiev phenomenon does not occur.

Now we present the second main result of [97].

Let $a, b \in R^1$, $a < b$. Suppose that $f : [a,b] \times X \times X \to R^1$ is a continuous function which satisfies the following assumptions:

(B1) There is an increasing function $\phi : [0,\infty) \to [0,\infty)$ such that

$$f(t,x,u) \geq \phi(||u||) \text{ for all } (t,x,u) \in [a,b] \times X \times X,$$

$$\lim_{t \to \infty} \phi(t)/t = \infty.$$

(B2) For each $M > 0$ there exist positive numbers δ, L and an integrable nonnegative scalar function $\psi_M(t)$, $t \in [a,b]$ such that for each $t \in [a,b]$, each $u \in X$, and each $x_1, x_2 \in X$ satisfying

$$||x_1||, ||x_2|| \leq M, \ ||x_1 - x_2|| \leq \delta$$

the following inequality holds:

$$|f(t,x_1,u) - f(t,x_2,u)| \le ||x_1 - x_2|| L(f(t,x_1,u) + \psi_M(t)).$$

(B3) For each $M > 0$ there is $L > 0$ such that for each $t \in [a,b]$ and each $x_1, x_2, u_1, u_2 \in X$ satisfying $||x_i||, ||u_i|| \le M$, $i = 1,2$ the following inequality holds:

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \le L(||x_1 - x_2|| + ||u_1 - u_2||).$$

Remark 1.7. It is not difficult to see that if (B2) holds with each ψ_M bounded, then f satisfies (A1)–(A3).

For each $z_1, z_2 \in X$ denote by $\mathcal{A}(z_1, z_2)$ the set of all $x \in W^{1,1}(a,b;X)$ such that $x(a) = z_1$, $x(b) = z_2$.

For each $x \in \mathcal{A}$ set

$$I(x) = \int_a^b f(t, x(t), x'(t))dt.$$

The next theorem is the second main result of [97].

Theorem 1.8. *Let $M > 0$. Then there exists $K > 0$ such that for each $z_1, z_2 \in X$ satisfying $||z_1||, ||z_2|| \le M$ and each $x(\cdot) \in \mathcal{A}(z_1, z_2)$ the following assertion holds:*

If mes$\{t \in [a,b] : \ ||x'(t)|| > K\} > 0$, then there exists $y \in \mathcal{A}(z_1, z_2)$ such that $I(y) < I(x)$ and $||y'(t)|| \le K$ for almost every $t \in [a,b]$.

Remark 1.9. (B3) implies that f is bounded on any bounded subset of $[a,b] \times X \times X$.

It is not difficult to see that the following proposition holds.

Proposition 1.10. *Let $\phi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\lim_{t\to\infty} \phi(t)/t = \infty$, $g : [a,b] \times X \to R^1$ be a continuous function such that*

$$g(t,u) \ge \phi(\|u\|) \text{ for all } (t,u) \in [a,b] \times X,$$

and let $h : [a,b] \times X \to [0,\infty)$ be a continuous function such that

$$\inf\{h(t,x) : \ (t,x) \in [a,b] \times X\} > 0.$$

Assume that for $\xi = g, h$ the following property holds:

For each $M > 0$ there is $L > 0$ such that for each $t \in [a,b]$ and each $x_1, x_2, u_1, u_2 \in X$ satisfying $||x_i||, ||u_i|| \le M$, $i = 1,2$ the following inequality holds:

$$|\xi(t,x_1) - \xi(t,x_2)| \le L||x_1 - x_2||.$$

Then (B1)–(B3) hold with the function

$$f(t,x,u) = h(t,x)g(t,u), \ (t,x,u) \in [a,b] \times X \times X.$$

The work [97] became a starting point of the author's research on nonoccurrence of Lavrentiev phenomenon for nonconvex nonautonomous variational and optimal control problems. The most important results which were obtained are presented in Chaps. 6–9 of this book.

In Chap. 6 we study nonoccurrence of the Lavrentiev phenomenon for a large class of nonconvex nonautonomous constrained variational problems. A state variable belongs to a convex subset H of a Banach space X with nonempty interior. Integrands belong to a complete metric space of functions $\mathcal{M}_B$ which satisfy a growth condition common in the literature and are Lipschitzian on bounded sets. We show nonoccurrence of the Lavrentiev phenomenon for most elements of $\mathcal{M}_B$ in the sense of Baire category. The results of Chap. 6 were obtained in [101].

In Chap. 7 we study nonoccurrence of the Lavrentiev phenomenon for a large class of nonconvex optimal control problems which is identified with the corresponding complete metric space of integrands $\mathcal{M}$ which satisfy a growth condition common in the literature and are Lipschitzian on bounded sets. We establish that for most elements of $\mathcal{M}$ (in the sense of Baire category) the infimum on the full admissible class of trajectory-control pairs is equal to the infimum on a subclass of trajectory-control pairs whose controls are bounded by a certain constant. The results of Chap. 7 were obtained in [100].

In Chap. 8 we show nonoccurrence of the Lavrentiev phenomenon for a class of nonconvex optimal control problems. We show that for most problems (in the sense of Baire category) the infimum on the full admissible class of trajectory-control pairs is equal to the infimum on a subclass of trajectory-control pairs with bounded controls. This result was obtained in [103].

In Chap. 9 we show nonoccurrence of gap for two large classes of infinite-dimensional linear control systems in a Hilbert space with nonconvex integrands. These classes are identified with the corresponding complete metric spaces of integrands which satisfy a growth condition common in the literature. For most elements of the first space of integrands (in the sense of Baire category) we establish the existence of a minimizing sequence of trajectory-control pairs with bounded controls. We also establish that for most elements of the second space (in the sense of Baire category) the infimum on the full admissible class of trajectory-control pairs is equal to the infimum on a subclass of trajectory-control pairs whose controls are bounded by a certain constant. The results of Chap. 9 were obtained in [104].

1.3 Turnpike Properties

Chapters 10–12 are devoted to turnpike theory and infinite horizon optimal control. The study of the existence and the structure of (approximate) solutions of variational and optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research [3–5, 10, 14, 15, 17, 18, 34, 36, 38, 43, 46, 50, 52, 62, 64, 68].

In this book we analyze the structure of solutions of the variational problems

$$\int_{T_1}^{T_2} f(t, z(t), z'(t))dt \to \min, \; z(T_1) = x, \; z(T_2) = y, \qquad (P)$$

$$z: \; [T_1, T_2] \to R^n \text{ is an absolutely continuous function,}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, and $f : [0, \infty) \times R^n \times R^n \to R^1$ belong to a space of integrands described in Chap. 11.

It is well known that the solutions of the problems (P) exist for integrands f which satisfy two fundamental hypotheses concerning the behavior of the integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex. For integrands f which do not satisfy the convexity assumption the existence of solutions of the problems (P) is not guaranteed and in this situation we consider *δ-approximate solutions*.

Let $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, $f : [0, \infty) \times R^n \times R^n \to R^1$ be an integrand and let δ be a positive number. We say that an absolutely continuous (a.c.) function $u : [T_1, T_2] \to R^n$ satisfying $u(T_1) = x$, $u(T_2) = y$ is a δ-approximate solution of the problem (P) if

$$\int_{T_1}^{T_2} f(t, u(t), u'(t))dt \leq \int_{T_1}^{T_2} f(t, z(t), z'(t))dt + \delta$$

for each a.c. function $z : \; [T_1, T_2] \to R^n$ satisfying $z(T_1) = x, \; z(T_2) = y$.

In Chaps. 10 and 11 we deal with the so-called turnpike property of the variational problems (P). To have this property means, roughly speaking, that the approximate solutions of the problems (P) are determined mainly by the integrand (cost function) and are essentially independent of the choice of interval and end point conditions, except in regions close to the end points.

Let us now give the precise definition of this notion.

We say that an integrand $f = f(t, x, u) \in C([0, \infty) \times R^n \times R^n)$ has the turnpike property if there exists a continuous function $X_f : [0, \infty) \to R^n$ (called the "turnpike") which satisfies the following condition:

For each bounded set $K \subset R^n$ and each $\epsilon > 0$ there exist constants $T > 0$ and $\delta > 0$ such that for each $T_1 \geq 0$, each $T_2 \geq T_1 + 2T$, each $x, y \in K$, and each δ-approximate solution $v : \; [T_1, T_2] \to R^n$ of the variational problem (P) the relation $|v(t) - X_f(t)| \leq \epsilon$ holds for all $t \in [T_1 + T, T_2 - T]$.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [77]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). Many turnpike results are collected in [99].

In this book we study the turnpike property of approximate solutions of the problems (P) with integrands f which belong to a complete metric space of functions $\mathfrak{M}$ to be described in Chap. 11. We do not impose any convexity assumption on f. This class of variational problems was studied in Chap. 2 of [99] for integrands f which belong to a subset $\mathfrak{M}_{co}$ of $\mathfrak{M}$. The subset $\mathfrak{M}_{co} \subset \mathfrak{M}$ consists of integrands $f \in \mathfrak{M}$ such that the function $f(t, x, \cdot) : R^n \to R^1$ is convex for any $(t, x) \in [0, \infty) \times R^n$. In Chap. 2 of [99] we showed that the turnpike property holds for a generic integrand $f \in \mathfrak{M}_{co}$. Namely we established the existence of a set $\mathcal{F}_{co} \subset \mathfrak{M}_{co}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}_{co}$ such that each $f \in \mathcal{F}_{co}$ has the turnpike property.

In this book we extend this turnpike result of [99] established for the space $\mathfrak{M}_{co}$ to the space of integrands $\mathfrak{M}$. We show the existence of a set $\mathcal{F} \subset \mathfrak{M}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{M}$ such that each $f \in \mathcal{F}$ has the turnpike property. We show that an integrand $f \in \mathcal{F}$ has a turnpike X_f which is a bounded continuous function. This result was obtained in [96].

In Chap. 10, given an $x_0 \in R^n$ we study the infinite horizon problem of minimizing the expression $\int_0^T f(t, x(t), x'(t))dt$ as T grows to infinity where $x : [0, \infty) \to R^n$ satisfies the initial condition $x(0) = x_0$. We analyze the existence and properties of approximate solutions for every prescribed initial value x_0. We also show that for every bounded set $E \subset R^n$ the $C([0, T])$ norms of approximate solutions $x : [0, T] \to R^n$ for the variational problem on an interval $[0, T]$ with $x(0), x(T) \in E$ are bounded by some constant which does not depend on T. The results of the chapter were obtained in [95].

In Chap. 11 we study the turnpike property of approximate solutions of variational problems with continuous integrands $f : [0, \infty) \times R^n \times R^n$ which belong to a complete metric space of functions $\mathfrak{M}$.

In Chap. 12 we establish a turnpike property of approximate solutions for a general class of discrete-time control systems without discounting and with a compact metric space of states. This class of control systems is identified with a complete metric space of objective functions. We show that for a generic objective function approximate solutions of the corresponding control system possess the turnpike property. This result was obtained in [107].

1.4 Examples

In this section we present examples of variational problems.

Example 1.11. Consider the variational problem

$$\int_0^1 [(x(t))^2 + (x'(t))^2]dt \to \min$$

$$x : [0, 1] \to R^1 \text{ is an a.c. function such that}$$

$$x(0) = 0, \; x(1) = 0$$

with the integrand $f(t, x, u) = x^2 + u^2$, $(t, x, u) \in R^3$. Clearly, the integrand f satisfies the growth condition (A) of Sect. 1.1, $f \in \mathfrak{M}_c$, and the function $x_*(t) = 0$, $t \in [0, 1]$ is the unique solution of the variational problem.

Assume that $\epsilon \in (0, 1)$ and an a.c. function $x : [0, 1] \to R^1$ satisfies

$$x(0) = 0, \ x(1) = 0$$

and

$$\int_0^1 [(x(t))^2 + (x'(t))^2] dt \le \epsilon.$$

Then it is not difficult to see that

$$\text{mes}\{t \in [0, 1] : \ |x(t)| + |x'(t)| \ge 2\epsilon^{1/4}\} \le \epsilon^{1/2}.$$

Example 1.12. Consider the variational problem

$$\int_0^1 [(x(t))^2 + ((x'(t))^2 - 1)^2] dt \to \min$$

$$x : [0, 1] \to R^1 \text{ is an a.c. function such that}$$

$$x(0) = 0, \ x(1) = 0$$

with the integrand $f(t, x, u) = x^2 + (u^2 - 1)^2$, $(t, x, u) \in R^3$. Clearly, the integrand f satisfies the growth condition (A) of Sect. 1.1 and $f \in \mathfrak{M}_c$.

Let n be a natural number. There exists an a.c. function $x_n : [0, 1] \to R^1$ such that for each integer $i \in [0, n-1]$, x_n is affine on the intervals $[in^{-1}, (2i+1)(2n)^{-1}]$ and $[(2i + 1)(2n)^{-1}, (i + 1)n^{-1}]$, and

$$x_n(in^{-1}) = 0, \ x_n((i + 1)n^{-1}) = 0, \ x_n((2i + 1)(2n)^{-1}) = (2n)^{-1}.$$

It is not difficult to see that

$$\int_0^1 [(x_n(t))^2 + ((x_n'(t))^2 - 1)^2] dt = \int_0^1 (x_n(t))^2 dt \le (2n)^{-1}.$$

This implies that the infimum of our integral functional over the set of admissible functions is zero. On the other hand, our variational problem does not have a solution.

Since the Lipschitz constant of x_n is 1 for any natural number n, the Lavrentiev phenomenon does not hold for our variational problem.

Example 1.13. Let

$$f(t, x, u) = (x - \cos(t))^2 + (u + \sin(t))^2, \ (t, x, u) \in R^1 \times R^1 \times R^1$$

and consider the family of the variational problems

$$\int_{T_1}^{T_2} [(v(t) - \cos(t))^2 + (v'(t) + \sin(t))^2] dt \to \min, \qquad (P)$$

$$v : [T_1, T_2] \to R^1 \text{ is an absolutely continuous function}$$
$$\text{such that } v(T_1) = y,\ v(T_2) = z,$$

where $y, z, T_1, T_2 \in R^1$ and $T_2 > T_1$. The integrand f depends on t, for each $t \in R^1$ the function $f(t, \cdot, \cdot) : R^2 \to R^1$ is convex, and for each $x, u \in R^1 \setminus \{0\}$ the function $f(\cdot, x, u) : R^1 \to R^1$ is nonconvex. Thus the function $f : R^1 \times R^1 \times R^1 \to R^1$ is also nonconvex and depends on t.

Assume that $y, z, T_1, T_2 \in R^1$, $T_2 > T_1 + 2$ and $\hat{v} : [T_1, T_2] \to R^1$ is an optimal solution of the problem (P). Note that the problem (P) has a solution since f is continuous and $f(t, x, \cdot) : R^1 \to R^1$ is convex and grows superlinearly at infinity for each $(t, x) \in [0, \infty) \times R^1$.

Define $v : [T_1, T_2] \to R^1$ by

$$v(t) = y + (\cos(1) - y)(t - T_1),\ t \in [T_1, T_1 + 1],$$
$$v(t) = \cos(t),\ t \in [T_1 + 1, T_2 - 1],$$
$$v(t) = \cos(T_2 - 1) + (t - T_2 + 1)(z - \cos(T_2)),\ t \in [T_2 - 1, T_2].$$

It is easy to see that

$$\int_{T_1+1}^{T_2-1} f(t, v(t), v'(t)) dt = 0$$

and

$$\int_{T_1}^{T_2} f(t, \hat{v}(t), \hat{v}'(t)) dt \le \int_{T_1}^{T_2} f(t, v(t), v'(t)) dt$$
$$= \int_{T_1}^{T_1+1} f(t, v(t), v'(t)) dt + \int_{T_2-1}^{T_2} f(t, v(t), v'(t)) dt$$
$$\le 2 \sup\{|f(t, x, u)| :\ t, x, u \in R^1,\ |x|, |u| \le |y| + |z| + 1\}.$$

Thus

$$\int_{T_1}^{T_2} f(t, \hat{v}(t), \hat{v}'(t)) dt \le c_1(|y|, |z|),$$

where

$$c_1(|y|, |z|) = 2 \sup\{|f(t, x, u)| :\ t, x, u \in R^1,\ |x|, |u| \le |y| + |z| + 1\}.$$

For any $\epsilon \in (0, 1)$ we have

$$\begin{aligned}
&\text{mes}\{t \in [T_1, T_2] : |\hat{v}(t) - \cos(t)| > \epsilon\} \\
&\le \epsilon^{-2} \int_{T_1}^{T_2} f(t, \hat{v}(t), \hat{v}'(t))dt \le \epsilon^{-2} c_1(|y|, |z|).
\end{aligned}$$

Since the constant $c_1(|y|, |z|)$ does not depend on T_2 and T_1 we conclude that if $T_2 - T_1$ is sufficiently large, then the optimal solution $\hat{v}(t)$ is equal to $\cos(t)$ up to ϵ for most $t \in [T_1, T_2]$. Moreover, we can show that

$$\{t \in [T_1, T_2] : |x(t) - \cos(t)| > \epsilon\} \subset [T_1, T_1 + \tau] \cup [T_2 - \tau, T_2],$$

where $\tau > 0$ is a constant which depends only on ϵ, $|y|$, and $|z|$.

Thus the integrand f has the turnpike property.

Chapter 2
Well-posedness of Optimal Control Problems Without Convexity Assumptions

In this chapter we prove generic existence results for classes of optimal control problems in which constraint maps are also subject to variations as well as the cost functions. These results were obtained in [87, 90]. More precisely, we establish generic existence results for classes of optimal control problems (with the same system of differential equations, the same boundary conditions and without convexity assumptions) which are identified with the corresponding complete metric spaces of pairs (f, U) (where f is an integrand satisfying a certain growth condition and U is a constraint map) endowed with some natural topology. We will show that for a generic pair (f, U) the corresponding optimal control problem has a unique solution.

In the theory developed here topologies on spaces of integrands and on spaces of integrand–map pairs are of great importance. Actually one space of integrand–map pairs, say $\mathcal{A}$, considered here is a topological product of a space of integrands and a space of multivalued maps. The values of these maps are elements of the space of all nonempty convex closed subsets of a finite-dimensional Euclidean space endowed with the Hausdorff distance. In the space of multivalued maps we consider the topology of uniform convergence. For the space of integrands we consider weak and strong topologies which induce weak and strong topologies on the space $\mathcal{A}$. We will prove the existence of a set $\mathcal{A}' \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets such that for each $(f, U) \in \mathcal{A}'$ the corresponding optimal control problem has a unique solution. In fact we will establish our result for various spaces of integrands: the space of the so-called $\mathcal{L} \bigotimes \mathcal{B}$-measurable integrands, the space of lower semicontinuous integrands and the space of continuous integrands, as well as their subspaces consisting of integrands $f(t, x, u)$ differentiable in u and subspaces consisting of integrands $f(t, x, u)$ differentiable in x and u. All these spaces are endowed with same weak topology. Their strong topology is always stronger then the topology of uniform convergence.

If we say that a function (set) is measurable we mean that it is Lebesgue measurable.

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2.1 Optimal Control Problems with Cesari Growth Condition

We use the following notations and definitions. Let $k \geq 1$ be an integer. We denote by $\text{mes}(E)$ the Lebesgue measure of a measurable set $E \subset R^k$, by $|\cdot|$ the Euclidean norm in R^k, and by $< \cdot, \cdot >$ the scalar product in R^k. We use the convention that $\infty - \infty = 0$. For any $f \in C^q(R^k)$ we set

$$||f||_{C^q} = ||f||_{C^q(R^k)} = \sup_{z \in R^k} \{|\partial^{|\alpha|} f(z)/\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}| : \tag{2.1}$$

$$\alpha_i \geq 0 \text{ is an integer, } i = 1, \dots, k, \; |\alpha| \leq q\},$$

where $|\alpha| = \sum_{i=1}^k \alpha_i$.

For each function $f : X \to [-\infty, \infty]$ where X is nonempty, we set $\inf(f) = \inf\{f(x) : x \in X\}$. For each set-valued mapping $U : X \to 2^Y \setminus \{\emptyset\}$ where X and Y are nonempty, we set

$$\text{graph}(U) = \{(x, y) \in X \times Y : \; y \in U(x)\}. \tag{2.2}$$

Let $m, n, N \geq 1$ be integers. We assume that Ω is a fixed bounded domain in R^m, $H(t, x, u)$ is a fixed continuous function defined on $\Omega \times R^n \times R^N$ with values in R^{mn} such that $H(t, x, u) = (H_i)_{i=1}^n$ and $H_i = (H_{ij})_{j=1}^m$, $i = 1, \dots n$, B_1 and B_2 are fixed nonempty closed subsets of R^n and $\theta^* = (\theta_i^*)_{i=1}^n \in (W^{1,1}(\Omega))^n$ is also fixed. Here

$$W^{1,1}(\Omega) = \{u \in L^1(\Omega) : \; \partial u/\partial x_j \in L^1(\Omega), \; j = 1, \dots m\}$$

and $W_0^{1,1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,1}(\Omega)$, where $C_0^\infty(\Omega)$ is the space of smooth functions $u : \; \Omega \to R^1$ with compact support in Ω [108].

If $m = 1$, then we assume that $\Omega = (T_1, T_2)$, where T_1 and T_2 are fixed real numbers for which $T_1 < T_2$.

For a function $u = (u_1, \dots u_n)$, where $u_i \in W^{1,1}(\Omega)$, $i = 1, \dots n$, we set

$$\nabla u_i = (\partial u_i/\partial x_j)_{j=1}^m, \; i = 1, \dots n, \; \nabla u = (\nabla u_i)_{i=1}^n.$$

Define set-valued mappings $\tilde{A} : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ and $\tilde{U} : \Omega \times R^n \to 2^{R^N} \setminus \{\emptyset\}$ by

$$\tilde{A}(t) = R^n, \; t \in \Omega, \; \tilde{U}(t, x) = R^N, \; (t, x) \in \Omega \times R^n. \tag{2.3}$$

For each $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ and each $U : \text{graph}(A) \to 2^{R^N} \setminus \{\emptyset\}$ for which $\text{graph}(U)$ is a closed subset of the space $\Omega \times R^n \times R^N$ with the product topology, we denote by $X(A, U)$ the set of all pairs of functions (x, u), where

$x = (x_1, \dots, x_n) \in (W^{1,1}(\Omega))^n$, $u = (u_1, \dots u_N) : \Omega \to R^N$ is Lebesgue measurable and the following relations hold:

$$x(t) \in A(t),\ t \in \Omega \text{ (a.e.)},\ u(t) \in U(t, x(t)),\ t \in \Omega \text{ (a.e.)}, \tag{2.4a}$$

$$\nabla x(t) = H(t, x(t), u(t)),\ t \in \Omega \text{ (a.e.)}, \tag{2.4b}$$

$$\text{if } m = 1, \text{ then } x(T_i) \in B_i,\ i = 1, 2, \tag{2.4c}$$

$$\text{if } m > 1 \text{ then } x - \theta^* \in (W_0^{1,1}(\Omega))^n. \tag{2.4d}$$

Note that in the definition of the space $X(A, U)$ we use the boundary condition (2.4c) in the case $m = 1$ while in the case $m > 1$ we use the boundary condition (2.4d). Both of them are common in the literature [12, 13, 17, 21].

We do this to provide a unified treatment for both cases. Note that we prove our generic result in the case $m = 1$ for a class of Bolza problems (with the same boundary condition (2.4c)) while in the case $m > 1$ it will be established for a class of Lagrange problems (with the same boundary condition (2.4d)).

To be more precise, we have to define elements of $X(A, U)$ as classes of pairs equivalent in the sense that (x_1, u_1) and (x_2, u_2) are equivalent if and only if $x_2(t) = x_1(t)$, $u_2(t) = u_1(t)$, $t \in \Omega$ (a.e.) If $m = 1$, then by an appropriate choice of representatives, $W^{1,1}(T_1, T_2)$ can be identified with the set of absolutely continuous functions $x : [T_1, T_2] \to R^1$, and we will henceforth assume that this has been done.

Let $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$, U : graph$(A) \to 2^{R^N} \setminus \{\emptyset\}$ and let graph(U) be a closed subset of the space $\Omega \times R^n \times R^N$ with the product topology.

For the set $X(A, U)$ defined above we consider the uniformity which is determined by the following base:

$$E_X(\epsilon) = \{((x_1, u_1), (x_2, u_2)) \in X(A, U) \times X(A, U) : \tag{2.5}$$
$$\text{mes}\{t \in \Omega :\ |x_1(t) - x_2(t)| + |u_1(t) - u_2(t)| \geq \epsilon\} \leq \epsilon\},$$

where $\epsilon > 0$. It is easy to see that the uniform space $X(A, U)$ is metrizable (by a metric ρ) (see [44]). In the space $X(A, U)$ we consider the topology induced by the metric ρ.

Next we define spaces of integrands associated with the maps A and U. By $\mathcal{M}(A, U)$ we denote the set of all functions f : graph$(U) \to R^1 \cup \{\infty\}$ with the following properties:

(i) f is measurable with respect to the σ-algebra generated by products of Lebesgue measurable subsets of Ω and Borel subsets of $R^n \times R^N$.
(ii) $f(t, \cdot, \cdot)$ is lower semicontinuous for almost every $t \in \Omega$.
(iii) For each $\epsilon > 0$ there exists an integrable scalar function $\psi_\epsilon(t) \geq 0$, $t \in \Omega$, such that $|H(t, x, u)| \leq \psi_\epsilon(t) + \epsilon f(t, x, u)$ for all $(t, x, u) \in$ graph (U).