BEAM PROPAGATION
METHOD FOR DESIGN OF
OPTICAL WAVEGUIDE
DEVICES
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OPTICAL WAVEGUIDE
DEVICES

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Universidad Autónoma de Madrid, Spain
To my beloved sisters:
Belín, María Cinta, María José, Lidia, Pilar and Zoila.
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The aim of this book is to provide the fundamentals and the applications of the beam-propagation method (BPM) implemented by finite-difference (FD) techniques, which is a widely used mathematical tool to simulate light wave propagation along axially varying optical waveguide structures. The content covers the background, variations of the method, numerical implementations and applications of the methodology to many practical examples. Thus, the book gives systematic and comprehensive reviews and tutorials on the analysis and design of integrated photonics devices based on optical waveguides using FD-BPM. It treats almost all aspects of BPM analysis, from fundamentals through the advancements developed by extension and modifications, to the most recent applications to specific integrated optical devices.

The book can be a text for postgraduate courses devoted to numerical simulation of integrated photonic devices. Also, it is suitable for supplementary or background reading in modern curricula graduate courses such as ‘Optoelectronics’, ‘Optical engineering’, ‘Optical-wave electronics’, ‘Photonics’ or ‘Integrated optics’. This book is also of interest for professional researchers and engineers in the area of integrated optics, optoelectronics and optical communications. Although BPM codes are commercially available, or even free, many engineers must develop their own software to suit their particular requirements. This book can serve both the building of home-made codes, as well for use of existing software by understanding the underlying approaches inherent in the BPM and its range of applicability.

Integrated photonics devices are based on optical waveguides with transversal dimensions of the order of microns. This means that the light propagation along these structures cannot be analysed in terms of ray optics; instead the light must be treated as electromagnetic waves. Hence, Chapter 1 presents the basics of the electromagnetic theory of light, starting from Maxwell’s equations in inhomogeneous media. Wave equations in terms of the transverse field components in inhomogeneous media are obtained, including the treatment of anisotropic media and second-order non-linear media. Using the slowly varying approximation, full vectorial equations for the electric and magnetic fields are obtained in Chapter 2, which are the basic differential equations for developing BPM algorithms. Finite-difference approximations
of the wave equations are then derived for the simplest case of scalar propagation in two-dimensional structures that allows us to study the stability and numerical dissipation of FD-BPM schemes. Chapter 3 develops full vectorial FD-BPM algorithms for the simulation of light propagation in 2D and 3D structures where the numerical implementations of the FD-BPM are detailed.

Extensions and modifications of the BPM approaches based on finite-difference techniques are presented in Chapter 4. These include wide-angle BPM, which relaxes the restriction of the application of BPM to paraxial waves and allows the simulation of light beams with large propagation angles respect of the longitudinal direction. BPM algorithms, which can handle multiple reflections known as bidirectional-BPM, are then discussed. Simulation of light propagation in active media, second-order non-linear media and anisotropic media are also topics covered in Chapter 4. The last sections include the description of time-domain (TD) simulation techniques based on finite differences, which can simulate the propagation of optical pulses and can manage backward waves due to reflections at waveguide discontinuities. Both the time-domain BPM and finite-difference time-domain (FDTD) are explained in detail. The different BPMs supply almost universal numerical tools for describing the performance of a great variety of integrated optical devices. Although particular devices have specific routes to be modelled with their own constraints, the great advantage of the BPM lies in the fact that, as few approximations have been made for its derivation, its applicability is quite wide and almost any integrated photonic device can be modelled by using it. The last chapter, Chapter 5, presents selected examples of integrated optical elements commonly used in practical integrated photonic devices, where their performance and relevant characteristics are analysed by the appropriate BPM approach.

Some appendices have been added at the end of the book. They include material related to BPM algorithms or BPM simulations of some integrated photonics devices, but which is not indispensable to the understanding of the different topics developed along the book chapters. The appendices include mathematical derivations of some formulae, physical phenomena descriptions and even relevant program listing.

Commonly accepted notation and symbols have been utilized throughout this book. However, some of the symbols have multiple meanings and therefore a list of symbols and their meanings is provided at the beginning of the book to clarify symbol usage. Also, a list of acronyms is given to help the reader.

A selection of BPM programs are made available free of charge for the readers at the website of the author (www.uam.es/personal_pdi/ciencias/glifante). Among others, this selection includes the programs ‘Vectorial mode solver for planar waveguides’, ‘Vectorial light propagation in 2D-structures’, ‘Vectorial light propagation in 3D-structures’ and ‘2D-light propagation in the time domain’.

Ginés Lifante Pedrola
Madrid, February 2015
List of Acronyms

ABC  absorbing boundary conditions
ADI  alternating direction implicit
ASE  amplified spontaneous emission
AWG  arrayed waveguide grating
BC   boundary condition
Bi-BPM bidirectional beam propagation method
BPM  beam propagation method
CCS  complementary coplanar strip
CFL  Courant–Friedrichs–Levy stability criterion
CN   Crank–Nicolson scheme
CS   coplanar strip
CW   continuous wave
DFR  distributed feedback reflector
EIM  effective index method
EM   electromagnetic
EO   electro-optic
ESW  equivalent straight waveguide
FD   finite difference
FD-BPM finite-difference beam-propagation method
FDTD finite-difference time domain
FE   finite element
FE-BPM finite-element beam-propagation method
FFT  fast Fourier transform
FFT-BPM fast Fourier transform beam-propagation method
FT   Fourier transform
FPR  free propagation region
FSR  free spectral range
FV-BPM full vectorial beam propagation method
IFD-NL-BPM iterative finite difference non-linear beam-propagation method
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>IL</td>
<td>insertion loss</td>
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<tr>
<td>Im-Dis-BPM</td>
<td>imaginary distance beam-propagation method</td>
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<td>IR</td>
<td>infrared</td>
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<tr>
<td>MMI</td>
<td>multimode interference</td>
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<tr>
<td>MPA</td>
<td>modal propagation analysis</td>
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<tr>
<td>MZ</td>
<td>Mach–Zehnder</td>
</tr>
<tr>
<td>MZI</td>
<td>Mach–Zehnder interferometer</td>
</tr>
<tr>
<td>NL</td>
<td>non-linear</td>
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<tr>
<td>NL-BPM</td>
<td>non-linear beam propagation method</td>
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<tr>
<td>OI</td>
<td>overlap integral</td>
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<td>PHASAR</td>
<td>phase-array</td>
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<td>PML</td>
<td>perfectly matched layer</td>
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<td>QPM</td>
<td>quasi-phase matching</td>
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<td>RE</td>
<td>rare earth</td>
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<td>SH</td>
<td>second harmonic</td>
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<td>SHG</td>
<td>second harmonic generation</td>
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<td>SV-BPM</td>
<td>semi-vectorial beam-propagation method</td>
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<td>SVE</td>
<td>slowly varying envelope</td>
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<td>SVEA</td>
<td>slowly varying envelope approximation</td>
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<td>TBC</td>
<td>transparent boundary conditions</td>
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<td>TD-BPM</td>
<td>time-domain beam-propagation method</td>
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<td>TE</td>
<td>transverse electric field</td>
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<td>TF/SF</td>
<td>total field/scattered field</td>
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<td>TM</td>
<td>transverse magnetic field</td>
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<td>UV</td>
<td>ultraviolet</td>
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<td>WDM</td>
<td>wavelength division multiplexing</td>
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List of Symbols

**Roman Symbols**

\( a_j \)  
tridiagonal system coefficient

\( a_\nu \)  
modal weight

\( A \)  
attenuation; also, waveguide cross section

\( A_x \)  
x dependent part of the operator \( P_{xx} \) (or \( Q_{xx} \))

\( A_y \)  
y dependent part of the operator \( P_{xx} \) (or \( Q_{xx} \))

\( A_{ij} \)  
spontaneous emission probability

\( b \)  
normalized propagation constant

\( b_j \)  
tridiagonal system coefficient

\( B_x \)  
x dependent part of the \( P_{yy} \) (or \( Q_{yy} \)) operator

\( B_y \)  
y dependent part of the \( P_{yy} \) (or \( Q_{yy} \)) operator

\( \mathbf{B}(r,t) \)  
magnetic flux density vector

\( c \)  
speed of light in free space

\( c_j \)  
tridiagonal system coefficient

\( C \)  
\( P_{xy} \) (or \( Q_{xy} \)) operator (cross-coupling term)

\( C_{M,N}(i,j) \)  
coefficient for the FDTD algorithm

\( d \)  
thickness, depth

\( d_{ijk} \)  
second-order non-linear tensor

\( d_{PML} \)  
PML thickness

\( D \)  
\( P_{yx} \) (or \( Q_{yx} \)) operator (cross-coupling term); also, distance, separation

\( D_{M,N}(i,j) \)  
coefficient for the FDTD algorithm

\( \mathbf{D} \)  
(complex amplitude of ) displacement vector

\( \mathcal{D}(r,t) \)  
eddy displacement vector

\( E_i \)  
i Cartesian component of \( E \)

\( E_{yx}, E_{yz} \)  
splitting (sub-components) of the magnetic field component \( E_y \) in FDTD

\( E(r) \)  
(complex amplitude of \( \mathcal{E}(r,t) \) for monochromatic waves

\( E_x(r) \)  
transverse component of \( E \)

\( \mathcal{E}(r,t) \)  
etric field

\( f(x) \)  
mode transversal profile
List of Symbols

\( f_v (x,y) \)  \( \text{eigenmode (eigenvector, transversal field distribution)} \)
\( F_j \)  \( \text{energy flux leaving the } j \text{-boundary} \)
\( \mathcal{F} \)  \( \text{operator for NL-BPM} \)
\( G \)  \( \text{differential operator (in 3D-scalar wave equation)} \)
\( G_x, G_y \)  \( \text{split of the differential operator } G \)
\( h \)  \( \text{Planck’s constant; also, height} \)
\( \mathcal{H} \)  \( \text{differential operator for TD-BPM} \)
\( H_i \)  \( i \text{ Cartesian component of } H \)
\( \mathcal{H}_x, \mathcal{H}_y \)  \( \text{split of the differential operator for TD-BPM} \)
\( H_{xy}, H_{yz} \)  \( \text{splitting (sub-components) of the magnetic field component } H_y \text{ in FDTD} \)
\( H(r) \)  \( \text{complex amplitude of } \mathcal{H}(r,t) \text{ for monochromatic waves} \)
\( H_x(r) \)  \( \text{transverse component of } H \)
\( \mathcal{H}_0(r) \)  \( \text{magnetic field amplitude for monochromatic waves} \)
\( \hat{H} \)  \( \text{matrix differential operator for } \Psi_t \text{ (or } \Phi_t \text{)} \)
\( i \)  \( \text{imaginary unity; also, integer} \)
\( I \)  \( \text{intensity (or irradiance)} \)
\( I_0 \)  \( \text{intensity of a monochromatic plane wave} \)
\( I_p \)  \( \text{pump intensity} \)
\( j \)  \( \text{integer} \)
\( \mathcal{J}(r,t) \)  \( \text{electric current density} \)
\( k \)  \( \text{wavenumber in the medium; also, integer} \)
\( k_0 \)  \( \text{wavenumber in free space} \)
\( K \)  \( \text{reference wavenumber} \)
\( \bar{K} \)  \( \text{complex-valued reference wavenumber} \)
\( L \)  \( \text{window size; also, length} \)
\( L_x, L_y \)  \( \text{transversal grid dimensions} \)
\( L_z \)  \( \text{longitudinal length} \)
\( L_c \)  \( \text{coupling length} \)
\( L_A, L_B \)  \( \text{pseudo-differential operators} \)
\( L_j \)  \( \text{operators for NL-BPM} \)
\( m \)  \( \text{integer} \)
\( M_m \)  \( \text{polynomial of degree } m \)
\( \mathcal{M} \)  \( \text{overall transfer matrix (for Bi-BPM)} \)
\( \mathcal{M}(r,t) \)  \( \text{magnetic current density} \)
\( n \)  \( \text{refractive index; also, integer} \)
\( n_0 \)  \( \text{reference refractive index} \)
\( n_c \)  \( \text{complex refractive index} \)
\( n_p \)  \( \text{refractive index of the PML medium} \)
\( N \)  \( \text{concentration of active ions} \)
\( N_e \)  \( \text{effective index of the symmetric mode} \)
\( N_{\text{eff}} \)  \( \text{effective index of the mode} \)
\( N_i \)  \( \text{population density in the } i \text{-th level} \)
\( N_n \)  \( \text{polynomial of degree } n \)
\( N_o \)  \( \text{effective indices of the anti-symmetric mode} \)
\( N_x, N_y \)  \( \text{number of transversal grid points} \)
\( O[.] \) \hspace{1cm} \text{approximation order in FD schemes}

\( p \) \hspace{1cm} \text{power exponent of the PML profile}

\( P(z) \) \hspace{1cm} \text{complex field amplitude correlation function}

\( P(\xi) \) \hspace{1cm} \text{Fourier transform of} \ P(z)

\( P_j \) \hspace{1cm} \text{propagation matrix (for bidirectional BPM)}

\( P_{ij} \) \hspace{1cm} \text{differential operator for the transverse SVE field} \ \psi_i \n
\( \mathcal{P}(r,t) \) \hspace{1cm} \text{polarization vector}

\( \mathcal{P}_{NL} \) \hspace{1cm} \text{non-linear polarization}

\( Q_{ij} \) \hspace{1cm} \text{differential operators for the transverse SVE field} \ \Phi_t \n
\( Q_m \) \hspace{1cm} \text{Von Neumann analysis parameter}

\( Q \) \hspace{1cm} \text{operator for wide band BPM}

\( r \) \hspace{1cm} \text{reflection coefficient}

\( r_j \) \hspace{1cm} \text{tridiagonal system coefficient}

\( r_0 \) \hspace{1cm} \text{maximum reflection coefficient at the PML region}

\( r \) \hspace{1cm} \text{position}

\( R \) \hspace{1cm} \text{reflectivity}

\( R_j, R_{ij} \) \hspace{1cm} \text{coefficient for FD schemes of BPM}

\( R_{ij} \) \hspace{1cm} \text{pump rate (or stimulated emission rate)}

\( S_{1_{ij}}-S_{4_{ij}} \) \hspace{1cm} \text{coefficients for FD schemes of BPM}

\( \mathcal{S} \) \hspace{1cm} \text{Poynting vector}

\( S \) \hspace{1cm} \text{complex Poynting vector}

\( t \) \hspace{1cm} \text{time; also, transmission coefficient}

\( T \) \hspace{1cm} \text{period}

\( T_j \) \hspace{1cm} \text{transmission coefficient}

\( T_{AB} \) \hspace{1cm} \text{interface matrix (for bidirectional BPM)}

\( u \) \hspace{1cm} \text{SVE-field component} \ \Psi_x \ (\text{or} \ \Phi_x)

\( u(x,y,z) \) \hspace{1cm} \text{SVE scalar optical field}

\( u_f \) \hspace{1cm} \text{SVE for the fundamental wave}

\( u_s \) \hspace{1cm} \text{SVE for the SH wave}

\( u^m_j \) \hspace{1cm} \text{discretized SVE optical field}

\( u^+_j \) \hspace{1cm} \text{discretized incident field} \ \psi^+_A \n
\( u^-_j \) \hspace{1cm} \text{discretized reflected field} \ \psi^-_A \n
\( u(r,t) \) \hspace{1cm} \text{temporal envelop of the electric field}

\( u_t \) \hspace{1cm} \text{SVE of the transverse electric field}

\( u_x, u_y, u_z \) \hspace{1cm} \text{unitary vectors along the} x-, y- \text{and} z\text{-axis}

\( v \) \hspace{1cm} \text{SVE-field component} \ \Psi_y \ (\text{or} \ \Phi_y); \text{also, propagation speed of an EM wave}

\( w \) \hspace{1cm} \text{width}

\( w(k,t) \) \hspace{1cm} \text{spatial frequencies}

\( w^+_j \) \hspace{1cm} \text{discretized forward field} \ \psi^+_B \n
\( W \) \hspace{1cm} \text{width}

\( W_{ij} \) \hspace{1cm} \text{stimulated emission rate}

\( W_{ij}^{NR} \) \hspace{1cm} \text{non-radiative probability}

\( W_{\nu} \) \hspace{1cm} \text{relative power carried by the} \ \nu\text{-th mode}

\( x \) \hspace{1cm} \text{Cartesian coordinate}
List of Symbols

\( X_j \) \hspace{1cm} \text{dimensionless operator}
\( y \) \hspace{1cm} \text{Cartesian coordinate}
\( z \) \hspace{1cm} \text{Cartesian coordinate}
\( Z \) \hspace{1cm} \text{total propagation length}
\( Z_{1ij} - Z_{4ij} \) \hspace{1cm} \text{coefficients for FD schemes of BPM}

\textbf{Greek Symbols}

\( \alpha \) \hspace{1cm} \text{Crank–Nicolson scheme parameter; also, absorption coefficient}
\( \alpha_{\text{eff}} \) \hspace{1cm} \text{effective attenuation coefficient (of PML)}
\( \tilde{\alpha}_s \) \hspace{1cm} \text{intrinsic propagation losses}
\( \beta \) \hspace{1cm} \text{propagation constant}
\( \beta_\nu \) \hspace{1cm} \text{propagation constant of the \( \nu \)th order eigenmode}
\( \chi_i \) \hspace{1cm} \text{polynomial coefficient}
\( \chi_L \) \hspace{1cm} \text{linear susceptibility}
\( \chi^{(2)} \) \hspace{1cm} \text{coefficient of second-order non-linear susceptibility}
\( \chi_{ijk} \) \hspace{1cm} \text{element of the second-order non-linearity susceptibility tensor}
\( \delta \) \hspace{1cm} \text{delta Kronecker function; also, ABC region thickness (or PML region)}
\( \Delta x, \Delta y \) \hspace{1cm} grid size
\( \Delta z \) \hspace{1cm} \text{longitudinal step size}
\( \Delta k \) \hspace{1cm} \text{mismatch parameter}
\( \Delta t \) \hspace{1cm} \text{time step}
\( \varepsilon \) \hspace{1cm} \text{scalar dielectric permittivity}
\( \varepsilon_r \) \hspace{1cm} \text{dielectric constant (relative dielectric permittivity)}
\( \varepsilon_0 \) \hspace{1cm} \text{dielectric permittivity of free space}
\( \varepsilon_{ij} \) \hspace{1cm} \text{element of the permittivity matrix}
\( \varepsilon \) \hspace{1cm} \text{permittivity tensor}
\( \phi(i) \) \hspace{1cm} \text{transversal field distribution of a waveguide mode for FDTD}
\( \gamma \) \hspace{1cm} \text{amplification factor (for Von Neumann analysis); also, damping factor (for bi-BPM)}
\( \Gamma \) \hspace{1cm} \text{correlation between optical fields; also, overlap integral}
\( \eta \) \hspace{1cm} \text{impermeability tensor}
\( \eta_0 \) \hspace{1cm} \text{free space impedance}
\( \phi(r) \) \hspace{1cm} \text{initial phase}
\( \varphi \) \hspace{1cm} \text{incident angle}
\( \Phi_x \) \hspace{1cm} \text{\( x \)-component of the SVE transversal magnetic field}
\( \Phi_y \) \hspace{1cm} \text{\( y \)-component of the SVE transversal magnetic field}
\( \Phi_t \) \hspace{1cm} \text{SVE field of} \( \mathbf{H}(r) \)
\( \kappa \) \hspace{1cm} \text{absorption index; also, coupling coefficient}
\( \kappa_{\text{max}} \) \hspace{1cm} \text{maximum value of} \( \kappa(x) \)
\( \kappa(x) \) \hspace{1cm} \text{absorption index profile (in ABC)}
\( \lambda \) \hspace{1cm} \text{wavelength}
\( \Lambda \) \hspace{1cm} \text{grating period}
\( \mu \) \hspace{1cm} \text{magnetic permeability}
\( \mu_0 \) \hspace{1cm} \text{magnetic permeability of free space}
\( \nu \) \hspace{1cm} \text{frequency}
\( \theta_i \) \hspace{1cm} \text{angle of reflection (or transmission)}
\( \rho \)  parameters for ABC region
\( \rho_i \)  magnetic conductivity of the PML
\( \rho(r,t) \)  charge density
\( \sigma(r) \)  electrical conductivity
\( \sigma(\rho), \sigma_i \)  electrical conductivity profile of the Bérenger layer
\( \sigma_{ij} \)  absorption (or emission) cross-section
\( \sigma_{\text{max}} \)  maximum conductivity of the PML
\( \tau \)  pulse temporal width; also, lifetime
\( \omega \)  angular frequency
\( \omega(k) \)  relation dispersion
\( \omega_s \)  angular frequency for the SH wave
\( \omega_f \)  angular frequency for the fundamental wave
\( \omega_0 \)  carrier frequency
\( \xi_i \)  polynomial coefficient
\( \xi_\nu \)  eigenvalue (relative propagation constant)
\( \psi(x, y, z) \)  generic scalar field
\( \psi^+ \)  forward field
\( \psi^- \)  backward field
\( \psi^+_A \)  incident field in region A
\( \psi^-_A \)  reflected field in region A
\( \psi^+_B \)  transmitted field in region B
\( \Psi \)  slowly varying electric field
\( \psi_i \)  SVE field of \( E_i(r) \)
\( \Psi_x \)  \( x \) component of the SVE transverse electric field \( \psi_i \)
\( \Psi_y \)  \( y \) component of the SVE transverse electric field \( \psi_i \)

**Mathematical Symbols**
\( \partial \)  partial differential
\( \nabla \)  gradient operator
\( \nabla \)  divergence operator
\( \nabla \times \)  curl operator
Electromagnetic Theory of Light

Introduction

Integrated photonics devices are based on optical waveguides with transversal dimensions of the order of microns, comparable to the wavelength of the optical radiation used in the integrated devices (visible and near infrared). This fact implies that the performance of the optical chips cannot be analysed in terms of ray optics, but instead the light must be treated as vectorial waves. Thus, to describe adequately the light propagation along the waveguide structures that define an integrated photonics device, the electromagnetic theory of light is required, which deals the light as optical waves in terms of their electric and magnetic fields. This treatment retains the vectorial character of the waves. Nevertheless, in some cases the vectorial nature of the electromagnetic waves can be simplified and a scalar treatment of the optical waves is enough for an accurate description of the light propagation through the optical waveguides.

Along this chapter the basics of the electromagnetic theory of light is described, which is the start point to derive the beam propagation equations to model the light propagation in optical waveguides and integrated photonic devices. First, the Maxwell's equations for light propagation in free space are presented in terms of the electric and magnetic field. Then, the electric displacement vector and the magnetic flux density vector are introduced to describe the optical propagation in material media. The constitutive relations allow then to establish a set of equations in terms of the electric and magnetic fields. Using these equations, the wave equations in inhomogeneous media are derived where the refractive index can be then defined. Then the wave equation for monochromatic waves in inhomogeneous media is obtained, where the temporal dependence of the fields is in the form of harmonic function. The especial cases of light propagation in absorbing media, anisotropic media and in second-order non-linear media are discussed, and wave equations for each case are derived. Finally, the wave equations in inhomogeneous isotropic and linear media in terms of the transverse field components are obtained, for both electric and magnetic fields. Also, wave equation for anisotropic media...
and second-order non-linear media are established in terms of the electric transverse field components. These equations will serve to derive in the subsequent chapters the beam propagation formalism.

1.1 Electromagnetic Waves

1.1.1 Maxwell’s Equations

Light is, in terms of classical theory, the flow of electromagnetic (EM) radiation through free space or through a material medium in the form of oscillating electric and magnetic fields. Although electromagnetic radiation occurs over an extremely wide range from gamma rays to long radio waves, the term ‘light’ is restricted to the part of the electromagnetic spectrum that covers from the vacuum ultraviolet (UV) to the far infrared. This part of the spectrum is often also called optical range. The EM radiation propagates in the form of two mutually perpendicular and coupled vectorial waves: the electric field \( \mathbf{E}(r,t) \) and the magnetic field \( \mathbf{H}(r,t) \). These two vectorial magnitudes are dependent on the position \( (r) \) and time \( (t) \). Therefore, to describe properly the light propagation in a medium, be it the vacuum or a material medium, it is necessary in general to know six scalar functions with their dependence of the position and the time. These functions are not independent but linked through Maxwell’s equations.

Maxwell’s equations form a set of four coupled equations involving the electric field vector and the magnetic field vector of the light and are based on experimental evidence, two of them being scalar equations and the other two vectorial equations. In their differential form, Maxwell’s equations for light propagating in the free space are:

\[
\nabla \cdot \mathbf{E} = 0; \\
\nabla \cdot \mathbf{H} = 0; \\
\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}; \\
\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},
\]

where the constants \( \epsilon_0 = 8.85 \times 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^4 \text{ A}^2 \) and \( \mu_0 = 4\pi \times 10^{-7} \text{ m kg s}^{-2} \text{ A}^{-2} \) represent respectively the dielectric permittivity and the magnetic permeability of free space and the \( \nabla \) and \( \nabla \times \) denote the divergence and curl operators, respectively.

The differential operator \( \nabla \) is defined as:

\[
\nabla \equiv \left( \frac{\partial}{\partial x} \mathbf{u}_x + \frac{\partial}{\partial y} \mathbf{u}_y + \frac{\partial}{\partial z} \mathbf{u}_z \right),
\]

where \( \mathbf{u}_x, \mathbf{u}_y \) and \( \mathbf{u}_z \) represent the unitary vectors along the \( x \)-, \( y \)- and \( z \)-axis, respectively. This differential operator acting to a scalar field gives rise to a vector (gradient). In particular, if \( \xi(x,y,z) \) represents a scalar field, we have:
\[ \nabla \xi(x,y,z) = \left( \frac{\partial \xi}{\partial x} \mathbf{u}_x + \frac{\partial \xi}{\partial y} \mathbf{u}_y + \frac{\partial \xi}{\partial z} \mathbf{u}_z \right). \] 

(1.3)

On the other hand, if \( \mathbf{A}(x,y,z) = A_x(x,y,z)\mathbf{u}_x + A_y(x,y,z)\mathbf{u}_y + A_z(x,y,z)\mathbf{u}_z \) is a vector field, the divergence operator \((\nabla \cdot )\) acts as follows:

\[ \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \]

(1.4)

which is a scalar magnitude. Finally, the curl differential operator \((\nabla \times )\) acting on the vector field \( \mathbf{A} \) gives another vector with the following components:

\[ \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{u}_x + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{u}_y + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{u}_z. \]

(1.5)

For the description of the electromagnetic field in a material medium it is necessary to define two additional vectorial magnitudes: the electric displacement vector \( \mathbf{D}(r,t) \) and the magnetic flux density vector \( \mathbf{B}(r,t) \). Maxwell’s equations in a material medium, involving these two magnitudes and the electric and magnetic fields, are expressed as:

\[ \nabla \cdot \mathbf{D} = \rho; \]

(1.6a)

\[ \nabla \cdot \mathbf{B} = 0; \]

(1.6b)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \]

(1.6c)

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \]

(1.6d)

where \( \rho(r,t) \) and \( \mathbf{J}(r,t) \) denote the charge density and the current density vector, respectively. If the medium is free of charges, which is the most common situation in optics, Maxwell’s equations simplify to the form:

\[ \nabla \cdot \mathbf{D} = 0; \]

(1.7a)

\[ \nabla \cdot \mathbf{B} = 0; \]

(1.7b)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \]

(1.7c)

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \]

(1.7d)
Now, in order to solve these differential coupled equations it is necessary to establish additional relations between the vectors $\mathbf{D}$ and $\mathbf{E}$, $\mathbf{J}$ and $\mathbf{E}$ as well as the vectors $\mathbf{H}$ and $\mathbf{B}$. These relations are called constitutive relations and depend on the electric and magnetic properties of the considered medium. In the most simple case of linear and isotropic media, the constitutive relations are given by:

$$
\mathbf{D} = \varepsilon \mathbf{E}; \quad (1.8a)
$$

$$
\mathbf{B} = \mu \mathbf{H}; \quad (1.8b)
$$

$$
\mathbf{J} = \sigma \mathbf{E}, \quad (1.8c)
$$

where $\varepsilon = \varepsilon(r)$ is the dielectric permittivity, $\mu = \mu(r)$ is the magnetic permeability and $\sigma = \sigma(r)$ is the electrical conductivity of the medium. Here, their dependence on the position vector $r$ has been explicitly indicated. If the medium is not linear, it is necessary to include additional terms involving power expansion of the electric and magnetic fields. Besides, in an isotropic medium (glasses for instance) these optical constants are scalar magnitudes and independent of the direction of the vectors $\mathbf{E}$ and $\mathbf{H}$, implying that the vectors $\mathbf{D}$ and $\mathbf{J}$ are parallel to the electric field $\mathbf{E}$ and the vector $\mathbf{B}$ is parallel to the magnetic field $\mathbf{H}$. By contrast, in an anisotropic medium (for instance, most of the dielectric crystals) the optical constants must be treated as tensorial magnitudes.

By using the constitutive relations for a linear and isotropic medium, Maxwell’s equations can be written in terms of the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ only:

$$
\nabla \cdot (\varepsilon \mathbf{E}) = 0; \quad (1.9a)
$$

$$
\nabla \cdot \mathbf{H} = 0; \quad (1.9b)
$$

$$
\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}; \quad (1.9c)
$$

$$
\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \quad (1.9d)
$$

A perfect dielectric medium is defined as a material in which the conductivity is very low and can be neglected ($\sigma \approx 0$). In this category fall most of the materials used for integrated optical devices, such as glasses, ferroelectric crystals, polymers or even semiconductors, while metals do not belong to this category because of their high conductivity. In addition, in most of materials (non-magnetic materials) and in particular, dielectric media, the magnetic permeability is very close to that of free space and the approximation $\mu \approx \mu_0$ holds. Then, in dielectric and non-magnetic media, Maxwell’s equations simplify in the form:

$$
\nabla \cdot (\varepsilon \mathbf{E}) = 0; \quad (1.10a)
$$

$$
\nabla \cdot \mathbf{H} = 0; \quad (1.10b)
$$

$$
\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}; \quad (1.10c)$$
∇ × \mathbf{H} = ε \frac{\partial \mathbf{E}}{\partial t}. \quad (1.10d)

In what follows, we will restrict ourselves to non-magnetic and low conductivity materials, where Maxwell’s equations (1.10a)–(1.10d) apply.

### 1.1.2 Wave Equations in Inhomogeneous Media

Combining the four Maxwell’s equations (1.10a)–(1.10d) it is possible to obtain an equation involving the electric field alone and another equation that involves only the magnetic field.

Taking the curl operation over the Eqs. (1.10c) and (1.10d) we obtain:

\[ \nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left( \mu_0 \frac{\partial \mathbf{H}}{\partial t} \right) ; \quad (1.11a) \]

\[ \nabla \times (\nabla \times \mathbf{H}) = \nabla \times \left( \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = (\nabla \varepsilon) \times \frac{\partial \mathbf{E}}{\partial t} + \varepsilon \left( \nabla \times \frac{\partial \mathbf{E}}{\partial t} \right) . \quad (1.11b) \]

Having in mind the vectorial identity \( \nabla \times \nabla \times \equiv \nabla (\nabla \cdot) - \nabla^2 \), these equations transform to:

\[ \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) ; \quad (1.12a) \]

\[ \nabla (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = (\nabla \varepsilon) \times \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial t} [\varepsilon \nabla \times \mathbf{E}] , \quad (1.12b) \]

where we have used the fact that the temporal and spatial derivatives commute and it is assumed that permittivity, \( \varepsilon \), is time independent. On the other hand, expanding the first Maxwell’s equation (1.10a):

\[ \nabla \cdot (\varepsilon \mathbf{E}) = (\nabla \varepsilon) \cdot \mathbf{E} + \varepsilon \nabla \cdot \mathbf{E} = 0 \Rightarrow \nabla \cdot \mathbf{E} = -\mathbf{E} \cdot \nabla \ln \varepsilon , \quad (1.13) \]

where we have used the relationship: \( \nabla \ln \varepsilon \equiv \frac{\nabla \varepsilon}{\varepsilon} \).

Introducing Eqs. (1.13) and (1.10d) into Eq. (1.12a), we have:

\[ \nabla^2 \mathbf{E} + \nabla (\mathbf{E} \cdot \nabla \ln \varepsilon) - \mu_0 \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 , \quad (1.14) \]

and similarly using Eqs. (1.10b) and (1.10c) into Eq. (1.12b), we obtain:

\[ \nabla^2 \mathbf{H} + (\nabla \ln \varepsilon) \times (\nabla \times \mathbf{H}) - \mu_0 \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 . \quad (1.15) \]

These last two differential equations are known as wave equations in inhomogeneous media, which are valid for linear, non-magnetic and isotropic material media. It is worth noting that,
although we have obtained a wave equation for the electric field $\mathbf{E}$ and another for the magnetic field $\mathbf{H}$, the solution of both equations are not independent, because the electric and magnetic fields are related through the Maxwell’s equations (1.10c) and (1.10d). The solutions of the wave equations are known as electromagnetic waves.

The electromagnetic waves transport energy and the flux of energy (measured in units of W/m$^2$) carried by the EM wave is given by the Poynting vector $\mathbf{S}$, defined as:

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}. \quad (1.16)$$

On the other hand, the intensity (or irradiance) $I$ of an EM wave, defined as the amount of energy passing through the unit area in the unit of time, is given by the time average of the Poynting vector modulus:

$$I = \langle |\mathbf{S}| \rangle. \quad (1.17)$$

The reason for using an averaged value instead of an instant value for defining the intensity of an EM wave is because the electric and magnetic fields associated to the EM wave oscillate at very high frequency and the apparatus used to detect that intensity (light detectors) cannot follow the instant values of the Poynting vector modulus.

1.1.3 Wave Equations in Homogeneous Media: Refractive Index

An optically homogeneous medium is defined as a material in which its optical properties are independent on the position. Then, for homogeneous dielectric media the second terms in Eqs. (1.14) and (1.15) vanish:

$$\nabla \ln \varepsilon = \nabla \frac{\varepsilon}{\varepsilon} = 0, \quad (1.18)$$

and the wave equations simplify on the forms:

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}; \quad (1.19a)$$

$$\nabla^2 \mathbf{H} = \mu_0 \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}. \quad (1.19b)$$

Each of these two vectorial wave equations can be split onto three scalar wave equations, expressed as:

$$\nabla^2 \xi = \mu_0 \varepsilon \frac{\partial^2 \xi}{\partial t^2}, \quad (1.20)$$

where the scalar variable $\xi(r,t)$ may represent each of the six Cartesian components of either the electric and magnetic fields. The solution of this scalar equation represents a wave that propagates with a speed $v$ (phase velocity) given by:
Therefore, the complete solution of the vectorial wave equations (1.19a) and (1.19b) represents an electromagnetic wave, where each of the Cartesian components of the electric and magnetic fields propagate in the form of waves of equal speed \( v \) in the homogeneous medium.

For propagation in free space (vacuum) and using the values for \( \varepsilon_0 \) and \( \mu_0 \) we obtain:

\[
c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \approx 3.00 \times 10^8 \text{ m/s},
\]

which corresponds to the speed of light in free space measured experimentally. It is worth noting that here the speed of light is obtained by using only values of electric and magnetic constants.

Usually, it is convenient to express the propagation speed \( v \) of the electromagnetic waves in a medium as a function of the speed of light in free space, \( c \), through the relation:

\[
v \equiv \frac{c}{n},
\]

where \( n \) represents the refractive index of the dielectric medium. Taking into account the relations (1.21) and (1.22), the refractive index can be related to the dielectric permittivity of the medium and that of the free space by:

\[
n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r},
\]

where we have introduced the magnitude relative dielectric permittivity, \( \varepsilon_r \), also called the dielectric constant, defined as the relation between the dielectric permittivity of the material medium and that of the free space. As we will see in the following chapters, the refractive index of a medium is the most important parameter for defining optical waveguide structures used in integrated photonic devices. As well as the refractive index of 1 corresponding to propagation through free space, the refractive index ranges from values close to 1.5 for glasses and some dielectric crystals (for instance, \( n(\text{SiO}_2) = 1.55 \)) to values close to 4 for semiconductor materials (for instance, \( n(\text{Si}) = 3.75 \)).

### 1.2 Monochromatic Waves

The temporal dependence of the electric and magnetic fields within the wave equations admits solutions on the form of harmonic functions. The electromagnetic waves with such sinusoidal dependence on the time variable are called monochromatic waves, which are characterized by their angular frequency \( \omega \) (in units of rad/s). In a general form, the electric and magnetic fields associated to a monochromatic wave can be expressed as:

\[
\mathbf{E}(r,t) = \mathbf{E}_0(r) \cos[\omega t + \phi(r)];
\]

\[
\mathbf{B}(r,t) = \mathbf{B}_0(r) \cos[\omega t + \phi(r)];
\]

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\[ H(r,t) = H_0(r) \cos[\omega t + \phi(r)], \]  

(1.25b)

where the field amplitudes \( E_0(r) \) and \( H_0(r) \), and the initial phase \( \varphi(r) \) have dependence on the position \( r \), and the time dependence of the fields is only in the cosine argument through \( \omega t \).

Usually, when dealing with monochromatic waves it is convenient to express the monochromatic fields using complex notation. Using this notation, the electric and magnetic fields are expressed as:

\[
\begin{align*}
E(r,t) &= \text{Re} \left[ E(r) e^{i\omega t} \right], \\
H(r,t) &= \text{Re} \left[ H(r) e^{i\omega t} \right],
\end{align*}
\]

(1.26a)

(1.26b)

where \( E(r) \) and \( H(r) \) denote the complex amplitudes of the electric and magnetic fields, respectively, \( i \) is the imaginary unity and \( \text{Re} \) stands for the real part. The angular frequency, \( \omega \), which characterizes the monochromatic wave, is related with the frequency, \( \nu \), and the period, \( T \), by:

\[
\omega = 2\pi \nu = 2\pi / T.
\]

(1.27)

The electromagnetic spectrum covered by light (optical spectrum) ranges from frequencies of \( 3 \times 10^5 \) Hz corresponding to the far infrared (IR), to \( 6 \times 10^{15} \) Hz corresponding to vacuum UV, being the frequency of visible light in the range of 430–770 THz.

The average of the Poynting vector for monochromatic waves as a function of the complex fields amplitudes takes the form:

\[
\langle S \rangle = \langle \text{Re} \{ E e^{i\omega t} \} \times \text{Re} \{ H e^{i\omega t} \} \rangle = \text{Re} \{ S \},
\]

(1.28)

where \( S \) is defined here as:

\[
S = \frac{1}{2} E \times H^*.
\]

(1.29)

which is called the complex Poynting vector. Using this definition, the intensity carried by a monochromatic EM wave can be expressed in a compact form as:

\[
I = |\text{Re} \{ S \}|.
\]

(1.30)

Maxwell’s equations (1.10a)–(1.10d) using the complex fields amplitudes \( E \) and \( H \) simplify notably in the case of monochromatic waves, because the partial derivatives with respect to the time can be directly obtained by multiplying by the factor \( i\omega \), resulting in:

\[
\begin{align*}
\nabla \cdot (\varepsilon E) &= 0; \\
\nabla \cdot H &= 0; \\
\nabla \times E &= -i\mu_0 \omega H; \\
\nabla \times H &= i\varepsilon \omega E,
\end{align*}
\]

(1.31a)

(1.31b)

(1.31c)

(1.31d)