Classic Texts in the Sciences

Jürgen Jost
Editor

Bernhard Riemann
On the Hypotheses Which Lie at the Bases of Geometry

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Editor
Jürgen Jost
Max Planck Institute for Mathematics in
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Leipzig, Germany

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Preface

This could be the plot of a novel: The main character is a shy, sickly, young mathematician, living in poor conditions at a German university in the middle of the nineteenth century. He does not succeed in establishing a closer contact with the greatest mathematical luminary of his time. He is working toward his habilitation degree (a prerequisite for becoming a candidate for a professorship at a university in Germany). Part of the process is a colloquium. For such a habilitation colloquium, the candidate must propose three topics from which the faculty can choose. The first two topics are derived from the significant technical contributions that he has already made. He finds it difficult to decide what he should choose for his third topic, partly because he believes that, as usual, the faculty will select the first topic on the list anyway. As the third one, he offers a rather vague natural philosophical theme. To his surprise and consternation, the faculty chooses that one. Instead of now familiarizing himself with the state of art of the discipline and, in particular the really significant prior discovery that shook the entire field, he immerses himself in the work of a rather obscure philosopher. But his lecture then penetrates as deeply as never before into a field that had occupied and challenged the greatest thinkers of mankind since classical antiquity, and it even hints at the greatest discovery of the physics of the following century. Even the contribution of the superstar of German science, who had independently approached the same subject from a different point of view, faded into insignificance in comparison with the depth of insight of our young mathematician. Other famous scientists entered the stage with grotesque errors of judgment on the topic and content of the habilitation lecture after it had been published by a friend after the untimely death of our hero. Subsequent generations of mathematicians worked out the ideas outlined in the brief lecture and confirmed their full validity and soundness and extraordinary range and potential.

However, this is not a novel, because something similar did actually occur. We hope and trust that readers forgive the author certain exaggerations, and, of course, in the following pages, everything will be represented correctly. The young mathematician was Bernhard Riemann, and the lecture was entitled "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" (On the hypotheses which lie at the bases of geometry). The mathematical genius was Carl Friedrich Gauss, the scientific superstar Hermann von Helmholtz, the fundamental prior mathematical discovery non-Euclidean geometry, the philosopher the now forgotten Johann Friedrich Herbart, the discovery in physics the theory of general relativity of Albert Einstein. The people who came up with completely wrong judgements included the psychologist Wilhelm Wundt and the philosopher Bertrand Russell. The friend who took care of the posthumous publication was the mathematician Richard Dedekind. The generations of subsequent mathematicians for whose research Riemann's ideas were a major inspiration include the author of these lines.

Typically, scientists read a scientific text from the point of view of the current state of science, interpret it in terms of subsequent developments, and seek at best unexplored potential for current scientific problems. Historians, however, want to determine the position of a text within the discourse of its time, reconstruct its origin and analyze the history of its reception. Although in the current debate on the role of the humanities, the importance of the historical sciences for understanding the present is emphasized, mathematicians are interested in the timeless content and not in the historical contingencies of scientific texts. Scientific projects that proved futile are either of no interest to the scientist or constitute annoying obstacles on a path that could have been straighter. For historians, in contrast, they can provide important insights into the history of ideas and the dynamics of discourses. For the scientist, texts whose effect has faded are without interest. For the historian this loss of interest is part of the reception history.

This edition of Riemann's "Ueber die Hypothesen, welche der Geometrie zugrunde liegen" tries to accept these challenges. The publisher is a professional scientist, not a historian of science. Therefore, the history is also sometimes read backwards. In particular, for this edition no thorough philological studies have been carried out.

Although there will also be a formal mathematical chapter, I shall mainly attempt to explain the basic concepts and basic ideas in words, even if this will inevitably lead to some loss of precision.

The current book is a somewhat expanded English translation of my original German work. In particular, instead of also translating Hermann Weyl's mathematical commentary that had been included in the German version, I have written a more detailed mathematical section. That section provides the mathematical background and puts Riemann's reasoning into the more general and systematic perspective achieved by his followers on the basis of his seminal ideas. Readers who are not so much interested in mathematical details may skip this section, since in another section, I have also explained Riemann's reasoning verbally as an alternative to mathematical deductions.

As mentioned before, I am not a historian of mathematics. Therefore, I am much indebted to some historians of mathematics, namely Erhard Scholz, Rüdiger Thiele and Klaus Volkert, for their very useful comments, corrections, suggestions and references. Of course, the responsibility for any shortcomings rests with me alone. I am also grateful to the Helmholtz expert Jochen Brüning for his insightful comments.

I thank Ingo Brüggemann, the librarian of the Max Planck Institute for Mathematics in the Sciences, and his staff for their valuable and efficient assistance in the acquisition of literature.

The largest amount of gratitude I owe to my friend, the late Olaf Breidbach, for his initiative in establishing the series in which this edition of Riemann's text may now appear, as well as for the many discussions on a wide range of scientific topics over several decades. After his untimely death, I am now alone in charge of this series of Classic Texts in the Sciences and its German language counterpart, which we had founded together with so much enthusiasm. I hope that I shall be able to preserve his spirit as one of the great historians of science of our times.

Leipzig, Germany Jürgen Jost
August 2015

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Introduction 1

A mathematical lecture without formulas, a geometric treatise without pictures or illustrations, a manuscript of only 16 pages that just came into being by chance, but a text that has shaped mathematics like few others works, which were all significantly longer, considerably more detailed, and much more carefully worked out. In this regard, we might mention the "*Methodus inveniendi*" by Leonhard Euler,[1] which founded the calculus of variations; Carl Friedrich Gauss' "*Disquisitiones arithmeticae*" that established mathematics as an independent discipline[2]; Georg Cantor's set theory, which introduced the modern conception of the infinite in mathematics; the theory of transformation groups of Sophus Lie, that is, the systematic study of symmetries that forms the mathematical basis for quantum mechanics; the programmatic writings of David Hilbert on the axiomatic foundation of various mathematical disciplines; or more recently the work of Alexander Grothendieck on the systematic unification of algebraic geometry and arithmetic. We are talking here of Bernhard Riemann's "*Ueber die Hypothesen, welche der Geometrie zugrunde liegen*" ("*On the hypotheses which lie at the bases of geometry*"), and this short script, written in 1854, but only published in 1868 after Riemann's death, whose wide ranging effects even take it beyond these works. This is because its position is at the intersection of mathematics, physics and philosophy, and it not only founds and establishes a central mathematical discipline, but also paves the way for the physics of the twentieth century and at the same time represents a timeless refutation of certain philosophical conceptions of space. In the present volume, this key text of mathematics will be edited, positioned in the controversies of its time, and its effects on the development of mathematics will be analyzed and compared to those of its opponents. Riemann's "*Ueber*

[1] I am also preparing an edition of this text for the current series.

[2] In the sense that it develops its problems autonomously and intrinsically, instead of obtaining them from physics or other sciences.

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die Hypothesen, welche der Geometrie zugrunde liegen" has shaped and transformed mathematics in a manner very different from, say, Euclid's Elements, the writings of Leibniz and Newton on the creation of the infinitesimalsimal calculus or the above-mentioned works. It has, in a manner no less fundamental than those, influenced the development of mathematics as a science. Moreover, this text is essential for Einstein's theory of General Relativity. More recently, it also provided the mathematical structure underlying quantum field theory and its developments in theoretical Elementary Particle Physics (superstring theory, quantum gravity etc.).

However, the history of its influence is not linear. Riemann's programmatic writing "*Ueber die Hypothesen, welche der Geometrie zugrunde liegen*" called to the scene the leading German physicist of his time, Hermann Helmholtz (later knighted, therefore von Helmholtz), the title of whose counter-essay "*Über die Thatsachen, die der Geometrie zugrunde liegen*" (*On the facts underlying geometry*) already pointed out a conflicting position and approach. (Though, in the text, the similarities with Riemann dominate[3] and its main thrust was not against Riemann, but against the Kantian concept of space.)[4] It would, however, be incorrect and misleading to view Helmholtz' work on the foundations of geometry simply as the by now obsolete and forgotten opposition of the established authority against the young genius, of the representative of a conservative scientific attitude against the protagonist of a novel scientific direction. Riemann's work was partly motivated by somewhat vague natural philosophical speculations—and in turn his text did have major implications for natural philosophy—whereas the origin of Helmholtz' considerations laid in sensory physiology—and his ideas remain highly relevant here. Moreover, Helmholtz influenced another fundamental mathematical theory, the theory of symmetry groups of Sophus Lie. Although Lie subjected the mathematical aspects of Helmholtz' work to a sharp criticism, he nevertheless adopted the latter's conceptual approach. Lie's theory of symmetry groups has become one of the cornerstones of quantum mechanics. The concepts of symmetry and invariance link the intuition of modern physics

[3]Helmholtz says that he had developed the essential elements of his consideration before learning of Riemann's work (which had been published with a four 10-year delay), but certainly later than Riemann, who had his lecture delivered and script written in 1854.

[4]It might appear natural to contrast this work here with Riemann's text. After giving this option serious thought, however, in the end I refrained from it, because this work of Helmholtz did not achieve the same depth and elegance as Riemann's. Moreover, among the various writings of Helmholtz on epistemological issues, this particular text is not the best and the clearest, and so, by the choice of that particular work, the important physiologist and physicist Helmholtz would have appeared in a wrong light. So if we had wanted to represent Helmholtz' theory here by one of his writings, then we should have selected another of his writings, namely " *Über den Ursprung und die Bedeutung der geometrischen Axiome*" (*On the origin and the importance of the geometrical axioms*) or his inaugural address as Rektor (president) of the University of Bonn "*Die Tatsachen in der Wahrnehmung*" (*The facts in perception*), but then we would have lost the close relationship with Riemann's habilitation address.

with the mathematical frame of geometry in the sense of Riemann and Einstein. In this sense, Helmholtz' text contained an aspect that was visionary for modern physics, even though this became clear only through the work of Lie and probably was quite different from what Helmholtz himself may have had in mind.

Riemann was probably motivated by somewhat vague nature-philosophical speculations—and his work then conversely had significant implications for the philosophy of nature. In contrast, Helmholtz's considerations firstly had their starting point in sensory physiology—and his ideas remain relevant here—and secondly, they also influenced a very important mathematical direction, namely the theory of symmetry groups of Sophus Lie. Even though Lie sharply criticized the mathematical details of Helmholtz' treatises, he took over the latter's conceptual approach. Lie's theory of symmetry groups became one of the essential foundations of quantum mechanics, and the concepts of symmetry and invariance connect the physical intuition of modern physics with the mathematical framework of geometry in the sense of Riemann and Einstein. In this sense, also Helmholtz' text played a pioneering role for modern physics. In contrast to Riemann's text, that text exerted its influence not directly, but only through the work of Lie, and probably this influence was rather different from what Helmholtz himself had imagined.

It is noteworthy that Riemann's "Hypotheses" as one of the key texts in mathematics proceeds without mathematical formulas (in the whole text, there is only a single formula which is of only marginal importance). This sets Riemann's text apart from other foundational mathematical works, like the sophisticated and deeply thought out symbolism of Leibniz or the formalization of the infinite of Cantor. Even his most important precursor, Gauss' "Disquisitiones generales circa superficies curvas", which founded modern differential geometry, the starting point of Riemannian Geometry, is different in this respect. At least in this case, the history of mathematics is not simply a progressive formalization, but it turns out that mathematical abstraction can in principle rise well above formulas .[5]

Bernhard Riemann has decisively shaped modern mathematics to such a degree that only the influence of Carl Friedrich Gauss is comparable with his. Not only did he found modern geometry with his habilitation lecture published here—and the most important

[5]Of course, the occasion of Riemann's text should be taken into account. It was a colloquium before the Faculty of Arts, and Riemann certainly wanted to pay respect to the lack of mathematical knowledge of most of the people in his audience. Among these, besides Gauss, who was not a professor of mathematics, but a professor of astronomy and director of the observatory, mathematics was represented only by the two Professors Ulrich (1798–1879) and Stern (1807–1894). However, other such lectures or writings, like Klein's Erlangen program with which he introduced himself to the faculty in Erlangen, certainly could be much more formalized, and if the faculty had chosen one of the other topics suggested by Riemann, the presentation would probably have been developed in mathematical formulas.

part of modern geometry is therefore called Riemannian geometry—but he has created a number of basic theories and introduced many fundamental concepts that guided and influenced many other areas of mathematics. His concept of a Riemann surface combined in an ingenious manner complex analysis and the theory of elliptic integrals. This work was at the same time the starting point for the development of topology, i.e., the investigation of forms and shapes independent of metric properties, in contrast to Riemannian geometry. It also had a decisive impact on modern algebraic geometry. On top of that, it introduced completely novel analytical tools in the theory of functions of a complex variable. The latter, even if initially, Weierstrass detected and pointed out essential analytical gaps that could be closed only later by Hilbert, paved the way for the modern calculus of variations and the existence theory for solutions of partial differential equations. Those in turn, as implemented and controlled with the methods of numerical analysis, constitute a fundamental tool of modern engineering. A novel and pathbreaking idea was that Riemann no longer tried to approach analytic functions in the complex plane through an analytical expression, but rather considered them as determined by their singularities (poles, i.e., points where they become infinite, or branch points). In this way he could assign to such a function a so-called Riemann surface and then determine the qualitative properties of the function in terms of the topology of that Riemann surface. This radiated in almost all areas of modern mathematics, and for example, even revolutionized number theory, the analytical expressions of which could then also be interpreted and treated by geometric methods. It was likewise a pathbreaking aspect of the theory of Riemann surfaces that Riemann not only looked at a single mathematical object, but conceptualized a class of objects through the variability of parameters. This led to the theory of moduli spaces which is basic for algebraic geometry. For this reason, Riemann surfaces also constitute the basic objects of the currently perhaps most promising theory, string theory, for the unification of the known physical forces. The so-called Riemann-Roch theorem (Gustav Roch (1839–1866) was an early deceased student of Riemann, who completed Riemann's work on these issues) was one of the guiding principles of mathematics in the second half of the twentieth century and resulted in the works of Hirzebruch, Atiyah-Singer and Grothendieck that produced key results of modern mathematics. The Riemann hypothesis, more than 150 years after its formulation, is still considered as the hardest and deepest open problem of all of mathematics.

On Riemann's Biography Bernhard Riemann, the son of a Lower Saxon protestant minister, lived from 1826 to 1866. He remained very attached to his family which put him in a difficult position due to many early deaths that led to unsecured financial circumstances. Like most of the great of the history of mathematics, he showed an extraordinary mathematical talent already as a schoolboy. After some hesitation, he followed this talent and studied mathematics instead of theology as desired by his father, in the scientific centers Göttingen and Berlin. His main academic teachers and role models were Carl Friedrich Gauss (1777–1855),[6] with whom he received his doctorate in 1851, and Peter Gustav Lejeune Dirichlet (1805–1859),[7] of whom he attended many lectures

[6]Gauss was born in Brunswick in modest circumstances. Since his outstanding mathematical talent was recognized early on, he was, however, generously supported by the Brunswick Duke. Already at a young age he made significant mathematical discoveries, such as on the question of the constructability of regular polygons. His *Disquisitiones Arithmeticae*, published in 1801, but written already some years earlier, are considered as the work that founded modern mathematics as an autonomous science. A spectacular success of his mathematical methods of error calculation was the rediscovery of the minor planet (asteroid) Ceres in the same year. This minor planet had been discovered by astronomers, but then again lost sight of until the Gaussian methods of path calculation would permit prediction of its position with high enough precision so that the astronomers knew to which position in the sky they had to turn their telescopes to find it. Since 1807, Gauss was a professor in Göttingen and the director of the observatory. Gauss is considered the greatest mathematician of all time, and he has influenced almost all areas of modern mathematics and even founded many of them. Together with the physicist Wilhelm Weber (1804–1891) he constructed the first telegraph. The mathematical methods developed by him are fundamental for astronomy and geodesy. Especially in old age, Gauss was difficult to approach, undoubtedly also due to a not very happy family life, and the shy Riemann could not establish direct personal contact with him. Riemann therefore acquired the mathematical theories and discoveries of Gauss by self-study. A recent biography of Gauss is Walter Kaufmann Bühler, *Gauss. A biographical study*, Berlin etc., Springer, 1981.

[7]Dirichlet was born in Düren in the Rhineland as a son of the local postmaster, whose father had immigrated from the Walloon region in present-day Belgium, where the Romanesque name comes from. During a stay in Paris from 1822 to 1827, as a foreigner, however, he was not allowed to attend the courses of the then leading French mathematician Augustin Louis Cauchy (1789–1857) at the Ecole Polytechnique. fortunately, he succeeded in gaining access to the circles of Jean-Baptiste Louis Fourier (1768–1830), who, starting from physical problems of thermodynamics, had introduced the famous series representations for periodic functions. Dirichlet proves a basic result about these series expansions. Alexander von Humboldt (1769–1859), who after his famous expeditions initially stayed in Paris and then held influential positions in Berlin, is impressed by him and supports and encourages him and brings him as a professor to Prussia, first to Breslau and then in 1829 to Berlin. Dirichlet and his friend and colleague Carl Gustav Jacob Jacobi (1804–1851) turn the University of Berlin, which had been founded in 1810 by Wilhelm von Humboldt (1767–1835) in the course of the reforms motivated and necessitated by the Napoleonic aggression against Prussia, into a center of mathematical research. Dirichlet's wife Rebecca was a granddaughter of the philosopher Moses Mendelssohn (1729–1786), a niece of the author Dorothea (von) Schlegel (1764–1839), who in turn was the wife of the writer and theorist of Romanticism Friedrich (von) Schlegel (1772–1829), and a sister of the composer Felix Mendelssohn Bartholdy (1809–1847), who as head of the Leipzig Gewandhaus Orchestra, initiated the rediscovery and renaissance of the baroque music of Bach and Händel. In this way, Dirichlet's life was intertwined with those of many other prominent personalities. Dirichlet was friendly and open towards Riemann, and Riemann could learn a lot from him. Dirichlet made in particular important contributions in number theory, and he founded the analytic direction of number theory. A historically oriented introduction can be found in W. Scharlau, H. Opolka, *From Fermat to Minkowski.* Lectures on the theory of numbers and its historical development, New York, Springer, [2]2010 (translated from the German). The principles applied by Dirichlet in the calculation of variations later played a central role in Riemann's studies on function theory and Riemann surfaces.

in Berlin. Dirichlet in 1855 became the successor of Gauss in Göttingen, and in 1859, Riemann in turn became his successor as a full professor in Göttingen after he had been appointed in 1857 as an associate professor. He was shy and sickly, but impressed the scientific world by the richness of his mathematical insight and the boldness and originality of his mathematical theories. He developed closer personal contacts outside of his family only with the younger mathematician Richard Dedekind (1831–1916).[8] He went through the steps of a standard academic career from the lectureship to a professorship in Göttingen. The salary that came with this professorship eased his financial situation considerably, especially because after the death of his parents and his brother he also took over the responsibility for three unmarried sisters. Health problems necessitated interruptions of this position by extended sojourns in Italy whose climate was more suitable for him, but where he succumbed to lung disease before reaching the age of 40, leaving behind his wife and a young daughter.

Riemann died neither as young as Niels Hendrik Abel (1802–1829) nor Evariste Galois (1811–1832), who in their short lives could create only one important mathematical theory (that of the Abelian integrals and group theory), nor did he reach the old age of the often grumpy and withdrawn Gauss. He had neither the almost inexhaustible vitality of Leonhard Euler (1707–1783) nor the active energy of Carl Gustav Jacob Jacobi (1804- 1851) and Felix Klein (1849–1925). He could not rely on a group of talented young students and collaborators as could David Hilbert (1862–1943), for the requisite institutional conditions were established in Germany only later by Felix Klein and others (and then destroyed again by the Nazis through expulsion and murder of Jewish mathematicians and the exile of those who were not Jewish but only dissenting). But Gauss and Riemann created the rise of mathematics in Germany and especially in Göttingen, that in the first place made such an institutionalization possible.

As far as the author knows, there is no detailed biography of Riemann written for general readers.[9] Otherwise, biographies of prominent mathematicians are not rare and

[8]On Dedekind see Winfried Scharlau (ed.), *Richard Dedekind. 1831|1981*, Braunschweig/Wiesbaden, Vieweg, 1981. The letters printed there also contain biographical material on Riemann, which can complement the picture in Dedekind's biography of Riemann in the latter's collected works.

[9]In Riemann's collected works edited by Heinrich Weber and Richard Dedekind, there is a 20 page biography of Riemann, written by his friend and colleague Dedekind. Hans Freudenthal wrote a short biography for the Dictionary of Scientific Biography. In addition to other short biographical sketches, there are the scientific biographies of Michael Monastyrsky and Detlef Laugwitz in which the development and impact of Riemann's scientific work is placed in the context of the circumstances of his life. The scientific biography of Laugwitz has been of great help to me at various places, and it also contains a detailed description of Riemann's life that is accessible to a general readership. Various other such analyses can be found in the reissue of Riemann's collected works edited by Raghavan Narasimhan. A systematic research of the unpublished scientific notes and sketches and the available biographical material on Riemann has been started by Erwin Neuenschwander. For some results, see Erwin Neuenschwander, *Riemanns Einführung in die Funktionentheorie*. Eine

in some countries constitute even an expression of national pride, like the biographies of the Norwegian mathematicians Niels Hendrik Abel and Sophus Lie (1842–1899) by Arild Stubhaug. In other cases, such as the biographies of David Hilbert and Richard Courant (1888–1972) by Constance Reid that are very popular among mathematicians, their lives under the often difficult and unfavorable circumstances of their times and the historical events affecting them also arouse interest. Since Riemann's life unfolded in a quiet time without dramatic personal or historical events, there is no material for a spectacular tale. Also, the cult of genius, for which Riemann should actually provide an excellent example, did not take him up as it had the younger mathematicians Abel and Galois or the artists Raphael, Mozart and Schiller who had a life span similar to that of Riemann.

The academic life of Bernhard Riemann appears unproblematic from today's perspective if we leave the precarious financial circumstances aside in which the so-called privatdozenten (private lecturers) had to live at that time. These privatdozenten were doctoral or post-doctoral level scientists who did not have a regular professorship. The ascent from lectureship to professorship was probably already in his time considered as the usual academic career. It should be pointed out, however, that in many cases there were deviations from this path, in both the positive and the negative sense. In any case, the modern university system had been founded by Wilhelm von Humboldt only half a century before Riemann, and the relocation of scientific research from the scientific academies and learned societies of the 18th to the universities of the nineteenth century and the establishment of appropriate academic career structures had needed some time. In particular, in the initial phase, the university system could therefore typically not form its next generation through internal university career paths, but had to recruit university teachers often from outside, from the group of high-school professors or that of the scientific practitioners, who worked in astronomical observatories, botanical gardens, pharmacies or other institutions. Conversely, aspiring scientists could therefore not necessarily build a purely academic career, but often had to take long biographical detours. Thus, on one side, there were gifted and talented scientists who never in their lives got a university position. On the other side, there were those who succeeded in gaining entry into an academic career from an outsider position, or conversely, those who from an early age were generously supported by noble sponsors or governments. The latter category included Gauss, who was supported by his ruler, the Duke of Brunswick, or Dirichlet, who at the initiative of Alexander von Humboldt was first sponsored in his studies in Paris and then given a professorship in Berlin. A well-known example outside of mathematics is the chemist Justus (von) Liebig (1803–1873), whom the Grand Duke of Hesse enabled his studies in Paris and then, as early as 1824, made a professor at Giessen

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quellenkritische Edition seiner Vorlesungen und einer Bibliographie zur Wirkungsgeschichte der Riemannschen Funktionentheorie. Abhandlungen der Akademie der Wissenschaften zu Göttingen, Math."=Phys. Klasse, Bd. 44, 1996. Various mathematical historical studies discuss the development of geometry before, through and after Riemann, but usually not from a biographical perspective. Sources can be found in the bibliography at the end of this book.

where Liebig then established the institution of the chemical laboratory for teaching and research.Notable examples of those, who through tough and sustained efforts managed external access to and then achieved central positions in the German scientific system, are the mathematician Karl Weierstrass (1815–1897), who had to spend many years as a grammar school teacher in Deutsch-Krone in former West Prussia and in Braunsberg in former East Prussia before he could gain scientific recognition by his mathematical work on elliptic integrals, or Hermann (von) Helmholtz (1821–1894), who first had to work as a military surgeon before he was able to start his academic career. Wilhelm Killing (1847–1923) conducted his important studies on the foundations of geometry and the infinitesimal transformation groups (Lie algebras) in the little time left by a huge teaching schedule that comprised all sciences and on top of which he even had to serve for some periods as the principal at the Lyceum Hosianum in Braunsberg in East Prussia, where his teacher Weierstrass had worked before. In 1892, Killing became a professor in Münster. There, however, his teaching and administrative tasks up to the office of the Rektor (University President) and his caritative commitments that were rooted in his catholicism took so much of his time that he could barely continue his mathematical research. Others, such as the mathematician Hermann Grassmann (1809–1877), were denied scientific recognition throughout their lives. Grassmann was a high school teacher in Stettin, and he founded linear algebra, which is fundamental in today's mathematical university education and is taught from the very first semester. (Grassmann was also a major Sanskrit researcher and studied in particular the Rigveda. In contrast to his mathematical work, these studies obtained the recognition of the academic world.) A well-known example outside of mathematics is of course the Augustinian monk Gregor Mendel (1822–1884), whose discovery of the quantitative laws of inheritance, one of the deepest insights throughout the history of biology, failed to gain the notice of the professional biologists, until after the turn of the century they were rediscovered by several researchers, in a weaker form first, and then became the foundation of the modern gene concept.

Previous editions (for details see the bibliography at the end):

- Collected Works (various editions, the most recent by Narasimhan);
- Weyl, with a detailed mathematical commentary