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Gian Italo Bischi
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Qualitative Theory of Dynamical Systems, Tools and Applications for Economic Modelling

Lectures Given at the COST Training School on New Economic Complex Geography at Urbino, Italy, 17–19 September 2015

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Editors
Gian Italo Bischi
Università di Urbino "Carlo Bo"
Urbino
Italy

Davide Radi
LIUC - Università Cattaneo
Castellanza
Italy

Anastasiia Panchuk
National Academy of Sciences of Ukraine
Kiev
Ukraine

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Preface

This volume contains the lessons delivered during the “Training School on qualitative theory of dynamical systems, tools and applications” held at the University of Urbino (Italy) from 17 September to 19 September 2015 in the framework of the European COST Action “The EU in the new complex geography of economic systems: models, tools and policy evaluation” (Gecomplexity). Gecomplexity is a European research network, inspired by the New Economic Geography approach, initiated by P. Krugman in the early 1990s, which describes economic systems as multilayered and interconnected spatial structures. At each layer, different types of decisions and interactions are considered: interactions between international or regional trading partners at the macrolevel; the functioning of (financial, labour, goods) markets as social network structures at mesolevel; and finally, the location choices of single firms at the microlevel. Within these structures, spatial inequalities are evolving through time following complex patterns determined by economic, geographical, institutional and social factors. In order to study these structures, the Action aims to build an interdisciplinary approach to develop advanced mathematical and computational methods and tools for analysing complex nonlinear systems, ranging from social networks to game theoretical models, with the formalism of the qualitative theory of dynamical systems and the related concepts of attractors, stability, basins of attraction, local and global bifurcations.

Following the same spirit, this book should provide an introduction to the study of dynamic models in economics and social sciences, both in discrete and in continuous time, by the methods of the qualitative theory of dynamical systems. At the same time, the students should also practice (and, hopefully, appreciate) the interdisciplinary “art of mathematical modelling” of real-world systems and time-evolving processes. Indeed, the set-up of a dynamic model of a real evolving system (physical, biological, social, economic, etc.) starts from a rigorous and critical analysis of the system, its main features and basic principles. Measurable quantities (i.e. quantities that can be expressed by numbers) that characterize its state and its behaviour must be identified in order to describe the system mathematically. This leads to a schematic description of the system, generally a simplified representation, expressed by words, diagrams and symbols. This task is, commonly, carried out by specialists of the real system, such as economists and social scientists. The following stage consists in the translation of the schematic model into a mathematical model, expressed by mathematical symbols and operators. This leads us to the mathematical study of the model by using mathematical tools, theorems, proofs, mathematical expressions and/or numerical methods. Then, these mathematical results must be translated into the natural language and terms typical of the system described, that is economic or biologic or physical terms, in order to obtain laws or statements useful for the application considered. This closes the path of mathematical modelling, but often it is not the end of the modelization process. In fact, if the results obtained are not satisfactory, in the sense that they do not agree with the observations or experimental data, then one needs to re-examine the model, by adding some details or by changing some basic assumptions, and start again the whole procedure. The chapters of this volume are mainly devoted to the mathematical methods for the analysis of dynamical models by using the qualitative theory of dynamical systems, developed through a continuous and fruitful interaction among analytical, geometric and numerical methods. However, several examples of model building are given as well, because this is the most creative stage, leading from reality to its formalization in the form of a mathematical model. This requires competence and fantasy, the reason why we used the expression "art of mathematical modelling".

The simulation of the time evolution of economic systems by using the language and the formalism of dynamical systems (i.e. differential or difference equations according to the assumption of continuous or discrete time) dates back to the early steps of the mathematical formalization of models in economics and social sciences, mainly in the nineteenth century. However, in the last decades, the importance of dynamic modelling increased because of the parallel trends in mathematics on one side and economics and social sciences on the other side. The two developments are not independent, as new issues in mathematics favoured the enhancement of understanding of economic systems, and the needs of more and more complex mathematical models in economics and social studies stimulated the creation of new branches in mathematics and the development of existing ones. Indeed, in recent mathematical research, a flourishing literature in the field of qualitative theory of nonlinear dynamical systems, with the related concepts of attractors, bifurcations, dynamic complexity, deterministic chaos, has attracted the attention of many scholars of different fields, from physics to biology, from chemistry to economics and sociology, etc. These mathematical topics become more and more popular even outside the restricted set of academic specialists. Concepts such as bifurcations (also called catastrophes in the Eighties), fractals and chaos entered and deeply modified several research fields.

On the other side, during the last decades, also economic modelling has been witnessing a paradigm shift in methodology. Indeed, despite its notable achievements, the standard approach based on the paradigm of the rational and representative agent (endowed with unlimited computational ability and perfect information) as well as the underlying assumption of efficient markets failed to explain many important features of economic systems and has been criticized on a number of grounds. At the same time, a growing interest has emerged in alternative approaches to economic agents' decision-making, which allow for factors such as bounded rationality and heterogeneity of agents, social interaction and learning, where agents' behaviour is governed by simpler "rules of thumb" (or "heuristics") or "trial and error" or even "imitations mechanisms". Adaptive system, governed by local (or myopic) decision rules of boundedly rational and heterogeneous agents, may converge in the long run to a rational equilibrium, i.e. the same equilibrium forecasted (and instantaneously reached) under the assumption of full rationality and full information of all economic agents. This may be seen as an evolutionary interpretation of a rational equilibrium, and some authors say that in this case, the boundedly rational agents are able to learn, in the long run, what rational agents already know under very pretentious rationality assumptions. However, it may happen that under different starting conditions, or as a consequence of exogenous perturbations, the same adaptive process leads to non-rational equilibria as well, i.e. equilibrium situations which are different from the ones forecasted under the assumption of full rationality, as well as to dynamic attractors characterized by endless asymptotic fluctuations that never settle to a steady state. The coexistence of several attracting sets, each with its own basin of attraction, gives rise to path dependence, irreversibility, hysteresis and other nonlinear and complex phenomena commonly observed in real systems in economics, finance and social sciences, as well as in laboratory experiments.

From the description given above, it is evident that the analysis of adaptive systems can be formulated in the framework of the theory of dynamical systems, i.e. systems of ordinary differential equations (continuous time) or difference equations (discrete time); the qualitative theory of nonlinear dynamical systems, with the related concepts of stability, bifurcations, attractors and basins of attraction, is a major tool for the analysis of their long-run (or asymptotic) properties. Not only in economics and social sciences, but also in physics, biology and chemical sciences, such models are a privileged instrument for the description of systems that change over time, often described as "nonlinear evolving systems", and their long-run aggregate outcomes can be interpreted as "emerging properties", sometimes difficult to be forecasted on the basis of the local (or step by step) laws of motion. As we will see in this book, a very important role in this theory is played by graphical analysis, and a fruitful trade-off between analytic, geometric and numerical methods. However, these methods built up a solid mathematical theory based on general theorems that can be found in the textbooks indicated in the references.

Chapter 1, by Gian Italo Bischi, Fabio Lamantia and Davide Radi, is the largest one, as it contains the basic lessons delivered during the Training School. It introduces some general concepts, notations and a minimal vocabulary about the mathematical theory of dynamical systems both in continuous time and in discrete time, as well as optimal control.

Chapter 2, by Anastasiia Panchuk, points out several aspects related to global analysis of discrete time dynamical systems, covering homoclinic bifurcations as well as inner and boundary crises of attracting sets.

Chapter 3, by Anna Agliari, Nicolò Pecora and Alina Szuz, describes some properties of the nonlinear dynamics emerging from two oligopoly models in discrete time. The target of this chapter is the investigation of some local and global bifurcations which are responsible for the changes in the qualitative behaviours of the trajectories of discrete dynamical systems. Two different kinds of oligopoly models are considered: the first one deals with the presence of differentiated goods and gradient adjustment mechanism, while the second considers the demand function of the producers to be dependent on advertising expenditures and adaptive adjustment of the moves. In both models, the standard local stability analysis of the Cournot-Nash equilibrium points is performed, as well as the global bifurcations of both attractors and (their) basins of attraction are investigated.

Chapter 4, by Ingrid Kubin, Pasquale Commendatore and Iryna Sushko, acquaints the reader with the use of dynamic models in regional economics. The focus is on the New Economic Geography (NEG) approach. This chapter briefly compares NEG with other economic approaches to investigation of regional inequalities. The analytic structure of a general multiregional model is described, and some simple examples are presented where the number of regions assumed to be small to obtain more easily analytic and numerical results. Tools from the mathematical theory of dynamical systems are drawn to study the qualitative properties of such multiregional model.

In Chap. 5, Fabio Lamantia, Davide Radi and Lucia Sbragia review some fundamental models related to the exploitation of a renewable resource, an important topic when dealing with regional economics. The chapter starts by considering the growth models of an unexploited population and then introduces commercial harvesting. Still maintaining a dynamic perspective, an analysis of equilibrium situations is proposed for a natural resource under various market structures (monopoly, oligopoly and open access). The essential dynamic properties of these models are explained, as well as their main economic insights. Moreover, some key assumptions and tools of intertemporal optimal harvesting are recalled, thus providing an interesting application of the theory of optimal growth.

In Chap. 6, Fabio Tramontana considers the qualitative theory of discrete time dynamical systems to describe the time evolution of financial markets populated by heterogeneous and boundedly rational traders. By using these assumptions, he is able to show some well-known stylized facts observed in financial markets that can be replicated even by using small-scale models.

Finally, in Chap. 7, Ugo Merlone and Paul van Geert consider some dynamical systems which are quite important in psychological research. They show how to implement a dynamical model of proximal development using a spreadsheet, statistical software such as R or programming languages such as C++. They discuss strengths and weaknesses of each tool. Using a spreadsheet or a subject-oriented statistical software is rather easy to start, hence being likely palatable for people with background in both economics and psychology. On the other hand, employing C++ provides better efficiency at the cost of requiring some more competencies. All the approaches proposed in this chapter use free and open-source software.

Urbino — Gian Italo Bischi
Kiev — Anastasiia Panchuk
Castellanza — Davide Radi

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Contents

Contributors

Anna Agliari Department of Economics and Social Sciences, Catholic University, Piacenza, Italy

Gian Italo Bischi DESP-Department of Economics, Society, Politics, Università di Urbino “Carlo Bo”, Urbino, PU, Italy

Pasquale Commendatore Department of Law, University of Naples Federico II, Naples, NA, Italy

Ingrid Kubin Department of Economics, Institute for International Economics and Development (WU Vienna University of Economics and Business), Vienna, Austria

Fabio Lamantia Department of Economics, Statistics and Finance, University of Calabria, Rende, CS, Italy; Economics—School of Social Sciences, The University of Manchester, Manchester, UK

Ugo Merlone Department of Psychology, Center for Cognitive Science, University of Torino, Torino, Italy

Anastasiia Panchuk Institute of Mathematics, National Academy of Sciences of Ukraine, Kiev, Ukraine

Nicolò Pecora Department of Economics and Social Sciences, Catholic University, Piacenza, Italy

Davide Radi School of Economics and Management, LIUC - Università Cattaneo, Castellanza, VA, Italy

Lucia Sbragia Department of Economics, Durham University Business School, Durham, UK

Iryna Sushko Institute of Mathematics, National Academy of Sciences of Ukraine, Kiev, Ukraine

Alina Szuz Independent Researcher, Cluj-Napoca, Romania

Fabio Tramontana Department of Mathematical Sciences, Mathematical Finance and Econometrics, Catholic University, Milan, MI, Italy

Paul van Geert Heymans Institute, Groningen, The Netherlands

Chapter 1
Qualitative Methods in Continuous and Discrete Dynamical Systems

Gian Italo Bischi, Fabio Lamantia and Davide Radi

Abstract This chapter gives a general and friendly overview to the qualitative theory of continuous and discrete dynamical systems, as well as some applications to simple dynamic economic models, and is concluded by a section on basic principles and results of optimal control in continuous time, with some simple applications. The chapter aims to introduce some general concepts, notations and a minimal vocabulary concerning the study of the mathematical theory of dynamical systems that are used in the other chapters of the book. In particular, concepts like stability, bifurcations (local and global), basins of attraction, chaotic dynamics, noninvertible maps and critical sets are defined, and their applications are presented in the following sections both in continuous time and discrete time, as well as a brief introduction to optimal control together with some connections to the qualitative theory of dynamical systems and applications in economics.

G.I. Bischi (✉)
DESP-Department of Economics, Society, Politics, Università di Urbino "Carlo Bo",
42 Via Saffi, 61029 Urbino, PU, Italy
e-mail: gian.bischi@uniurb.it

F. Lamantia
Department of Economics, Statistics and Finance, University of Calabria,
3C Via P. Bucci, 87036 Rende, CS, Italy
e-mail: fabio.lamantia@unical.it

F. Lamantia
Economics—School of Social Sciences, The University of Manchester,
Arthur Lewis Building, Manchester, UK

D. Radi
School of Economics and Management, LIUC - Università Cattaneo,
22 C.so Matteotti, 21053 Castellanza, VA, Italy
e-mail: dradi@liuc.it

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1.1 Some General Definitions

In this section we introduce some general concepts, notations and a minimal vocabulary about the mathematical theory of dynamical systems. A *dynamical system* is a mathematical model, i.e., a formal, mathematical description, of a system evolving as time goes on. This includes, as a particular case, systems whose state remains constant, that will be denoted as systems at equilibrium.

The first step to describe such systems in mathematical terms is the characterization of their "state" by a finite number, say n, of measurable quantities, denoted as "state variables", expressed by real numbers $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. For example in an economic system these numbers may be the prices of n commodities in a market, or the respective quantities, or they can represent other measurable indicators, like level of occupation, or salaries, or inflation. In an ecologic system these n numbers used to characterize its state may be the numbers (or densities) of individuals of each species, or concentration of inorganic nutrients or chemicals in the environment. In a physical system[1] the state variables may be the positions and velocities of the particles, or generalized coordinates and related momenta of a mechanical system, or temperature, pressure etc. in a thermodynamic system.

This ordered set of real numbers can be seen as a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, i.e., a "point" in an n-dimensional space, and this allows us to introduce a "geometric language", in the sense that a 1-dimensional dynamical system is represented by point along a line, a 2-dimensional one by a point in a Cartesian plane and so on.

Sometimes only the values of the state variables included in a subset of $\mathbb{R}^n$ are suitable to represent the real system. For example only non-negative values of x_i are meaningful if x_i represents a price in an economic system or the density of a species in an ecologic one, or it can be that in the equations that define the system a state variable x_i is the argument of a mathematical function that is defined in a given domain, like a logarithm, a square root or a rational function. As a consequence, only the points in a subset of $\mathbb{R}^n$ are admissible states for the dynamical system considered, and this leads to the following definition.

Definition 1.1 The *state space* (or *phase space*) $M \subseteq \mathbb{R}^n$ is the set of admissible values of the state variables.

As a dynamical system is assumed to evolve with time, these numbers are not fixed but are functions of time $x_i = x_i(t)$, $i = 1, \ldots, n$, where t may be a real number (*continuous time*) or a natural number (*discrete time*). The latter assumption may sound quite strange, whereas it represents a common assumption in systems where changes of the state variables are only observed as a consequence of events occurring at given time steps (event-driven time). For example, it is quite common in economic and social sciences where in many systems the state variables can change as a consequence of human decisions that cannot be continuously revised, e.g., after

[1] Physics is the discipline where the formalism of dynamical system has been first introduced, since 17th century, even if the modern approach, often denoted as qualitative theory of dynamics systems, has been introduced in the early years of the 20th century.

production periods (the typical example is output of agricultural products) or after the meetings of an administration council or after the conclusions of contracts etc. (decision-driven time).

So, in the following we will distinguish these two cases, according to the domain of the state functions: $x_i : \mathbb{R} \to \mathbb{R}$ or $x_i : \mathbb{N} \to \mathbb{R}$, i.e., the continuous or discrete nature of time. In any case, the purpose of dynamical systems is the following: given the state of the system at a certain time t_0, compute the state of the system at time $t \neq t_0$. This is equivalent to the knowledge of an operator

$$\mathbf{x}(t) = \mathbf{G}\left(t, \mathbf{x}(t_0)\right) \ , \tag{1.1}$$

where boldface symbols represent vectors, i.e., $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in M \subseteq \mathbb{R}^n$ and $\mathbf{G}(\cdot) = (G_1(\cdot), \ldots, G_n(\cdot)) : M \to M$. If one knows the *evolution operator* $\mathbf{G}$ then from the knowledge of the *initial condition* (or *initial state*) $\mathbf{x}(t_0)$ the state of the system at any future time $t > t_0$ can be computed, as well as at any time of the past $t < t_0$. Generally we are interested in the forecasting of future states, especially in the asymptotic (or long-run) evolution of the system as $t \to +\infty$, i.e., the fate, or the destiny of the system. However, even the flashback may be useful in some cases, like in detective stories when the investigators from the knowledge of the present state want to know what happened in the past.

The vector function $\mathbf{x}(t)$, i.e., the set of n functions $x_i(t)$, $i = 1, \ldots, n$ obtained by (1.1), represents the parametric equations of a *trajectory*, as t varies. In the case of continuous time $t \in \mathbb{R}$ the trajectory is a curve in the space $\mathbb{R}^n$, that can be represented in the $n+1$-dimensional space $(\mathbb{R}^n, t)$, and denoted as *integral curve*, or in the state space (also denoted as "phase space") $\mathbb{R}^n$, see Fig. 1.1. In the latter case the direction of increasing time is represented by arrows, and the curve is denoted as *phase curve*.

In the case of discrete time a trajectory is a sequence (i.e., a countable set) of points, and the time evolution of the system jumps from one point to the successive one in the sequence. Sometimes line segments can be used to join graphically the points, moving in the direction of increasing time, thus getting an ideal piecewise smooth curve by which the time evolution of the system is graphically represented.

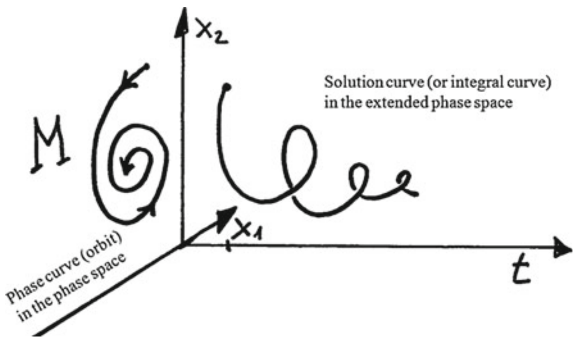

Fig. 1.1 Solution curve and its projection in the phase space

An *equilibrium* (*stationary state* or *fixed point*) $\mathbf{x}^* = \left(x_1^*, \ldots, x_n^*\right)$ is a particular trajectory such that all the state variables are constant

$$\mathbf{x}(t) = \mathbf{G}\left(t, \mathbf{x}^*\right) = \mathbf{x}^* \text{ for each } t > t_0 \; .$$

An equilibrium is a trapping point, i.e., any trajectory through it remains in it for each successive time: $\mathbf{x}(t_0) = \mathbf{x}^*$ implies $\mathbf{x}(t) = \mathbf{x}^*$ for $t \geq t_0$. This definition can be extended to any subset of the phase space:

Definition 1.2 A set $A \subseteq M$ is *trapping* if $\mathbf{x}(t_0) \in A$ implies $\mathbf{x}(\mathbf{t}) = \mathbf{G}\left(t, \mathbf{x}(t_0)\right) \in A$ for each $t > t_0$.

This can also be expressed by the notation $\mathbf{G}(t, A) \subseteq A$, where

$$\mathbf{G}(t, A) = \{\mathbf{x}(t) \in M : \exists t \geq t_0 \text{ and } \mathbf{x}(t_0) \in A \text{ so that } \mathbf{x}(t) = \mathbf{G}\left(t, \mathbf{x}(t_0)\right)\} \; .$$

So, any trajectory starting inside a trapping set cannot escape from it. We now define a stronger property, in the sense that it concerns particular kinds of trapping sets.

Definition 1.3 A closed set $A \subseteq M$ is *invariant* if $\mathbf{G}(t, A) = A$, i.e., each subset $A' \subset A$ is not trapping.

In other words, any trajectory starting inside an invariant set remains there, and all the points of the invariant set can be reached by a trajectory starting inside it. Notice that an equilibrium point is a particular kind of invariant set (let's say the simplest). However, we will see many other kinds of invariant sets, where interesting cases of nonconstant trajectories are included.

We now wonder what happens if we start a trajectory from an initial condition close to an invariant set, i.e., in a neighborhood of it. The trajectory may enter the invariant set (and then it remains trapped inside it) or it may move around it or it may go elsewhere, far from it. This leads us to the concept of stability of an invariant set (Fig. 1.2).

Definition 1.4 (*Lyapunov stability*) An invariant set A is *stable* if for each neighborhood U of A there exists another neighborhood V of A with $V \subseteq U$ such that any trajectory starting from V remains inside U.

In other words, Lyapunov stability means that all the trajectories starting from initial conditions outside A and sufficiently close to it remain around it. Instability is the negation of stability, i.e., an invariant set A is unstable if a neighborhood $U \supset A$

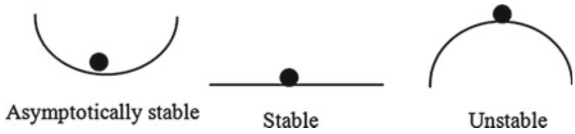

Fig. 1.2 Analogies with the gravity field

exists such that initial conditions taken arbitrarily close to A exist that generate trajectories that exit U. The following definition is stronger.

Definition 1.5 (*Asymptotic stability*) An invariant set A is *asymptotically stable* (and it is often called *attractor*) if:

(i) A is stable (according to the definition given above);
(ii) $\lim_{t\to+\infty} \mathbf{G}(t, \mathbf{x}) \in A$ for each initial condition $\mathbf{x} \in V$.

In other words, asymptotic stability means that the trajectories starting from initial conditions outside A and sufficiently close to it not only remain around it, but tend to it in the long run (i.e., asymptotically), see the schematic pictures in Fig. 1.3. At a first sight, the condition (ii) in the definition of asymptotic stability seems to be stronger than (i), hence (i) seems to be superfluous. However it may happen that a neighborhood $U \supset A$ exists such that initial conditions taken arbitrarily close to A generate trajectories that exit U and then go back to A in the long run (see the last picture in Fig. 1.3).

Of course, all these definitions expressed in terms of neighborhoods can be restated by using a norm (and consequently a distance) in $\mathbb{R}^n$, for example the euclidean norm $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$ from which the distance between two points $\mathbf{x} = (x_1, \ldots, x_n)$ and

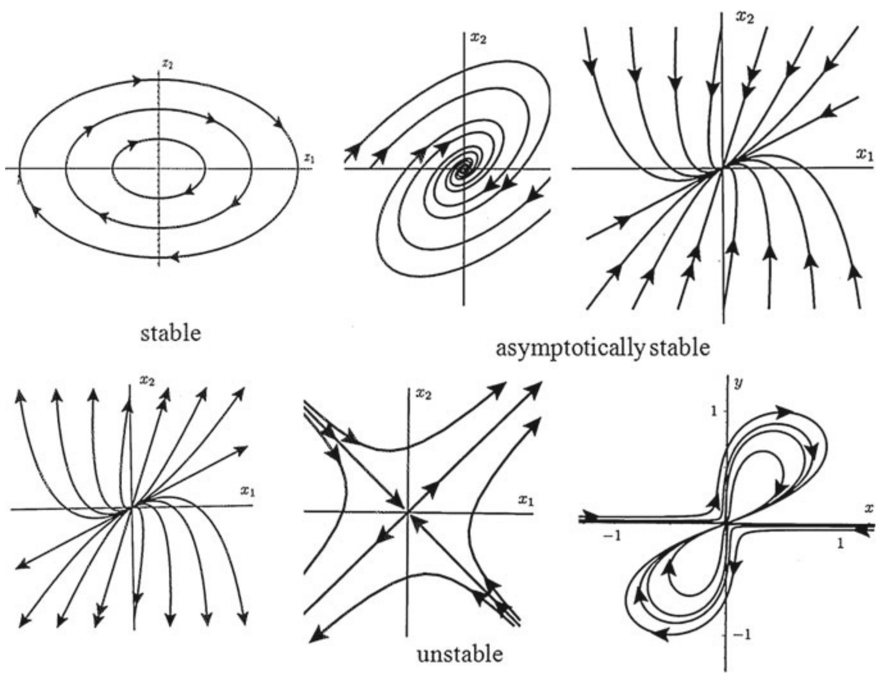

Fig. 1.3 Qualitative examples of stable, asymptotically stable and unstable equilibria

$\mathbf{y} = (y_1, \ldots, y_n)$ can be defined as $\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. As an example we can restate the definitions given above for the particular case of an equilibrium point.

Let $\mathbf{x}(t) = \mathbf{G}(t, \mathbf{x}(t_0))$, $t \geq 0$, a trajectory starting from the initial condition $\mathbf{x}(t_0) = \mathbf{G}(t_0, \mathbf{x}(t_0))$ and $\mathbf{x}^*$ an equilibrium point $\mathbf{x}^* = \mathbf{G}(t, \mathbf{x}^*)$ for $t \geq 0$. The equilibrium $\mathbf{x}^*$ is stable if for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\|\mathbf{x}(t_0) - \mathbf{x}^*\| < \delta_\varepsilon \Longrightarrow \|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$ for $t \geq 0$. If in addition $\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}^*\| = 0$ then $\mathbf{x}^*$ is asymptotically stable. Instead, if an $\varepsilon > 0$ exists such that for each $\delta > 0$ we have $\|\mathbf{x}(t) - \mathbf{x}^*\| > \varepsilon$ for some $t > 0$ even if $\|\mathbf{x}(t_0) - \mathbf{x}^*\| < \delta$, then $\mathbf{x}^*$ is unstable.

These definitions are local, i.e., concern the future behavior of a dynamical system when its initial state is in an arbitrarily small neighborhood of an invariant set. So, they can be used to characterize the behavior of the system under the influence of small perturbation from an equilibrium or another invariant set. In other words, they give an answer to the question: given a system at equilibrium, what happens when small exogenous perturbation move its state slightly outside the equilibrium state? However, in the study of real systems we are also interested in their global behavior, i.e., far from equilibria (or more generally from invariant sets) in order to consider the effect of finite perturbations and to answer questions like: how far can an exogenous perturbation shift the state of a system from an equilibrium remaining sure that it will spontaneously go back to the originary equilibrium? This kind of questions leads to the concept of basin of attraction.

Definition 1.6 (*Basin of attraction*) The *basin of attraction* of an attractor A is the set of all points $\mathbf{x} \in M$ such that $\lim_{t\to+\infty} \mathbf{G}(t, \mathbf{x}) \in A$, i.e.,

$$B(A) = \left\{ \mathbf{x} \in M \text{ such that } \lim_{t\to+\infty} \mathbf{G}(t, \mathbf{x}) \in A \right\} .$$

If $B(A) = M$ then A is called *global attractor*. Generally the extension of the basin of a given attractor gives a measure of its robustness with respect to the action of exogenous perturbations. However this is a quite rough argument, because a greater extension of the basin of an attractor may does not imply greater robustness if the attractor is close to a basin boundary. Moreover, when basins are considered, one realizes that in some cases stable equilibria may be even more vulnerable than unstable ones (see Fig. 1.4).

Other important indicators should be critically considered. For example, how fast is the convergence towards an attractor? Even if an invariant set is asymptotically stable and it has a large basin, an important question concerns the speed of convergence, i.e., the amount of time which is necessary to reduce the extent of a perturbation. In some cases this time interval may be too much for any practical purpose. These arguments lead us to the necessity of a deep understanding of the global behavior of a dynamical system in order to give useful indications about the performance of the real system modeled. The main problem is that, generally, the operator $\mathbf{G}$ that allows to get an explicit representation of the trajectories of the dynamical system for any initial condition in the phase space, is not known, or cannot be expressed in terms of elementary functions, or its expression is so complicated that it cannot be

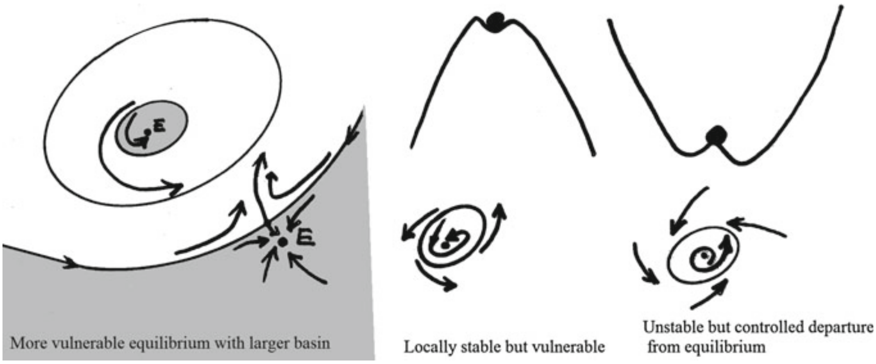

Fig. 1.4 Stability and vulnerability

used for any practical purpose. In general a dynamical system is expressed in terms of *local evolution equations*, also denoted as *dynamic equations* or *laws of motion*, that state how the dynamical system changes as a consequence of small time steps. In the case of continuous time the evolution equations are expressed by the following set of *ordinary differential equations* (ODE) involving the time derivative, i.e., the speeds of change, of each state variable

$$\begin{aligned} \frac{dx_i(t)}{dt} &= f_i(x_1(t), \ldots, x_n(t); \alpha) \ , \quad i = 1, \ldots, n \ , \\ x_i(t_0) &= \overline{x}_i \ , \end{aligned} \tag{1.2}$$

where the time derivative at the left hand side represents, as usual, the speed of change of the state variable $x_i(t)$ with respect to time variations, the functional relations give information about the influence of the same state variable x_i (self-control) and of the other state variables x_j, $j \neq i$ (cross-control) on such rate of change, and $\alpha = (\alpha_1, \ldots \alpha_m)$, $\alpha_i \in \mathbb{R}$, represents m real parameters, fixed along a trajectory, which can assume different numerical values in order to represent exogenous influences on the dynamical systems, e.g., different policies or effects of the outside environment. The modifications induced in the model after a variation of some parameters α_i are called *structural modifications*, as such changes modify the shape of the functions f_i, and consequently the properties of the trajectory.

The set of equations (1.2) are "differential equations" because their "unknowns" are functions $x_i(t)$ and they involve not only $x_i(t)$ but also their derivatives. In the theory of dynamical systems it is usual to replace the Leibniz notation $\frac{dx}{dt}$ of the derivative with the more compact "dot" notation $\dot{x}$ introduced by Newton. With this notation, the dynamical system (1.2) is indicated as

$$\dot{x}_i = f_i(x_1, \ldots, x_n; \alpha), \quad i = 1, \ldots, n \ , \tag{1.3}$$

Differential equations of order greater than one, i.e., involving derivatives of higher order, can be easily reduced to systems of differential equations of order one in the form (1.2) by introducing auxiliary variables. For example the second order differential equation (involving the second derivative $\ddot{x} = \frac{d^2x}{dt^2}$)

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0 \tag{1.4}$$

with initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ can be reduced to the form (1.3) by defining $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, so that the equivalent system of two first order differential equations becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \, , \\ \dot{x}_2 &= -bx_1 - ax_2 \end{aligned}$$

with $x_1(0) = x_0, x_2(0) = v_0$. If along a trajectory the parameters explicitly vary with respect to time, i.e., some $\alpha_i = \alpha_i(t)$ are functions of time, then the model is called *nonautonomous*. Also a nonautonomous model can be reduced to an equivalent autonomous one in the normal form (1.2) of dimension $n+1$ by introducing the dynamic variable $x_{n+1} = t$ whose time evolution is governed by the added first order differential equation $\dot{x}_{n+1} = 1$.

In the case of discrete time, the evolution equations are expressed by the following set of *difference equations* that inductively define the time evolution as a sequence of discrete points starting from a given initial condition

$$\begin{aligned} x_i(t+1) &= f_i(x_1(t), \ldots, x_n(t); \alpha), \quad i = 1, \ldots, n \, , \\ x_i(0) &= \overline{x}_i \end{aligned} \tag{1.5}$$

Also in this case a higher order difference equation, as well as a nonautonomous difference equation, can be reduced to an expanded system of first order difference equations. For example, the second order difference equations

$$x(t+1) + ax(t) + bx(t-1) = 0$$

starting from the initial conditions $x(-1) = x_0$, $x(0) = x_1$ can be equivalently rewritten as

$$\begin{aligned} x(t+1) &= -ax(t) - by(t) \, , \\ y(t+1) &= x(t) \, , \end{aligned}$$

where $y(t) = x(t-1)$, with initial conditions being $x(0) = x_1$, $y(0) = x_0$. Analogously, a nonautonomous difference equation

$$x(t+1) = f(x(t), t)$$

becomes

$$x(t+1) = f(x(t), y(t)),$$
$$y(t+1) = y(t) + 1,$$

where $y(t) = t$.

So, the study of (1.2) and (1.5) constitutes a quite general approach to dynamical systems in continuous and discrete time respectively. They are local representations of the evolution of systems that change with time. Their qualitative analysis consists in the study of existence and main properties of attracting sets, their basins, and their qualitative changes as the control parameters are let to vary. We refer the reader to standard textbooks and the huge literature about difference and differential equations in order to study their general properties and methods of solutions. The aim of this lecture note is just to give a general overview of the basic elements for a qualitative understanding of the long run behavior of some dynamic models. We will first consider the case of continuous time, then the case of discrete time by stressing the analogies and differences between these two time scales, and finally we shall give some concepts and results about optimal control analysis.

1.2 Continuous-Time Dynamical Systems

In this section we consider dynamic equations in the form (1.2), starting from problems with $n = 1$, i.e., 1-dimensional models where the state of the system is identified by a single dynamical variable, then we move to $n = 2$ and finally some comments on $n > 2$. For each case, we will first consider linear models, for which an explicit expression of the solution can be obtained, and then we move to nonlinear models for which we will only give a qualitative description of the equilibrium points, their stability properties and the long-run (or asymptotic) properties of the solutions without giving their explicit expression. We will see that such qualitative study (also denoted as qualitative or topological theory of dynamical systems, a modern point of view developed in the 20th century) essentially reduces to the solution of algebraic equations and inequalities, without the necessity to use advanced methods for solving integrals. We start with a sufficiently general (for the goals of these lecture notes) theorem of existence and uniqueness of solutions of ordinary differential equations.

Theorem 1.1 (Existence and Uniqueness) *If the functions f_i have continuous partial derivatives in M and $x(t_0) \in M$, then there exists a unique solution $x_i(t)$, $i = 1, \dots, n$, of the system (1.2) such that $x(t_0) = \bar{\mathbf{x}}$, and each $x_i(t)$ is a continuous function.*

Indeed, the assumptions of this theorem may be weakened, by asking for bounded variations of the functions f_i in the equations of motion (1.2), such as the so called Lipschitz conditions. However the assumptions of the previous Theorem are suitable for our purposes. Moreover, other general theorems are usually stated to define the conditions under which the solutions of the differential equations have a regular behavior. We refer the interested reader to more rigorous books, see the bibliography for details.

1.2.1 *One-Dimensional Dynamical Systems in Continuous Time*

1.2.1.1 The Simplest One: A Linear Dynamical System

Let us consider the following dynamic equation

$$\dot{x} = \alpha x \quad \text{with initial condition} \quad x(t_0) = x_0 \ . \tag{1.6}$$

It states that the rate of growth of the dynamic variable $x(t)$ is proportional to itself, with proportionality constant α (a parameter). If $\alpha > 0$ then whenever x is positive it will increase (positive derivative means increasing). Moreover, as x increases also the derivative increases, so it increases faster and so on. This is what, even in the common language, is called "exponential growth", i.e., "the more we are, the more we increase". Instead, whenever x is negative it will decrease (negative derivative) so it will become even more negative and so on. This is a typical unstable behavior.

On the contrary, if $\alpha < 0$ then whenever x is positive it will decrease (and will tend to zero) whereas when x is negative the derivative is positive, so that x will increase (and tend to zero). A stabilizing behavior.

In this case an explicit solution can be easily obtained to confirm these arguments. In fact, it is well known, from elementary calculus, that the only function whose derivative is proportional to the function itself is the exponential, so $x(t)$ will be in the general form $x(t) = k\mathrm{e}^{\alpha t}$, where k is an arbitrary constant that can be determined by imposing the initial condition $x(t_0) = x_o$, hence $k\mathrm{e}^{\alpha t_0} = x_0$, from which $k = x_0\mathrm{e}^{-\alpha t_0}$. After replacing k in the general form we finally get the (unique) solution

$$x(t) = x_0\mathrm{e}^{\alpha(t-t_0)} \ . \tag{1.7}$$

The same solution can be obtained by a more standard integration method, denoted as separation of the variables: from $\frac{dx}{dt} = \alpha x$ we get $\frac{dx}{x} = \alpha dt$ and then, integrating both terms we get

$$\int_{x_0}^{x(t)} \frac{dx}{x} = \int_{t_0}^{t} \frac{dx}{x} \Longrightarrow \ln x(t) - \ln x_0 = \alpha(t - t_0) \Longrightarrow \ln \frac{x(t)}{x_0} = \alpha(t - t_0) \ ,$$

from which (1.7) is obtained by taking exponential of both members. Some graphical representations of (1.7), with different values of the parameter α and different initial conditions, are shown in Fig. 1.5 in the form of integral curves, with time t represented along the horizontal axis and the state variable along the vertical one. Among all the possible solutions there is also an equilibrium solution, corresponding to the case of vanishing time derivative $\dot{x} = 0$ (equilibrium condition). In fact, from (1.6) we can see that the equilibrium condition corresponds to the equation $\alpha x = 0$ which, for $\alpha \neq 0$, gives the unique solution $x^* = 0$. Indeed, the trajectory starting from the

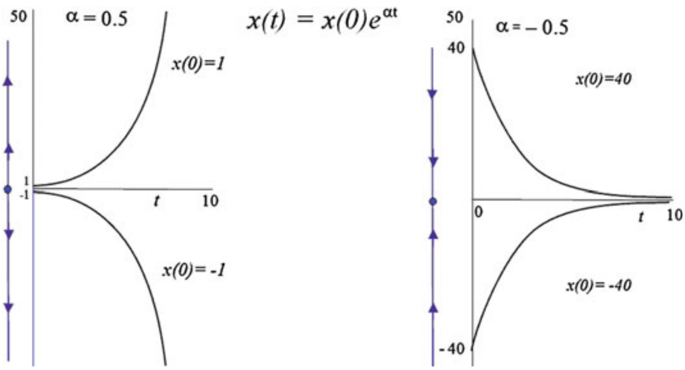

Fig. 1.5 Integral curves and phase portraits of $\dot{x} = \alpha x$

initial condition $x_0 = 0$ is given by $x(t) = 0$ for each t, i.e., starting from $x_0 = 0$ the system remains there forever. However, as shown in Fig. 1.5, different behaviors of the system can be observed if the initial condition is slightly shifted from the equilibrium point, according to the sign of the parameter α. In fact if $\alpha > 0$ (left panel) then the system amplifies this slight perturbation and exponentially departs from the equilibrium (unstable, or repelling, equilibrium) whereas if $\alpha < 0$ (right panel) then the system recovers from the perturbation going back to the equilibrium after a given return time (asymptotically stable, or attracting, equilibrium).

This qualitative analysis of existence and stability of the equilibrium can be obtained even without any computation of the explicit analytic solution (1.7), by solving the equilibrium equation $\alpha x = 0$ and by a simple algebraic study of the sign of the right hand side of the dynamic equation (1.6) around the equilibrium, as shown in Fig. 1.6. This method simply states that if the right hand side of the dynamic equation (hence $\dot{x}$) is positive then the state variable increases (arrow towards positive direction of the axis), if $\dot{x} < 0$ then x decreases (arrow towards negative direction).

This 1-dimensional representation (i.e., along the line) is the so called phase diagram of the dynamical system, where the invariant sets are represented (the equilibrium in this case) together with the arrows that denote tendencies associated with any point of the phase space (and consequently stability properties). Of course, the knowledge of the explicit analytic solution gives more information, for example the time required to move from one point to another. For example, in the case $\alpha < 0$, corresponding to stability of the equilibrium $x^* = 0$, we can state that after a

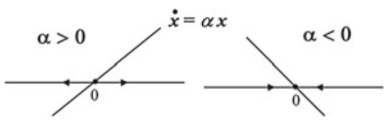

Fig. 1.6 Graphic of the the line $y = \alpha x$ and the corresponding one-dimensional phase diagram

displacement of the initial condition at distance $d = \|x_0 - x^*\|$ from the equilibrium, the time required to reduce such a perturbation at the fraction d/e (where e is the Neper constant $e \simeq 2.7$) is $T_r = -1/a$, an important stability indicator known as *return time*. As it can be seen, as the parameter α goes to 0 the return time tends to infinity. In fact, if $\alpha = 0$ all the points are equilibrium points, i.e., any initial condition generates a constant trajectory that remains in the same position forever.

As an example, let us consider the dynamic equation that describes the growth of a natural population. If $x(t)$ represents the number of individuals in a population (of insects, or bacteria, or fishes or humans), $n > 0$ represents the natality (or birth) rate and $m > 0$ represents the mortality (or death) rate then a basic balance equation used in any population model states that

$$\dot{x} = nx - mx = (n - m)\,x$$

which is of the form (1.6) with $\alpha = n - m$. Of course in this case, due to the meaning of the model, only non-negative values of the state variable x are admissible. The qualitative analysis of this model states that if natality is greater than mortality then the population exponentially increases, if the two rates are identical the population remains constant and if mortality exceeds natality the population exponentially goes to extinction. A quite reasonable result. We now introduce a modification in the simple population growth model by introducing a constant immigration (emigration) term $b > 0\ (< 0)$

$$\dot{x} = \alpha x + b\ . \tag{1.8}$$

Now the equilibrium condition $\dot{x} = 0$ becomes $\alpha x + b = 0$ from which the equilibrium is $x^* = -b/a$. If $\alpha < 0$ and $b > 0$ (endogenously decreasing population with constant immigration) then the equilibrium is positive and stable (as $\dot{x} < 0$ for $x > x^*$ and $\dot{x} > 0$ for $x < x^*$). Instead, for $\alpha > 0$ and $b < 0$ (endogenously increasing population with constant emigration) the equilibrium is positive and unstable. We conclude by noticing that the dynamic model (1.8) is called linear nonhomogeneous (or affine) and can be reduced into the form (1.6) by a change of variable (a translation). In fact, let us define the new dynamic variable $X = x - x^* = x + b/a$. This change of variable corresponds to a translation that brings the new zero coordinate into the equilibrium point. If we replace $x = X - b/a$ into (1.8) we get $\dot{X} = \alpha X$. Then we have the linear model (1.6) in the dynamic variable $X(t)$, with initial condition $X(t_0) = x_0 + b/a$, whose solution is $X(t) = X(t_0)\mathrm{e}^{\alpha(t-t_0)}$. Going back to the original variable, by using the transformation $X = x + b/a$, we obtain

$$x(t) = \left(x_0 + \frac{b}{\alpha}\right)\mathrm{e}^{\alpha(t-t_0)} - \frac{b}{\alpha}\ .$$

This is a first example of conjugate dynamical systems, as the models (1.6) and (1.8) can be transformed one into the other by an invertible change of coordinates. We will give a more formal definition of conjugate (or qualitative equivalent) dynamic models in the following chapters.

As an example, let us now consider a dynamic formalization of a partial market of a single commodity, under the Walrasian assumption that the price of the good increases (decreases) if the demand is higher (lower) than supply. The simplest dynamic equation to represent this assumption is given by

$$\dot{p} = f(p) = k\left[D(p) - S(p)\right] , \tag{1.9}$$

where $q = D(p)$ represents the demand function, i.e., the quantity demanded by consumers when the price of the good considered is p, $q = S(p)$ represents the supply function, i.e., the quantity of the good that producers send to the market when the price is p, $k > 0$ is a constant that gives the speed by which the price reacts to a disequilibrium between supply and demand. The standard occurrence is that supply function $S(p)$ is increasing and demand function is decreasing, as shown in Fig. 1.7. The equilibrium point p^* is located at the intersection of demand and supply curves, and it is stable because the derivative of p is positive on the left and negative on the right of p^*, so that p^* is always reached in the long run even if the initial price $p(0)$ is not an equilibrium one (or equivalently if the price has been displaced from the equilibrium price). An analytic solution of the dynamic equations can be obtained under the assumption that demand and supply functions are linear

$$D(p) = a - bp\ , \quad S(p) = a_1 + b_1 p\ ,$$

where all the parameters a, b, a_1, b_1 are positive. In fact, in this case the dynamic equation is a linear differential equation with constant coefficients

$$\dot{p} = -k\,(b + b_1)\,p + k\,(a - a_1)$$

which is in the form (1.8) and has equilibrium point $p^* = (a - a_1)/(b + b_1)$. As we will see in the next sections, a similar analysis, based on the linearization of the model around the equilibrium point, is possible by computing the slopes of the functions (i.e., their derivatives) at the equilibrium point.

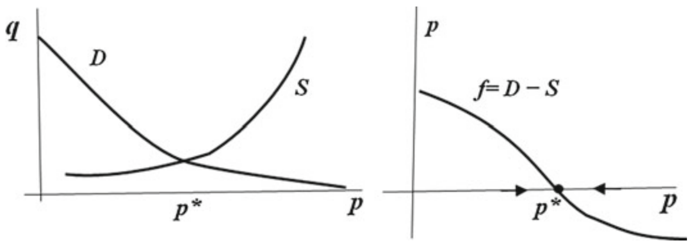

Fig. 1.7 Qualitative graphical analysis of price dynamics with standard demand and supply functions

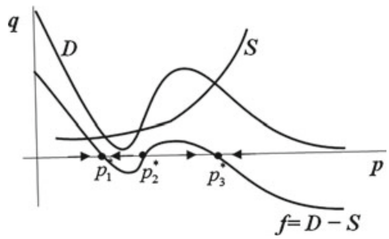

Fig. 1.8 Qualitative analysis of (1.9) with bimodal demand function

Let us now consider a different demand curve, obtained by assuming that consumers exhibit a nonstandard behavior for intermediate prices. In the situation shown in Fig. 1.8, even if demanded quantity is high for low prices and low for high prices, like in the standard case, we assume that for intermediate prices consumers prefer to buy the good at higher price because they use price as a quality indicator. Such assumption leads to a"bimodal" shape of the demand function (i.e., with two inversion points, a relative minimum and relative maximum) that may intersect the supply curve in three points, like in Fig. 1.8, and consequently three coexisting equilibrium prices, say $p_1^* < p_2^* < p_3^*$. By using the qualitative analysis, we can see that the time derivative of the price $p(t)$ is positive whenever $p < p_1^*$ or $p_2^* < p < p_3^*$, i.e., where $D(p) > S(p)$. This leads to a situation of *bistability* as both the lowest equilibrium price p_1^* and the highest one p_3^* are asymptotically stable, each with its own basin of attraction, whereas the intermediate unstable equilibrium price p_2^* separates the basins, i.e., it acts as a watershed located on the boundary between the two basins.

1.2.1.2 Qualitative Analysis and Linearization Procedure for the Logistic Model

The population model described in the Sect. 1.2.1.1 is quite unrealistic as it admits unbounded population growth, which is impossible in a finite world. As already noticed by Malthus [27], when the population density becomes too high, scarcity of food or space (overcrowding effect) causes mortality, proportional to the population density. So an extra mortality term, say sx, should be added to the natural mortality m, and thus the model becomes

$$\dot{x} = f(x) = nx - (m + sx)x = \alpha x - sx^2 \tag{1.10}$$

which is a nonlinear dynamic model. Also in this case, after separation of the variables, an analytic solution can be found by integrating a rational function. In fact, after some algebraic transformations of the rational function the following solution is obtained

$$x(t) = \frac{\alpha x_0 e^{\alpha t}}{\alpha + sx_0\,(e^{\alpha t} - 1)}\ , \tag{1.11}$$

whose graph (for different initial conditions) is shown in Fig. 1.9.

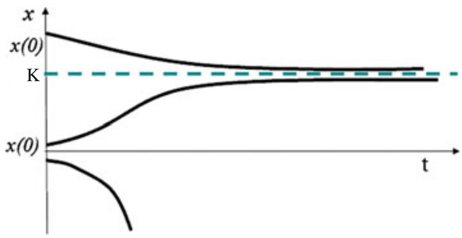

Fig. 1.9 Graph of function (1.11) with three different initial conditions $x(0)$

As it can be seen from the graph of $x(t)$ in (1.11), all solutions starting from a positive initial condition asymptotically converge to the attracting equilibrium $K = \alpha/s$ (usually called carrying capacity in the language of ecology) represented by the horizontal asymptote. Another equilibrium point exists, given by the extinction equilibrium $Q = 0$, which is repelling.

However, the possibility to find an analytic solution by integrating a nonlinear differential equation is a rare event, so we now try to infer the same conclusions without finding the explicit solution, i.e., by using qualitative methods. As usual, the first step is the localization of the equilibrium points, solutions of the equilibrium condition $\dot{x} = 0$, i.e., $f(x) = x(\alpha - sx) = 0$, from which the two solutions $x_0^* = 0$ and $x_1^* = \alpha/s$ are easily computed. In order to determine their local stability properties, it is sufficient to notice that the graph of the right hand side of (1.10), see Fig. 1.10, has negative slope around the equilibrium x_1^*, so that $\dot{x}$ is positive on the left and negative on the right, and vice versa at the equilibrium x_0^*, as indicated by the arrows along the x axis (the 1-dimensional state space of the system). This can be analytically determined even without the knowledge of the whole graph of the function, as it is sufficient to compute the sign of the x-derivative of the right hand side at each equilibrium point. In fact, it is well known that the derivative computed in a given point of the graph represents the slope of the graph (i.e., of the line tangent to the graph) at that point. So, the local behavior of the dynamical system in a neighborhood of an equilibrium point, hence its local stability as well, is generally the same as the one of the linear approximation (i.e., the tangent). This rough argument will be explained more formally in the next sections. In the particular case of the logistic model (1.10) the derivative is $\frac{df}{dx} = f'(x) = \alpha - 2sx$, and computed at the two equilibrium points becomes $f'(0) = \alpha > 0, f'(\alpha/s) = -\alpha < 0$, hence $Q = 0$ is

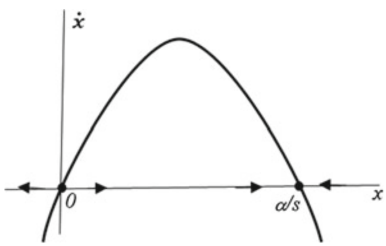

Fig. 1.10 Qualitative dynamic analysis of logistic equation (1.10)

unstable, $K = \alpha/s$ is stable. Moreover the parameter α can be seen as an indicator of how fast the system will go back to the stable equilibrium after a small displacement, as the *return time* for the linear approximation is $T_r = 1/\alpha$.

Before ending this part, we notice that the equilibrium points $x_0^* = 0$ and $x_1^* = \alpha/s$ are two (constant) solutions of (1.10), whose graphs in the plane (t, x) are horizontal lines. Thus, by the theorem of existence and uniqueness of a solution stated above, any other (nonconstant) solution $x(t)$ of (1.10) cannot cross these two horizontal lines. From (1.10) by a simple second-degree inequality, it is easy to see that $\dot{x} > 0$ occurs whenever $x \in (0, \alpha/s)$. Moreover, being $\frac{d^2x}{dt^2} = \frac{d\dot{x}}{dt} = \alpha\dot{x} - 2sx\dot{x} = \dot{x}(\alpha - 2sx)$, we deduce that $x(t)$ is strictly decreasing and concave whenever $x(0) \in (-\infty, 0)$ and that $x(t)$ is strictly decreasing and convex whenever $x(0) \in (\alpha/s, +\infty)$. Finally, when $x(0) \in (0, \alpha/s)$, $x(t)$ is strictly increasing and from convex becomes concave when $x(t) = \alpha/(2s)$, see again Fig. 1.9.

1.2.1.3 Qualitative Analysis of One-Dimensional Nonlinear Models in Continuous Time

The qualitative method used to understand the dynamic properties of the logistic equation can be generalized to any one-dimensional dynamic equation in continuous time

$$\dot{x} = f(x) \tag{1.12}$$

It consists, first of all, in the localization of the equilibrium points according to the equilibrium condition $\dot{x} = 0$, i.e., the solutions of the equation $f(x) = 0$. As a consequence of the Theorem of uniqueness, oscillations are not possible for a 1-dimensional dynamical system in continuous time, hence for a system starting from any initial condition which is not an equilibrium, only increasing or decreasing solutions can be obtained. Hence just four different phase portraits characterize the dynamic behavior of the 1-dimensional system around an equilibrium, as shown in Fig. 1.11.

Of course, if an initial condition coincides with an equilibrium point, i.e., $x(0) = x^*$ and $f(x^*) = 0$, then the unique solution is $x(t) = x^*$ for $t \geq 0$. In other words, starting from an equilibrium point, the system remains there forever. The natural question arising is what happens if the initial condition is taken close to an equilibrium point, i.e., if the system is slightly perturbed from the equilibrium considered. Will the distance from the equilibrium increase or will the perturbation be reduced so that the system spontaneously goes back to the originary equilibrium? An answer to this question is easy in the case of hyperbolic equilibria, defined as equilibrium points with nonvanishing derivative, i.e., $f'(x^*) \neq 0$. In fact, if x^* is one of such solutions

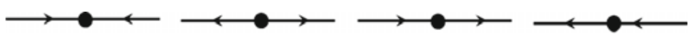

Fig. 1.11 The four different phase diagrams around an equilibrium point