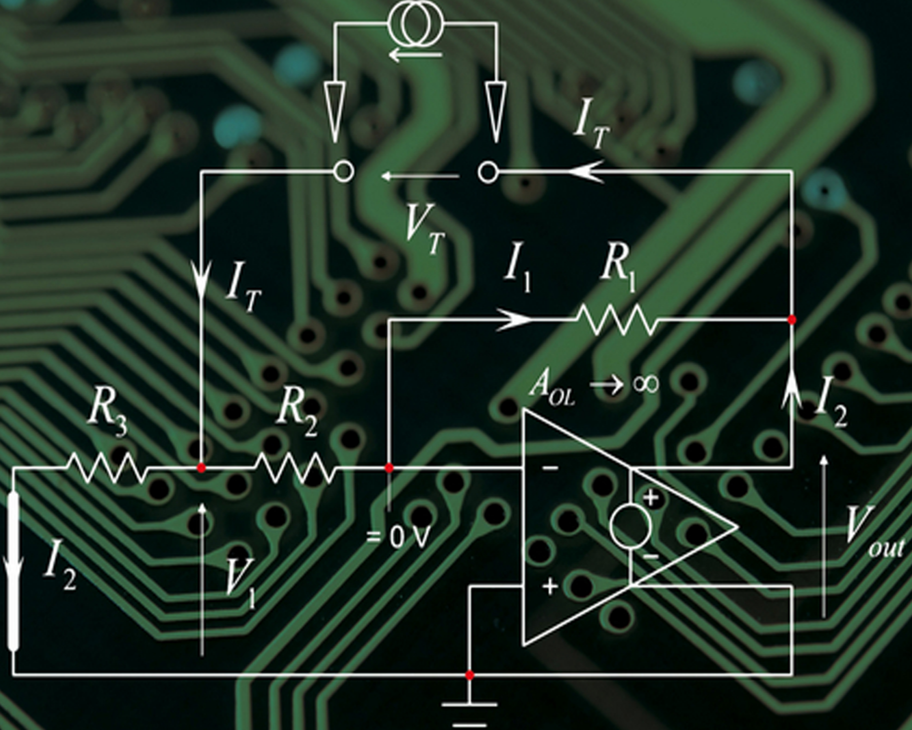
LINEAR CIRCUIT TRANSFER FUNCTIONS

AN INTRODUCTION TO FAST ANALYTICAL TECHNIQUES



CHRISTOPHE P. BASSO

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AN INTRODUCTION TO FAST ANALYTICAL TECHNIQUES

Christophe P. Basso
ON Semiconductor, Toulouse, France

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About the Author

Christophe Basso is a Technical Fellow at ON Semiconductor in Toulouse, France, where he leads an application team dedicated to developing new offline PWM controller's specifications. He has originated numerous integrated circuits among which the NCP120X series has set new standards for low standby power converters.

Further to his 2008 book *Switch-Mode Power Supplies: SPICE Simulations and Practical Designs*, published by McGraw-Hill, he released a new title in 2012 with Artech House, *Designing Control Loops for Linear and Switching Power Supplies: a Tutorial Guide*. He holds 17 patents on power conversion and often publishes papers in conferences and trade magazines including How2Power and PET.

Christophe has over 20 years of power supply industry experience. Prior to joining ON Semiconductor in 1999, Christophe was an application engineer at Motorola Semiconductor in Toulouse. Before 1997, he worked at the European Synchrotron Radiation Facility in Grenoble, France, for 10 years. He holds a BSEE equivalent from the Montpellier University (France) and a MSEE from the Institut National Polytechnique of Toulouse (France). He is an IEEE Senior member.

When he is not writing, Christophe enjoys snowshoeing in the Pyrenees.

Preface

First as a student and later as an engineer, I have always been involved in the calculation of transfer functions. When designing power electronics circuits and switch mode power supplies, I had to apply my analytical skills on passive filters. I also had to linearize active networks when I needed the control-to-output dynamic response of my converter. Methods to determine transfer functions abounded and there are numerous textbooks on the subject. I started in college with mesh-node analysis, and at some point ended up using state variables. If all paths led to the correct result, I often struggled rearranging equations to make them fit a friendly format. Matrices were useful for immediate numerical results but, when trying to extract a meaningful symbolic transfer function, I was often stuck with an intractable result. What matters with a transfer function formula is that you can immediately distinguish poles, zeros and gains without having to rework the expression. This is the idea behind the term *low-entropy*, a concept forged by Dr. Middlebrook.

Simulation gives you an idea where poles and zeros hide by interpreting the phase and magnitude plots with minimum-phase functions. However, inferring which terms really affect a pole or a zero position from a Bode plot is a different story. Fortunately, if the transfer function is written the right way, then you can immediately identify which elements contribute to the roots and assess how they impact the dynamic response. As some of these parasitics vary in production or drift with temperature, you have to counteract their effects so that reliability is preserved during the circuit's life. The typical example is when you are asked to assess the impact of a parasitic term variation on a product you have designed: if a new capacitor or a less expensive inductor is selected by the buyers, will production be affected? Is there a chance that stability will be jeopardized in some operating conditions? Implementing the classical analysis method will surely deliver a result describing the considered circuit, but extracting the information you need from the final expression is unlikely to happen if the equations you have are disorganized or in a *high-entropy* form.

This is where Fast Analytical Circuit Techniques (FACTs) come into play. The acronym was formed by Dr. Vatché Vorpérian, who formalized the technique you are about to discover here. Before him, Dr. Middlebrook published numerous papers and lectured on his Extra-Element Theorem (EET), later generalized to the *N* extra-element theorem by one of his alumni. Since Hendrik Bode in the 40's, authors have come up with techniques aiming to simplify linear circuit analysis through various approaches. All of them were geared towards determining the transfer function at a pace quicker than what traditional methods could provide. Unfortunately, while traveling and visiting customers world-wide, I have found that, despite all the available documentation, FACTs were rarely adopted by engineers or students. When describing examples in my seminars and showing the method at work in small-signal analysis, I could sense interest from the audience through questions and comments. However, during the discussions I had later on with some of the engineers or students, they confessed that they tried to acquire the skill but gave up because of the intimidating mathematical formalism and the complexity of the examples. If one needs to be rigorous when tackling electrical analysis, perhaps a different approach and pace could make people feel at ease when learning the method. This is what I strived to do with this new book, modestly shedding a different light on the subject by progressing with simple-to-understand examples and clear explanations. As a student, I too struggled to apply these fast analytical circuits techniques to real-world problems; as such, I identified the obstacles and worked around them with success. Thus, the seeds for this book were sown.

This book consists of five chapters. The first chapter is a general introduction to the technique, explaining what transfer functions are and how time constants characterize a circuit. The second chapter digs into transfer function definitions and polynomial forms, introducing the low-Q approximation, and how to organize 2nd and 3rd-order denominators or numerators. The third chapter uses the superposition theorem to gently introduce the extra-element theorem. Numerous examples are given to illustrate its usage in different 1st-order configurations. The fourth chapter deals with the 2-extra element theorem, generalized and applied to 2nd-order networks. Numerous examples illustrated with Mathcad® and SPICE punctuate the explanations. Finally, the fifth chapter tackles 3rd- and 4th-order circuits, all illustrated with examples. Each chapter ends with 10 fully documented problems. There is no secret; mastering a technique requires patience and practice, and I encourage you to test what you have learned after each chapter through these problems.

I have adopted the same casual writing style already used in my previous books, as readers' comments show that the way I present things better explains complex matters. Please let me know if my approach still applies here and if you enjoy reading this new book. As usual, feel free to send me your comments or any typos you may find at cbasso@wanadoo.fr. I will maintain an errata list in my personal webpage as I did for the previous books (http://cbasso.pagesperso-orange.fr/Spice.htm). Thank you, and have fun determining transfer functions!

Christophe Basso
May 2015

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A book like this one could not have been written and published without the help of many contributing friends. My warmest thanks and love first go to my sweet wife Anne who endured my ups and downs when determining some of the book transfer functions: equations time is over and we can now enjoy the long and warm evenings of summer to come!

I was fortunate to share my work with my ON Semiconductor colleagues and friends who played a crucial role in reviewing my pages and challenging the method. Stéphanie Cannenterre reviewed and practiced numerous book exercises. She now masters the method: well done! Dr. José Capilla raced with me several times to determine a transfer function with his Driving Point Impedance method and I recognize his skills in doing so. Special thanks go to my friend Joël Turchi with whom I spent endless hours debating the method or discussing the validity of an equation. Merci Joël for your kindness and invaluable support for this book!

Two people did also accompany me from the beginning of the writing process. Mon ami Canadien Alain Laprade from ON Semiconductor in East Greenwich who developed an addicted relationship to the FACTs and kindly reviewed all my work. Monsieur Feucht from Innovatia did also a tremendous work in correcting my pages but also kindly polished my English. I am not exactly a novelist and cannot hide my French origins Dennis!

I want to warmly thank the following reviewers for their kind help in reading my pages during the 2015 summer: Frank Wiedmann (Rhode & Schwarz), Thierry Bordignon, Doug Osterhout (both are with ON Semiconductor), Tomas Gubek – děkuji! (FEI), Didier Balocco (Fairchild), Jochen Verbrugghe, Bart Moeneclaey (both are with Ghent University), Bruno Allard (INSA Lyon), Vatché Vorpérian (JPL), Luc Lasne (Bordeaux University) and Garrett Neaves (Freescale Semiconductor).

Last but not least, I would like to thank Peter Mitchell at Wiley & Sons UK for giving me the opportunity to publish my work.

1

Electrical Analysis – Terminology and Theorems

This first chapter is an introduction to some of the basic definitions and terms you must understand in order to perform electrical analysis with efficiency and speed. By electrical analysis, I imply finding the various relationships that characterize a particular electrical network. To excel in this field, as in any job, you need to master a few tools. Obviously, they are innumerable and I am sure you have learned a plethora of theorems during your student life. Some names now seem distant simply because you never had a chance to exercise them. Or you actually did but implementation was so obscure and complex that you left quite a few of them aside. This situation often happens in an engineer's life where real-case experience helps clean up what you have learned at school to only retain techniques that worked well for you. Sometimes, when what you know fails to deliver the result, it is a good opportunity to learn a new procedure, better suited to solve your current case. In this chapter, I will review some of the founding theorems that I extensively use in the examples throughout this book. However, before tackling definitions and examples, let us first understand what the term *transfer function* designates.

1.1 Transfer Functions, an Informal Approach

Assume you are in the laboratory testing a circuit encapsulated in a box featuring two connectors: one for the input, the second for the output. You do not know what is inside the box, despite the transparent case in the picture! You now inject a signal with a function generator to the input connector and observe the output waveform with an oscilloscope. Using the right terminology, you *drive* the circuit input and observe its *response* to the stimulus. The input waveform represents the *excitation* denoted u and it generates a *response* denoted y. In other words, the excitation variable propagates through the box, undergoes changes in phase, amplitude, perhaps induces distortion etc. and the oscilloscope reproduces the response on its screen.

The waveform displayed by the oscilloscope is a *time-domain* graph in which the horizontal axis x is graduated in seconds while the vertical axis y indicates the signal *amplitude* (positive or negative). Its dimension depends on the observed variable (volts, amperes and so on). The input waveform is denoted in lower case as it is an *instantaneous* signal, observed at a time – the *instant* t – $u(t)$. A similar notation applies to the output signal, $y(t)$. In Figure 1.1, you see a low duty ratio square-wave injected in the box engendering a rather distorted waveform on the output.

Linear Circuit Transfer Functions: An Introduction to Fast Analytical Techniques, First Edition. Christophe P. Basso.

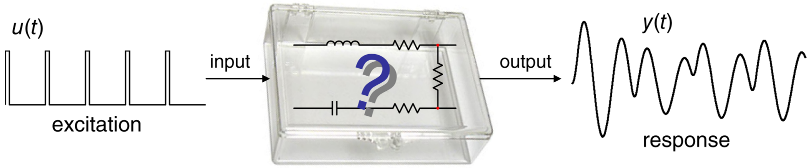


Figure 1.1 A black box featuring an input and an output signal. What is the relationship linking output and input waveforms?

This ringing signal tells us that the box could associate resonant elements, probably capacitors and inductors but not much more than that. If we change the excitation, what type of shape will we obtain? Knowing what is inside the box will let us predict its response to various types of excitation signals.

There are several available ways to characterize an electrical linear circuit. One of them is called *harmonic analysis*. The input signal is replaced by a sinusoidal waveform and you observe how the stimulus propagates through the box to form the response. This is shown in Figure 1.2:

The excitation level must be of reasonable amplitude – understand *small* – so that the response signal is not distorted. The input signal dc bias must also be set accounting for the physical constraints imposed by the active circuit so that upper- or lower-rail saturation is avoided. In other words, the box internal circuitry is not *overdriven* and remains *linear* during the analysis. Linearity is confirmed if the output signal is sinusoidal with the same frequency as the input sine and only varies in amplitude and phase while you ac-sweep the network. This is a so-called *small-signal* analysis. In the Laplace domain, you perform such harmonic analysis when you set $s = j\omega$ in which $\omega = 2\pi f$ represents the angular frequency expressed in radians per seconds (rads/s). Laplace analysis with $s = j\omega$ applies to linear circuits only.

Should you increase the input signal amplitude or change the operating bias point, slewing or clipping may happen. In this case, you explore the box *large-signal* or *nonlinear* response. This is a characterization different than the small-signal approach and it offers another insight into the circuit operation. Let us keep linear and once the right input amplitude is found, i.e. a signal of comfortable amplitude is observed on the oscilloscope screen, the frequency is varied step by step while output amplitude/phase couples are recorded in an array. At each frequency point f, we store the ratio of the response amplitude $Y(f)$ in volts to the excitation amplitude $U(f)$ in volts also. At each frequency point f, we save the phase information linking both input and output waveforms. As U and Y are complex variables affected by a magnitude and a phase, we can write:

$$A_v(s) = \frac{Y(s)}{U(s)} \tag{1.1}$$

A_v represents a *transfer function*, a mathematical relationship linking a response signal Y to an excitation signal U. Please note that the excitation signal U resides in the transfer function

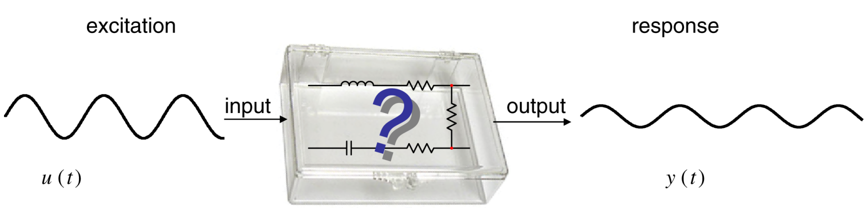


Figure 1.2 The black box is now driven by a sinusoidal stimulus for a small-signal analysis.

denominator while the response Y sits in the numerator. It will always be this way throughout the book.

The transfer function is a complex variable characterized by a magnitude noted $|A_v(f)|$ and an argument, $\angle A_v(f)$ also noted $\arg A_v(f)$. The ratios $Y(f)/U(f)$ we have stored correspond to the transfer function magnitude (also called *modulus*) observed at a frequency f while the phase difference between Y and U represents the transfer function argument or phase at the considered frequency. The transfer function magnitude dimension depends on the observed variables as we will later see. Here, because volts are involved for both variables, the transfer function magnitude is *dimensionless* or *unitless*. Furthermore, $|A_v|$ can only be greater than or equal to zero. It is what makes the difference between an amplitude which can take on any value, positive, null or negative and a magnitude which can only be zero or positive. If it is 0, there is no output signal. If $|A_v|$ is less than 1, we talk about *attenuation*. Now, if $|A_v|$ is greater than 1, it is designated as a *gain*. If the magnitude can only be a null or positive number, what about a gain of −2 then? It simply characterizes a stage offering a gain of 2, lagging or leading the excitation signal phase by 180°.

1.1.1 Input and Output Ports

It is convenient to represent our box as a two-port circuit. A *port* is a pair of connections that can input or output signals such as voltage and current. Figure 1.3 shows an illustration of this principle where you see two connecting ports, one input and one output.

Under some conditions, a port can take on the input and output roles at the same time. Imagine you want to measure the output impedance of the box. To realize this measurement, you classically implement Figure 1.4 where a current across the output terminals is injected while the voltage across the same terminals is observed This is what is called a *single injection*, i.e. one stimulus and one response. In this experiment, the box input port is shorted (see Appendix 1A). The excitation variable is the current $I_{out}(s)$ injected into the port while the response is the voltage $V_{out}(s)$ collected across the port's terminals. The output impedance Z obtained from the ratio of the port voltage to the injected

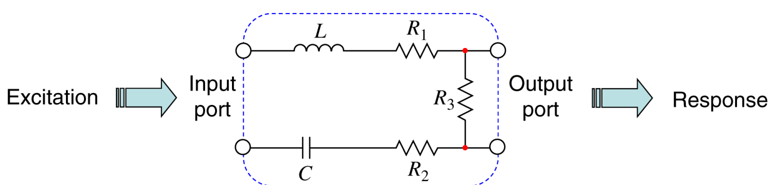


Figure 1.3 The input port receives the excitation signal while the output port delivers the response.

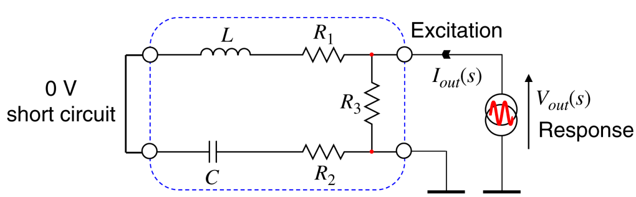


Figure 1.4 A port can be both an input and an output at the same time. Here, an output impedance measurement.

current is a transfer function. It has the dimension of an impedance expressed in ohms:

$$Z_{out}(s) = \frac{V_{out}(s)}{I_{out}(s)} \tag{1.2}$$

I_{out}, the excitation signal lies in the denominator while the response, V_{out}, stands in the numerator. We will come back on this important peculiarity.

If input and output connectors are fixed, physical ports, which let you respectively inject and observe signals, nothing prevents one from creating other observation ports as needed. Simply remove a resistor, a capacitor or an inductor and its connecting points become a new port. This port can now be used as a new input stimulus or as an output variable you want to observe. As already mentioned, this newly created port can also play the role of an input and output port at the same time. In that case, the box originally featuring one input and one output, becomes a two-input/two-output system as illustrated in Figure 1.5 in which the inductor has been removed. Using adequate terminology, we analyze the system by performing a *double-injection*: two stimuli – inputs 1 and 2 – giving two responses, outputs 1 and 2.

In this example, the voltage across the removed inductor terminals is the response while the injected current is the excitation signal. By dividing the port voltage by the injected current, we have the resistance offered by the port terminals when the element initially connected has been removed. In other words, we 'look' at the resistance offered by the inductor port as shown in Figure 1.6 where the symbol R? and the arrow imply this exercise. Expressed in a different manner, we find the equivalent *output resistance* exhibited by the port when 'driving' the inductor, hence the name *driving point* resistance or *driving point impedance* abbreviated as DPI. Combining resistance and inductance gives us a time constant τ ('tau') associated with this inductive element:

$$\tau = \frac{L}{R} \tag{1.3}$$

To conduct this exercise and find the resistance R, you can directly look at the sketch and infer the resistive series-parallel arrangement without solving a single equation. This exercise is called *network inspection*: you simply observe the network in certain conditions (for instance in dc, or when V_{in} is set to 0) and find resistance values by observing how components are connected together. For example, in Figure 1.6, what resistance do you 'see' looking into the inductor port while capacitor C is

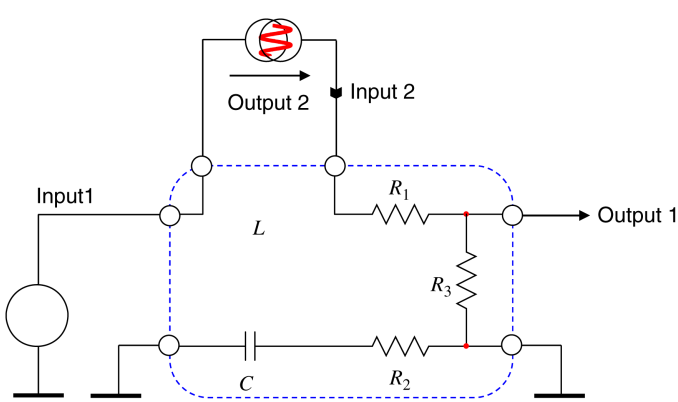


Figure 1.5 If you remove a component from this circuit, its connections become a connecting port. You can bias this port and consider it as a new input, or as a new output, or both of them at the same time.

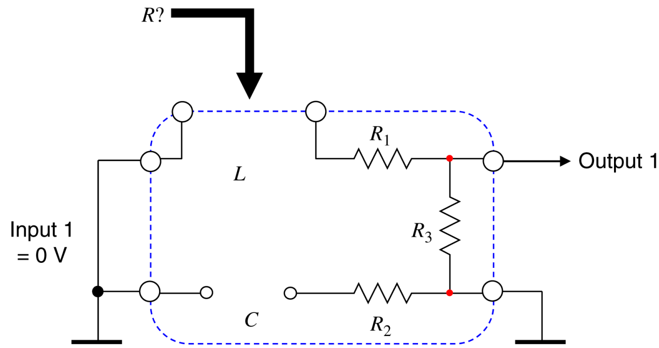


Figure 1.6 Removing the inductor lets you look at the port output resistance that drives the inductor. Associating the port resistance and the inductance leads to a time constant. Here, the resistance seen at the inductor port is $R_1 + R_3$.

disconnected for the exercise? R_1 appears first and then R_3 in series goes to ground and returns to the inductor left terminal via the shorted input source. R_2 is open and plays no role:

$$R = R_1 + R_3 \tag{1.4}$$

Applying (1.3) with (1.4) gives the definition for the time constant involving L:

$$\tau_1 = \frac{L}{R_1 + R_3} \tag{1.5}$$

A similar exercise can be conducted with the capacitor to also unveil the resistance R that drives this element. In this case, the time constant associated with the capacitance is simply:

$$\tau = RC \tag{1.6}$$

Assuming a shorted inductance in this particular illustration, what resistance value do you see in Figure 1.7 when looking into the capacitor port? The left terminal is grounded while the second

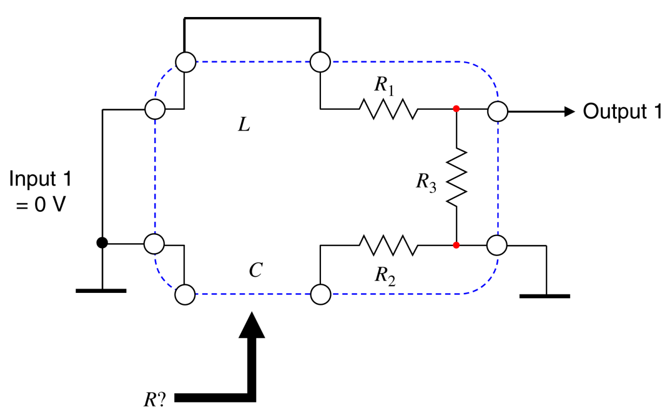


Figure 1.7 Removing the capacitor lets you conduct a similar exercise to unveil the time constant associated with this component. In this case, the resistance seen at the capacitor port is R_2.

terminal also goes to ground via R_2. R_1 and R_3 play no role since their series combination goes from one ground to the other one. Therefore:

$$R = R_2 \tag{1.7}$$

The time constant involving the capacitor is simply:

$$\tau_2 = R_2 C \tag{1.8}$$

We have two storage elements, C and L, and there are two time constants. For each storage element, there is an associated time constant.

Rather than looking into a capacitive or an inductive port, we could also remove a resistor and define what resistance drives it, the exercise remains the same. Sometimes, looking into the port to 'see' the resistance is not as straightforward, especially when controlled sources are involved. In this case, you need to add a test current generator as in Figure 1.5 and define the voltage generated across the considered terminals. The resistance offered by the port being the port voltage divided by the test current generator. This test generator will later be labeled I_T and the voltage across its terminals V_T.

What we just described is part of the technique foundations we will later describe: find resistances offered across the connecting terminals of resistive, capacitive or inductive elements once they have been temporarily removed from the circuit under certain conditions. Breaking a complex passive or active circuit into a succession of *simple* configurations where time constants are unveiled will help us characterize a network featuring poles and zeros. The Extra Element Theorem (EET) and later, the *n* Extra Element Theorem (*n*EET), make an extensive usage of these methods and it is important to understand this prerequisite. Appendix 1A will refresh our memory regarding available methods to derive output impedances while Appendix 1B collects several examples to let you exercise your skills at finding these resistances.

1.1.2 Different Types of Transfer Function

Depending where you inject the excitation and where you observe the response, you can define six types of transfer functions as detailed in [1]. For the sake of simplicity, input and output ports are ground-referenced but could also be differential types. The first one, is the voltage gain A_v already encountered in the above lines and it appears in Figure 1.8 together with an operational amplifier (op amp) in an inverting configuration. In all the following illustrations, the op amp is considered a perfect element (infinite open-loop gain, infinite bandwidth, zero output and infinite input impedances). You sweep the input voltage with a sinusoid, the stimulus, and observe the voltage at the op amp output, the response. In Laplace notation, you compute A_v as:

$$A_v(s) = \frac{V_{out}(s)}{V_{in}(s)} \tag{1.9}$$

A_v is dimensionless, sometimes expressed in [V]/[V].

The second one is the current gain, A_i, this time involving input and output currents as shown in Figure 1.9. The excitation signal is now the input current I_{in} while the observed variable is the output current I_{out}:

$$A_i(s) = \frac{I_{out}(s)}{I_{in}(s)} \tag{1.10}$$

A_i is dimensionless, sometimes expressed in [A]/[A].

The third transfer function is called a *transadmittance* – short name for transfer admittance – and is denoted Y_t. You observe the output current while the input is excited by a voltage source.

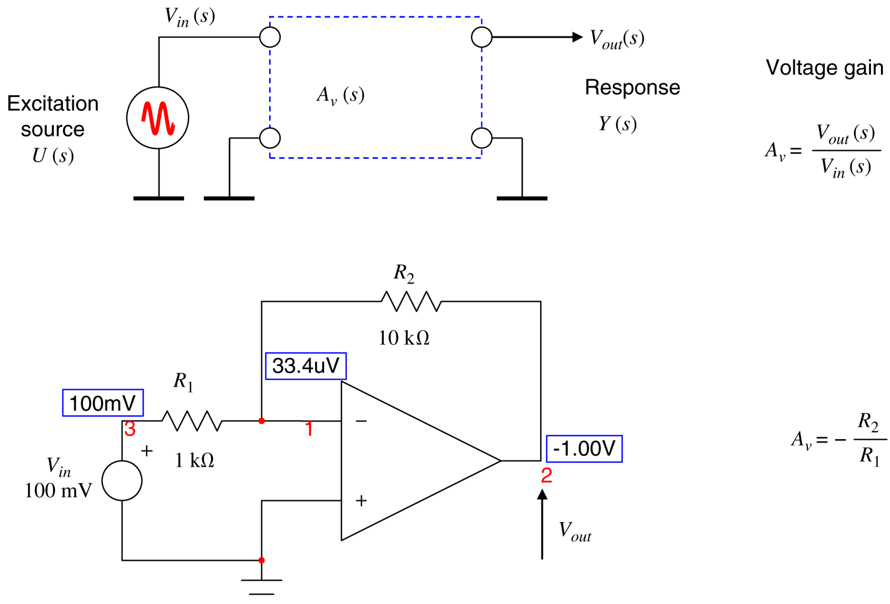


Figure 1.8 The voltage gain A_v is the first transfer function and links the output voltage to the input voltage.

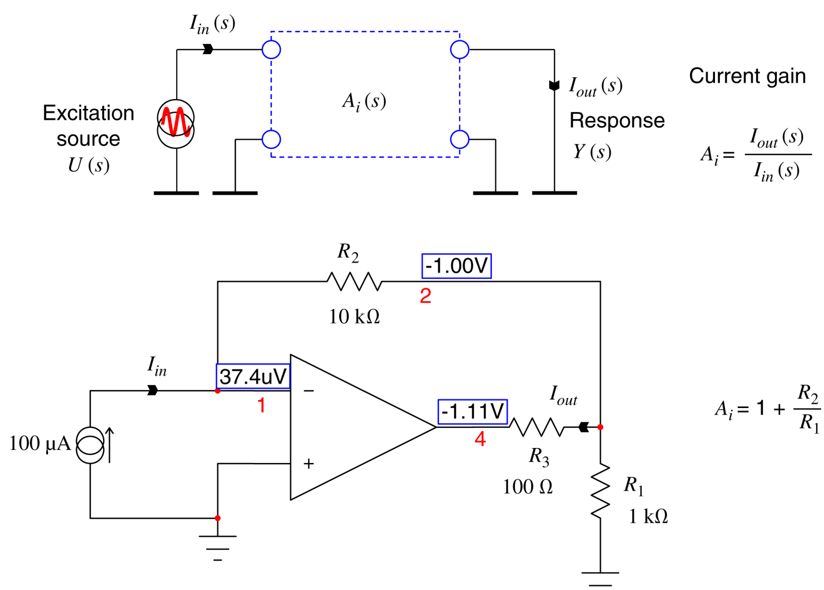


Figure 1.9 The current gain A_i is the second transfer function and links the output current to the input current.

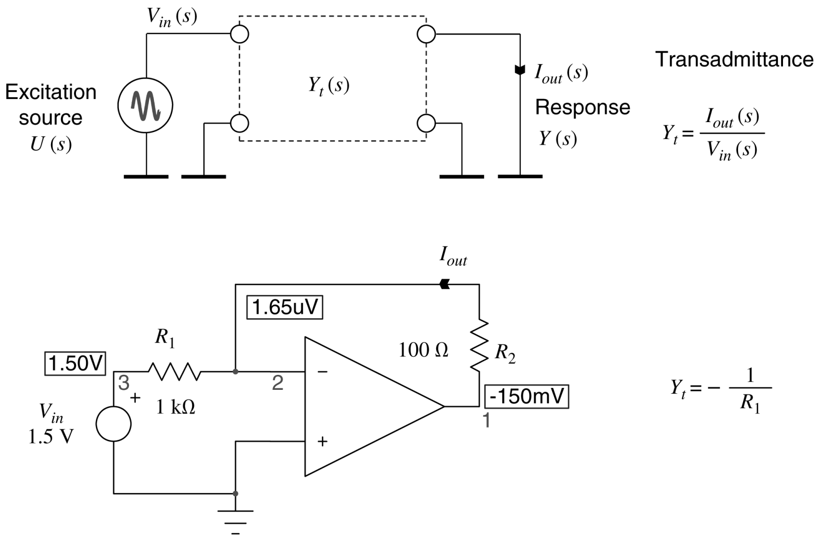


Figure 1.10 The transadmittance Y_t links the output current to the input voltage. Here the current in R_2 is imposed by V_{in} and reaches 1.5 mA. The transadmittance gain is −0.001 A/V or −1 mS.

The measurement configuration is shown in Figure 1.10. The definition is as follows:

$$Y_t(s) = \frac{I_{out}(s)}{V_{in}(s)} \tag{1.11}$$

If the two preceding gains were dimensionless, the transadmittance is expressed in ampere per volt, [A]/[V] or siemens [S]. Similarly, we can define the fourth transfer function in which, this time, the input is excited by a current source while the output voltage is the response (Figure 1.11). The ratio of these two variables is designated as a *transimpedance* – short name for transfer impedance – denoted Z_t and expressed in volt per ampere, [V]/[A] or ohm [Ω]:

$$Z_t(s) = \frac{V_{out}(s)}{I_{in}(s)} \tag{1.12}$$

Transimpedance amplifiers are often used in case you want to amplify a photodiode current for instance. You will find in [2] a design example of such a circuit.

In the four previous transfer functions, the involved quantities – excitation and response signals – appear at two different places in the network. We conveniently considered the box input and output terminals for the examples, but definitions apply equally for relationships between any ports in the network. For the two remaining transfer functions, impedance Z and admittance Y, excitation and response signals are observed at the same port terminals. It is therefore important to distinguish how we create the excitation signal and what is considered the response signal. You can argue that it is not a problem to reverse excitation and response because impedance and admittances are reciprocal to each other. However, if we want to stick to our transfer function definition in which the excitation waveform lies in the denominator while the response appears in the numerator, then, for a *driving point impedance* (DPI) function $Z_{dp}(s)$, the excitation signal is a current source and for a driving point admittance function $Y_{dp}(s)$, the stimulus is a voltage source.

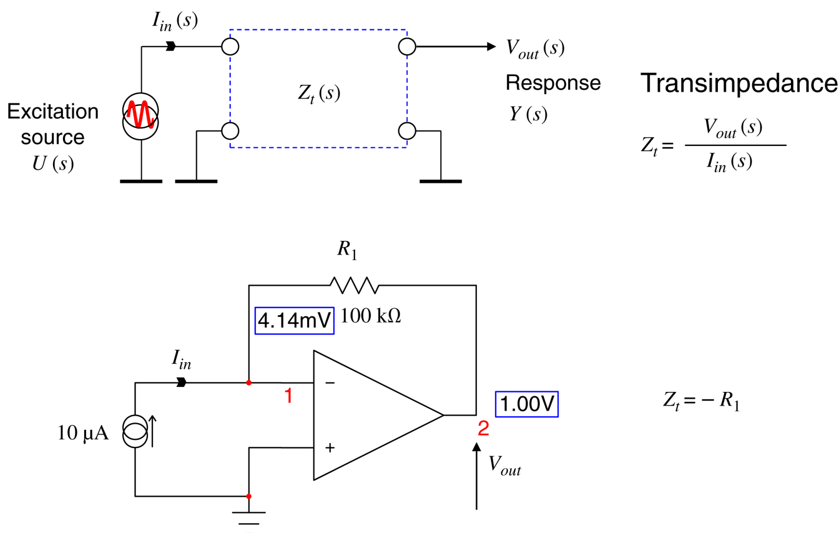


Figure 1.11 The transimpedance Z_t links the output voltage to the input current. In the op amp example, resistor R_1 brings a transimpedance gain of $-100\,\text{kV/A}$.

The 5th transfer function is thus the port input impedance $Z(s)$ whose generalized transfer function is given below:

$$Z_{dp}(s) = \frac{V_1(s)}{I_1(s)} \tag{1.13}$$

If you consider V_{in} and I_{in} or V_{out} and I_{out}, you respectively measure the network input and output impedances by injecting a test current in the port and measuring the voltage across the port terminals. Figure 1.12 shows sources arrangement for this specific measurement. The dimension of an impedance is ohm, [Ω].

Finally, the 6th transfer function is the *admittance*, the inverse of an impedance. You measure an admittance by exciting the concerned port with a voltage source which produces a current, the response (Figure 1.13). The generalized transfer function of an admittance is:

$$Y_{dp}(s) = \frac{I_1(s)}{V_1(s)} \tag{1.14}$$

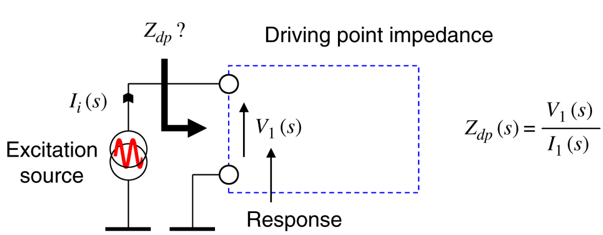


Figure 1.12 Impedances have the dimension of ohms. The excitation signal is a current.

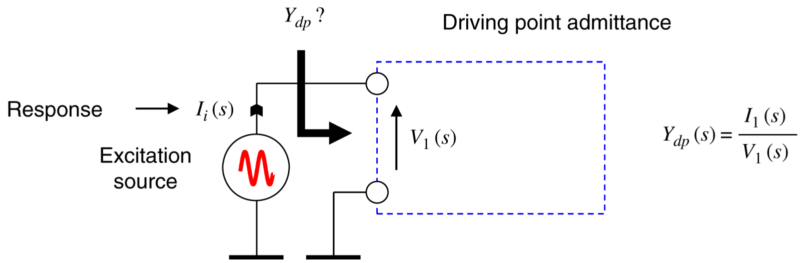
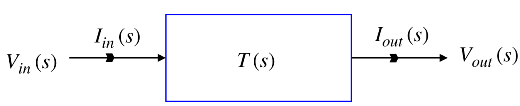


Figure 1.13 Admittances have the dimension of Siemens. The excitation signal is a voltage.

If you consider I_{in} and V_{in} or I_{out} and V_{out}, you respectively measure the network input and output admittances.

Admittances are expressed in siemens, abbreviated [S]. Old notations such *mhos*, ℧ or Ω^{-1} are no longer in use in the International System of units (SI, after the French *Système International d'unités*).

As explained, when determining a port impedance, the excitation signal is a current source. In certain configurations, it is sometimes more convenient to actually calculate the admittance instead by exciting the circuit with a voltage source. The final result is simply reversed to obtain the impedance we are looking for. We will see an application of this principle in an example later on. Figure 1.14 below summarizes the 6 transfer functions we just described.



$$T(s) = \frac{N(s)}{D(s)}$$

$N(s)$ ← Response signal
$D(s)$ ← Stimulus signal

$$A_v(s) = \frac{V_{out}(s)}{V_{in}(s)} \quad \text{voltage gain} \qquad A_i(s) = \frac{I_{out}(s)}{I_{in}(s)} \quad \text{current gain}$$

$$Y_t(s) = \frac{I_{out}(s)}{V_{in}(s)} \quad \text{transadmittance} \qquad Z_t(s) = \frac{V_{out}(s)}{I_{in}(s)} \quad \text{transimpedance}$$

$$Y_{in}(s) = \frac{I_{in}(s)}{V_{in}(s)}, \quad Y_{out}(s) = \frac{I_{out}(s)}{V_{out}(s)} \quad \text{admittance}$$

$$Z_{in}(s) = \frac{V_{in}(s)}{I_{in}(s)}, \quad Z_{out}(s) = \frac{V_{out}(s)}{I_{out}(s)} \quad \text{impedance}$$

Figure 1.14 There are six different transfer functions, 4 of them have a stimulus and a response at different locations – different ports – while two of them, Z_{dp} and Y_{dp}, have stimulus and response at the same port.

1.2 The Few Tools and Theorems You Did Not Forget . . .

In the litany of theorems and analysis tools I had been taught during my university years, there are a few I did not forget because I exercise them almost every day in my engineer's job. Voltage and current dividers are the first in the tools list. They are of tremendous help when it comes to simplifying circuits and a quick refresh is given below. Among theorems, the first one is Thévenin's theorem, after the French electrical engineer, Charles Léon Thévenin, in 1883. The second is the dual of Thévenin's theorem, Norton's theorem, after the American electrical engineer, Edward Lawry Norton who described the theorem in his 1926 technical memorandum. The third one is obviously the superposition theorem whose extension will lay the foundations for the EET and, later, the *n*EET. Superposition and the EET are thoroughly detailed in Chapter 3.

Let's have a look at a few examples applying these tools, showing how Thévenin and Norton can help us simplify circuits in a quick and efficient way.

1.2.1 The Voltage Divider

This is one of the most useful tools I employ when analyzing electrical circuits. It works with all passive elements in dc or ac (direct or alternating voltages/currents) and the Thévenin theorem makes an extensive use of it. Figure 1.15 shows its simple representation.

The circulating current I_1 is the input voltage V_{in} divided by the total resistive path, $R_1 + R_2$:

$$I_1 = \frac{V_{in}}{R_1 + R_2} \tag{1.15}$$

The voltage across R_2 is the resistance value multiplied by current I_1:

$$V_{out} = I_1 R_2 \tag{1.16}$$

Substituting (1.15) in (1.16), we have:

$$V_{out} = V_{in} \frac{R_2}{R_1 + R_2} \tag{1.17}$$

If we divide both sides of the equation by V_{in}, we have the transfer function linking V_{out} to V_{in}:

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2} \tag{1.18}$$

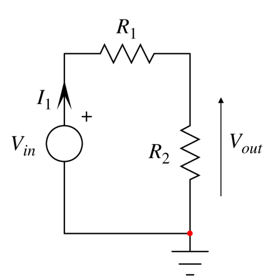
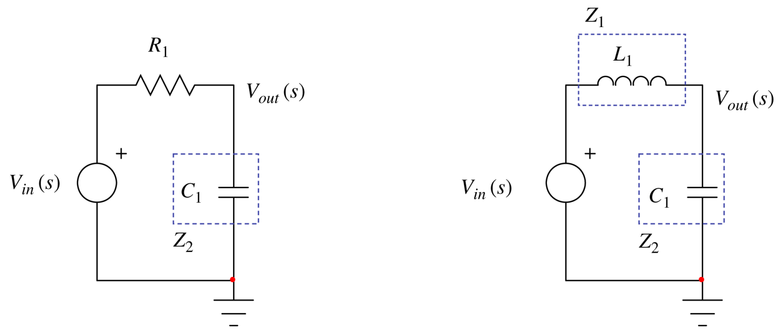


Figure 1.15 A resistive divider is a great tool to simplify circuits.



$$Z_2(s) = \frac{1}{sC_1}$$

$$A_v(s) = \frac{Z_2(s)}{Z_2(s) + R_1} = \frac{\frac{1}{sC_1}}{\frac{1}{sC_1} + R_1} = \frac{1}{1 + sR_1C_1}$$

$$Z_1(s) = sL_1 \qquad Z_2(s) = \frac{1}{sC_1}$$

$$A_v(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{\frac{1}{sC_1}}{\frac{1}{sC_1} + sL_1} = \frac{1}{1 + s^2L_1C_1}$$

Figure 1.16 The divider equation works with passive elements such as capacitors and inductors.

When you see networks such as those of Figure 1.16, you can immediately apply (1.18) without writing a single line of algebra. In this example, (1.18) is updated with impedances rather than resistances:

$$A_v(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \tag{1.19}$$

Please note that (1.18) and (1.19) only work if R_2 or Z_2 are unloaded. Should you have another circuit connected across R_2 or Z_2 respectively in Figure 1.15 and Figure 1.16, (1.18) and (1.19) no longer work.

1.2.2 The Current Divider

This is another example of a very useful tool often involved in electrical analysis. Consider Figure 1.17a circuit in which you need to find the current flowing in R_3.

The total current I_1 is V_{in} divided by the resistive path connected to the source:

$$I_1 = \frac{V_{in}}{R_1 + R_2||R_3} \tag{1.20}$$

In this expression, the '||' operator refers to the paralleling of R_2 and R_3:

$$R_2||R_3 = \frac{R_2R_3}{R_2 + R_3} \tag{1.21}$$

Mathematically, the parallel operator has precedence over the addition: $R_2||R_3$ is first computed and then added to R_1.

The original sketch can then be updated to a simpler one as shown in Figure 1.17b. Kirchhoff's current law (KCL) tells us that the sum of the currents entering a junction equals the sum of currents

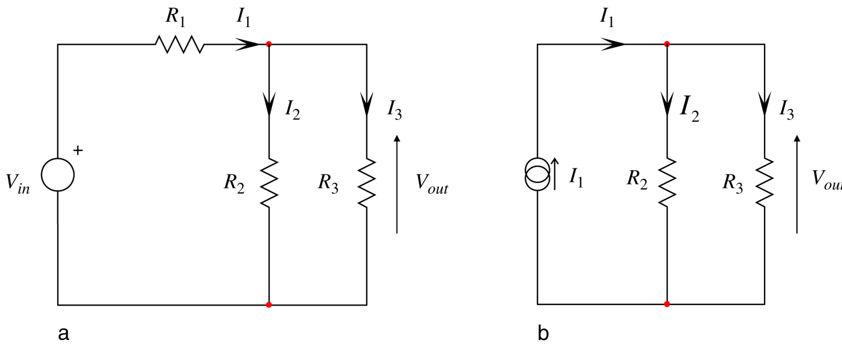


Figure 1.17 The current divider is another great simple tool.

leaving it. Thus:

$$I_1 = I_2 + I_3 \tag{1.22}$$

Currents I_2 and I_3 are defined by the voltage across their terminals, V_{out}:

$$I_3 = \frac{V_{out}}{R_3} \tag{1.23}$$

$$I_2 = \frac{V_{out}}{R_2} \tag{1.24}$$

Extracting V_{out} from (1.23) and (1.24) then equating results gives another relationship linking I_3 and I_2:

$$R_3 I_3 = R_2 I_2 \tag{1.25}$$

Extracting I_2 from (1.22) and substituting it in (1.25) leads to:

$$R_3 I_3 = R_2 (I_1 - I_3) \tag{1.26}$$

Rearranging and factoring leads to the relationship linking I_3 and I_1:

$$I_3 = I_1 \frac{R_2}{R_2 + R_3} \tag{1.27}$$

This is the current divider expression which helps us get the current into R_2 or R_3 when I_1 splits between these elements. Figure 1.18 gives another representation. The current flowing in R_2 equals

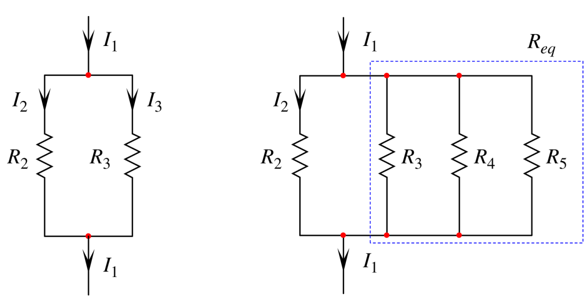


Figure 1.18 The current divider is easily generalized to paralleled resistors.

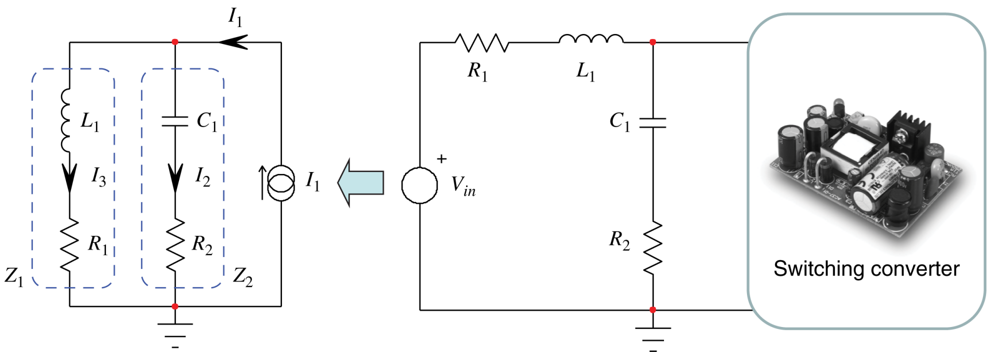


Figure 1.19 Passive elements arranged to form a filter: how much current flows in L_1?

the main current I_1 multiplied by the resistance 'facing' R_2 (thus R_3) and divided by the sum of resistances, $R_2 + R_3$. The right side of Figure 1.18 generalizes the concept where more resistors are connected in parallel with R_3. If $R_{eq} = R_3 \| R_4 \| R_5$ then the current in R_2 is simply:

$$I_2 = I_1 \frac{R_{eq}}{R_{eq} + R_2} \tag{1.28}$$

This technique works equally well with energy-storing components as represented in Figure 1.19. This is a typical Electromagnetic Interference (EMI) filter found in switching converters. I_1 illustrates the converter current signature – its high-frequency input current – C_1 is the front-end capacitor while L_1 is the filtering inductor. With a perfect filter, all the alternating current would flow in C_1 while only direct current flows in L_1, providing the dc source with the right isolation to the switching current. Reality differs and what you need is the current really flowing in L_1 and check what attenuation this configuration brings. Apply the current divider expression to Figure 1.19 circuit and you have

$$\frac{I_3(s)}{I_1(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{R_2 + \dfrac{1}{sC_1}}{R_2 + \dfrac{1}{sC_1} + R_1 + sL_1} = \frac{1 + sR_2C_1}{1 + sC_1(R_1 + R_2) + s^2L_1C_1} \tag{1.29}$$

We did not write a single equation to derive this transfer function, we just *inspected* the figure and applied the current division law. This technique is called solving for a transfer function by *inspection*.

1.2.3 Thévenin's Theorem at Work

Any 2-port *linear* system made of resistors, capacitors, inductors, dependent/independent current/voltage sources can be represented by an equivalent Thévenin model. This equivalent circuit is made of a complex generator V_{th} associated with a complex output impedance Z_{th}. When solving complex networks transfer functions, or if the current or voltage at a given point is needed, the idea is to apply Thévenin's theorem and break the complex circuit into a simpler representation with a Thévenin equivalent circuit in place. This idea behind Thévenin's approach is to model the I-V characteristics 'seen' by the load. You remove the load and model the equivalent source that drives it, affected by a certain output impedance/resistance. As such, Thévenin's and Norton's equivalent circuits do not