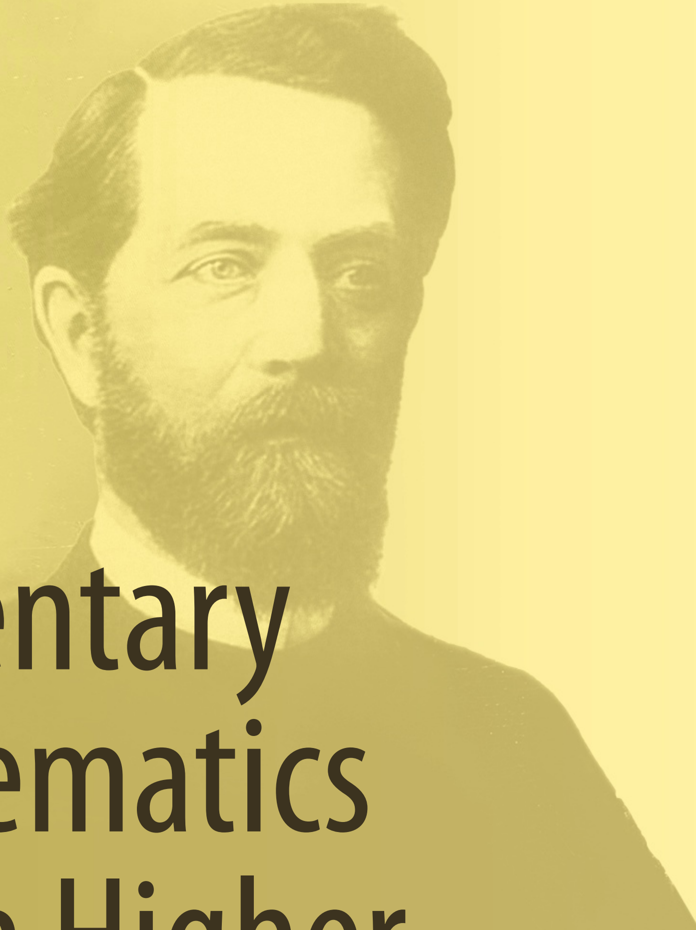
Felix Klein



Elementary Mathematics from a Higher Standpoint

Volume II: Geometry

Springer

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ISBN 978-3-662-49443-1 ISBN 978-3-662-49445-5 (eBook)
DOI 10.1007/978-3-662-49445-5
Library of Congress Control Number: 2016943431

Translation of the 4th German edition „Elementarmathematik vom höheren Standpunkte aus“, vol. 2 by Felix Klein, Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Band 15, Verlag von Julius Springer, Berlin 1926. A previous English language edition, Felix Klein "Elementary Mathematics from an Advanced Standpoint – Geometry", translated by E. R. Hedrick and C. A. Noble, New York 1939, was based on the 3rd German edition and published by Dover Publications.

Printed on acid-free paper

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The registered company is Springer-Verlag GmbH Berlin Heidelberg

Preface to the 2016 Edition

In general, for the preface for this volume II, I can refer to the preface for volume I – especially regarding the necessary correction of „advanced“ to „higher“, regarding the notion of *elementary mathematics* and of *elementarisation*, and regarding the need for a revised translation.

I should like to highlight the special features of this volume. While the first volume is dedicated to the analytic side of mathematics, this volume complements it by exposing geometry. But the second volume is complementary to the first one in two more respects. It is characteristic for the first volume that Klein – in treating arithmetic, algebra, and analysis – had always emphasised a geometric approach to the concepts: to provide a geometric interpretation and to reveal the key function of *Anschauung* in developing and in understanding the analytic concepts. Here, for geometry, Klein carefully elaborates the analytic side of the geometric concepts. His major aim is to show the ultimate unity of mathematics.

The second complimentary function is Klein's masterful manner of elaborating the process of elementarisation for the whole of geometry. In fact, it was his key scientific achievement to have changed the character of geometry: until his times, there existed side by side a number of – one might say – different geometries, having been established by the one or the other mathematician, for some specific objective and continuing in a rather dispersed manner and without seeking mutual relations or developments. Klein's *Erlanger Programm* of 1872 is emblematic for the radical change of this situation, for having succeeded in a new, "higher" unity of geometry. All the enormous variety of geometrical theories, approaches and subdisciplines had become unified – thanks to rebuilt foundations and, consequently, new "elements". Particularly illuminating is to see how the development of non-Euclidean geometries became a part of the processes leading to this unified architecture. As Klein has put it, based on Cayley: "Projective geometry is all geometry". Evidently, this elementarisation provided an exemplary methodology for realising a form of mathematics teacher training, which provided future teachers with a proper standpoint for their action later in their profession.

The structure of the second volume is different from the first one. While issues of teaching were there always integrated into the various conceptual topics, Klein decided to give the conceptual exposition as a coherent presentation and to discuss questions of teaching in a separate chapter. In this manner, Klein was able to realise this illuminating way of revealing the essential unity of geometry and its restructured elements. Evidently, this last chapter on teaching was very dear to Klein and an integral part of his conception. It remains therefore absolutely incomprehensible why the two American translators, Earle Raymond Hedrick and Charles Albert Noble, decided in 1939 to omit this chapter – and without any notice.

This chapter analyses the teaching of geometry in various European countries, first at the time of the first international reform movement, and then for the interwar period, which is otherwise not well explored. One gets revealing insight in the characteristic differences of methodology and organisation of school mathematics and of geometry teaching. Moreover, a number of issues of teaching geometry are discussed.

Regarding the first translation, their second volume reveals analogous problems mentioned in my preface to the first volume: wrong or inconsistent mathematical terminology and misunderstandings of the German text. To give just one example: on p. [4], the German text speaks, in a geometric context, of "Inhaltsbestimmung". Well, "Bestimmung" is "determination", but "Inhalt"? When no context is given, there are two meanings for this German term: either "area" (or "volume") or "content". Although the context is clearly geometric, they translated with "content". Even more strangely, they continue at first with "area" – but after a few pages, they again use "content".

In this volume, too, the reader will find, again in square brackets and in bold, *the page numbering of the original edition*. Cross-references in notes and in the text *refer to this numbering*, as well as *the name index and the subject index* (that is, the original text has not been changed in this respect).

In the present translation I have added, when possible, the first names of the persons mentioned. In the German edition, as it was customary at that time, the first names were indicated only with the initials. The bibliographic references in the notes have also been completed, when needed.

In the notes of English version of 1939, Hedrick and Noble had sometimes added references for recent pertinent American publications; these have been maintained. Several additional notes have been introduced; they are marked by square brackets.

As in volume I, the German names of the nine grades of secondary schools have been maintained, for greater exactness: *Sexta*, *Quinta*, *Quarta*, *Unter-Tertia*, *Ober-Tertia*, *Unter-Sekunda*, *Ober-Sekunda*, *Unter-Prima*, *Ober-Prima*.

I am thanking Leo Rogers for his careful re-reading of the book, and the various colleagues whom I asked advice, in particular Geoffrey Howson.

We are grateful to Dover Publication to have authorised the use of their book "Elementary Mathematics from an Advanced Standpoint", translated by E. R. Hedrick and C. A. Noble, for a revised new edition.

Gert Schubring

Preface to the First Edition

In the preface to Part I of these lecture notes (Arithmetic, Algebra, Analysis) I expressed a doubt as to whether Part II, devoted to geometry, could appear soon. Nevertheless it has been possible to complete it, thanks to the diligence of Mr. Hellinger.

Concerning the origin and purpose of this series of lecture courses I have nothing especial to add to what was said in the foreword to Part I. However, a comment seems necessary concerning the new form, which this second part has assumed.

This form is, in fact, quite unlike that of Part I. I made up my mind to give, above all, a *comprehensive view* of the field of geometry, of such a range as I should wish every teacher in a secondary school would master; the discussions about geometry *teaching* were pushed into the background and were placed in connected form at the end, insofar as there was room left, but now in a connected form.

The choice of this new order was motivated partly by the desire to avoid a stereotyped form. There were, however, more important and deeper reasons. In geometry we possess no such homogeneous textbooks corresponding to the general level of the science, such as they exist in algebra and analysis, thanks to the prototype of the French *cours*. We find, rather, one aspect treated here, another particular aspect there, of this extensive subject, just as it has been developed by one or another group of researchers. In contrast to this, it seemed to be demanded by the pedagogic and the general scientific purposes, which I am intending that I attempt a more unified presentation.

I close with the wish that the two complementary parts of my *Elementary Mathematics from a Higher Standpoint* which are herewith completed may find the same friendly reception in the teaching world as the lectures on the organisation of mathematics teaching by Mr. Schimmack and myself, which appeared last year.

Göttingen, Christmas, 1908 **Klein**

Preface to the Third Edition

In virtue of the overall plan for the new edition of my lithographed lectures, which I explained in the preface to the third edition of the first volume, the text and presentation of the present second volume, have remained unaltered, except for small changes in detail and a few insertions.[1]

The two supplements, which concern literature about scientific and pedagogic aspects, which was not considered in the original text, were prepared by Mr. Seyfarth, after repeated conferences with me. He assumed again the major portion of the burden entailed by the publication. Messrs. Ernst Hellinger, H. Vermeil, and Alwin Walther assisted him in the proof reading. Mr. Vermeil undertook the preparation of the two indexes. I am obliged to these gentlemen, and also to the publisher Julius Springer, who showed at all occasions much cooperation in realising my proposals.

Göttingen, May, 1925 **Klein**

[1] Newly added remarks are indicated by square brackets.

Contents

Introduction [1]

Aim and Form of this Lecture Course

Gentlemen! The lecture course, which I now begin, will be an immediate continuation of, and a supplement to, my course of last winter.[2] My purpose now, as it was then, is to summarise all the mathematics that you studied during your student years, insofar as this could be of interest for the future teacher, and, in particular, to show its importance for the practice of school teaching. I carried out this plan, during the winter semester, for *Arithmetic*, *Algebra*, and *Analysis*. During the current semester, attention will be given to *geometry*, which was then left aside. In this lecture course, comprehension of our considerations will be independent of knowledge of the preceding lecture course. Moreover, I shall give the whole a somewhat different tone: In the foreground I shall place, let me say, the *encyclopaedic approach* – you will be offered a *survey of the entire field of geometry* into which you can arrange, as into a rigid frame, all the separate items of knowledge which you have acquired in the course of your study, in order to have them at hand when occasion to apply them arises. Only afterward shall that *interest in mathematics teaching* appear by itself, which was always my emphasis last winter.

I should like to refer to a *vacation course* for teachers of mathematics and physics, which was given here in Göttingen during the Easter vacation in 1908. In it I gave an account of my winter lecture course. In connection with this, and also with the talk by Professor *Otto Behrendsen* of the local Gymnasium, there arose an interesting and stimulating discussion concerning the reorganisation of teaching arithmetic, algebra, and analysis, and more particularly about the introduction of differential and integral calculus into the schools.[3] The participants showed an extremely gratifying interest in these questions and, in general, in our efforts to bring the university into living touch with the schools. I hope that my present lecture **[2]** course also may exert an influence in this direction. May they contribute their part toward the elimination of the old complaint, which we have had to hear continu-

[2] [Appeared as Volume I of this series of lecture notes on *Elementary Mathematics from a Higher Standpoint*, Berlin, 1924, 3rd edition. The quotation "Part I" refers to the third edition.]

[3] See the report by Rudolf Schimmack, *Ueber die Gestaltung des mathematischen Unterrichts im Sinne der neueren Reformideen*, Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, vol. 39 (1908), pp. 513–527, (also printed separately, Leipzig, 1908).

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_1

ally – and often justly – from the schools: higher education provides, indeed, much of a special nature, but it leaves the beginning teacher entirely without orientation as to many important general things which he could really use later.

Concerning now the *topics of this lecture course*, let me say that, as in the preceding course, I shall now and then have to presuppose knowledge of important theorems from all of the fields of mathematics, which you have studied, in order to lay emphasis upon a *general survey of the whole*. To be sure, I shall always try to assist your memory by brief statements, so that you can easily orient yourself in the literature. On the other hand, I shall draw attention, more than is usually done, and as I did in Part I, to the *historical development of the science*, to the accomplishments of its great pioneers. I hope, by discussions of this sort, to further, as I like to say, your *general mathematical culture*: alongside of knowledge of details, as these are supplied by the special lecture courses, there should be a grasp of subject-matter and of historical contexts.

The Efforts for "Fusion"

Allow me to make a last general remark, in order to avoid a misunderstanding, which might arise from the nominal separation of this "geometric" part of my lectures from the first arithmetic part. In spite of this separation, I advocate here, as always in such general lecture courses, a tendency which I like best to designate by the catchphrase "*fusion of arithmetic and geometry*" – meaning by arithmetic, as is usual in the schools, the field which includes not merely the theory of integers, but also the whole of algebra and analysis. Some are inclined, especially in Italy, to use the word "*fusion*" as a catchphrase for efforts, which are restricted to geometry. In fact, it has long been the custom in secondary as well as in higher education, first to study geometry of the plane and then, entirely separated from it, the geometry of space. On this account, space geometry is unfortunately often slighted, and the noble faculty of space intuition, which we possess originally, is stunted. In contrast to this, the "fusionists" wish to treat the plane and space together, in order not to restrict our thinking artificially to two dimensions. This endeavour also meets my approval, but I am thinking, at the same time, of a still more far-reaching fusion. Last semester I endeavoured always to enliven the abstract discussions of arithmetic, algebra, and analysis by means of figures and graphic methods, which
[3] bring the things nearer to the individual and often only thus succeed in making him understand, for the first time, why he should be interested in them. Similarly, I shall now, from the very beginning, accompany space intuition, which, of course, will hold first place, with analytic formulas, which facilitate in the highest degree the precise formulation of geometric facts.

You will most easily see what I am meaning when I turn now to our subject; at first a series of simple geometric fundamental forms will be considered.

First Part: The Simplest Geometric Formations

I. Line segment, Area, Volume as Relative Quantities

Definition by Determinants; Interpretation of Signs

You will notice by the heading of this section that I am following the intention announced above, of examining simultaneously the corresponding magnitudes on the straight line, in the plane, and in space. At the same time, however, we shall take into account the principle of fusion by making use at once of the *rectangular system of coordinates* for the purpose of analytic formulation.

If we have a *line segment*, let us think of it as laid upon the x-axis. If the abscissas of its endpoints are x_1 and x_2, its *length* is $x_1 - x_2$, and we may write this difference in the form of the determinant

$$(1,2) = x_1 - x_2 = \frac{1}{1} \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}.$$

Similarly, the *area of a triangle* in the x-y-plane which is formed by the three points 1, 2, 3, with coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), will be

$$(1,2,3) = \frac{1}{1 \cdot 2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Finally, we have, for the *volume of the tetrahedron* made by the four points 1, 2, 3, 4, with coordinates (x_1, y_1, z_1), ..., (x_4, y_4, z_4), the formula

$$(1,2,3,4) = \frac{1}{1 \cdot 2 \cdot 3} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

We say ordinarily that the length, or, as the case may be, the area or the volume, **[4]** is equal to the *absolute value* of these several magnitudes, whereas, actually, our formulas furnish, over and above that, a *definite sign*, which depends upon the order in which the points are taken. We shall make it a fundamental rule always

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_2

to take into account those signs, which the analytic formulas supply, in geometry. We must accordingly inquire as to the *geometric significance of the sign in these determinations of areas*.

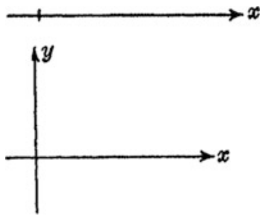

Figure 1

It is important, therefore, how we choose the *system of rectangular coordinates*. Let us, then, at the outset, adopt a convention, which is, of course, arbitrary, but which must be binding in all cases. In the case of *one dimension*, we shall think of the positive x-axis as always pointing to the right. In the plane, the positive x-axis will be directed toward the right, the positive y-axis upward (see Fig. 1). If we were to let the y-axis point downward, we should have an essentially different coordinate system, one which would be a reflection of the first and not superimposable upon it by mere motion in the plane, i.e., without extending into space. Finally, the *coordinate system in space* will be obtained from the one in the plane by adding to the latter a z-axis directed positively to the *front* (see Fig. 2). A choice of the z-axis pointing positively to the rear would give, again, an essentially different coordinate system, one which could not be made to coincide with ours by any movement in space.[4]

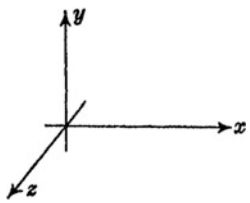

Figure 2

If we always adhere to these conventions, we shall find the *interpretation of our signs in simple geometric properties of the succession of points as these are determined by their numbering*.

[4] These two systems are distinguished as "right-handed" and "left-handed" because they correspond respectively to the position of the first three fingers of the right and left hand. (See Vol. I, p. [70])

For the *segment* (1, 2) this property is obvious: *The expression* $x_1 - x_2$ *for its length is positive or negative according as point* 1 *lies to the right or to the left of point* 2.

In the case of the *triangle*, we obtain: *The formula for the area has the positive or the negative sign according as passing around the triangle from the vertex* 1 *to* 3 *via 2 turns out to be counterclockwise or the reverse*. We shall prove this by taking, first, a conveniently placed special triangle, calculating directly the determinant, which expresses its area, and then, through an argument about continuity, resolve **[5]**
the general case. We consider that triangle which has, as its first vertex, the unit point on the x-axis ($x_1 = 1$, $y_1 = 0$), as its second the unit point on the y-axis ($x_2 = 0$, $y_2 = 1$), and as its third the origin ($x_3 = 0$, $y_3 = 0$). According to our convention about the system of coordinates, we must pass around this triangle in the counterclockwise sense (see Fig. 3), and our formula for its area yields the positive value:

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = +\frac{1}{2}\,.$$

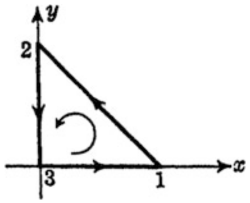

Figure 3

Now we can bring the vertices of this triangle, by continuous deformation, into coincidence with those of any other triangle travelled around in the same sense, and we can do this in such a way that the three vertices of the triangle shall at no time become collinear. In this process, our determinant changes its value continuously, and since it vanishes only when the points 1, 2, 3 are collinear, it must always remain positive. This establishes the fact that the area of any triangle whose boundary is travelled around in counterclockwise sense is positive. If we interchange two vertices of the original triangle, we see at once that every triangle, which is travelled around *in clockwise sense* has a negative area.

We can now treat the *tetrahedron* in analogous fashion. We start, again, with a conveniently placed tetrahedron. As first, second, and third vertices, we choose, in order, the unit points on the x-, y-, and z-axes, and as fourth vertex the origin (see Fig. 4). Its volume is therefore

$$\frac{1}{6}\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = +\frac{1}{6}\,.$$

It follows, as before, that every tetrahedron, which can be obtained from this one by continuous deformation while the four vertices never become co-planar (i.e., during which the determinant never vanishes), has positive volume. But one can characterise all these tetrahedrons by the sense in which that face-triangle $(2,3,4)$ is travelled around when it is looked at from the vertex 1. In this way we obtain the result: *The volume of the tetrahedron* $(1,2,3,4)$ *which our formula yields is positive if the vertices* $2,3,4$, *looked at from vertex* 1, *follow one another in counterclockwise sense*; *otherwise it is negative*.

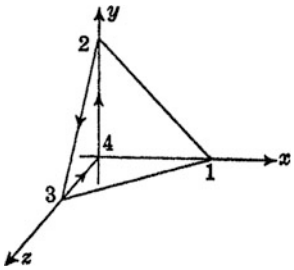

Figure 4

[6] We have thus, from our analytic formulas, actually deduced geometric rules which permit us to assign a definite sign to any segment, any triangle, any tetrahedron, if the vertices are given in a definite order. Great advantages are thus gained over the ordinary elementary geometry, which considers length and area as absolute magnitudes. Indeed, we can establish general simple theorems even there where elementary geometry must distinguish numerous cases according to the particular form of the figure.

Simple Applications; in Particular the Cross-Ratio

Let me begin with a very primitive example, the *ratio of the segments* made by three points on a line, say the x-axis. Denoting the three points by 1, 2, and 4 (see Fig. 5), as is the most convenient in view of what is to follow, we see that the ratio in question will be given by the formula

$$S = \frac{x_1 - x_2}{x_1 - x_4},$$

and it is clear that this quotient is positive or negative according as the point 1 lies outside or inside the segment (2, 4). If one gives, as is customary in elementary expositions, only the absolute value

$$|S| = \frac{|x_1 - x_2|}{|x_1 - x_4|},$$

we must always either refer expressly to the figure, or state in words whether we have in mind an inside or an outside point, which is, of course, more cumbersome. *The introduction of the sign thus takes account of the different possible orders of the points on the line*, a fact to which we shall often have to refer during this lecture course.

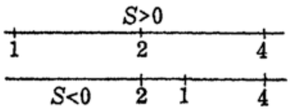

Figure 5

If we now add a fourth point 3, we can set up the *cross ratio* of the four points, that is,

$$D = \frac{x_1 - x_2}{x_1 - x_4} : \frac{x_3 - x_2}{x_3 - x_4} = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)} .$$

This expression has again a definite sign, and we see at once that $D < 0$ when the pair of points 1 and 3, on the one hand, and the pair 2 and 4, on the other hand, mutually separate one another; and that $D > 0$ in the opposite case, i.e., when 1 and 3 lie both outside or both inside the segment 2, 4. (See Figs. 6 and 7.) Thus there are always two essentially different positions, which yield the same absolute **[7]**
value D. If this absolute value alone is given, we must, moreover, give expressly the determination of the position. For example, if one defines harmonic points by the equation $D = 1$, as is still the custom, unfortunately, in the schools, one must include in the definition the demand of a separate position of the two pairs of points, whereas in our plan the *one* demand $D = -1$ is sufficient.

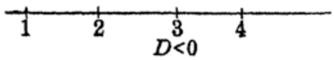

Figure 6

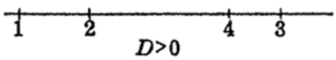

Figure 7

This practice of taking account of the sign is especially useful in *projective* geometry, in which, as you know, the cross-ratio plays a leading role. There we have the familiar theorem that four points on a line have the same cross-ratio as the four points, which arise when we project the given points from an arbitrary centre upon another line (perspective). If we now consider the cross-ratio as a relative magnitude, affected by a sign, the converse of this theorem holds without exception: If each of two sets of four points lies on one of two lines, and if they have the same

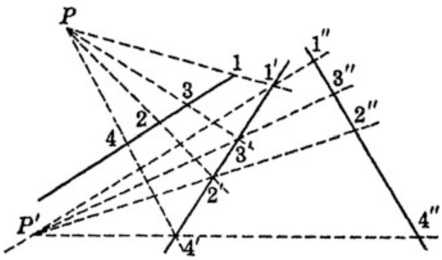

Figure 8

cross-ratio, they can be derived one from the other by projection, either single or repeated. For example, in Fig. 8, the sets 1, 2, 3, 4, and 1″, 2″, 3″, 4″ by projection from the centres P and P'. If, however, one knows only the absolute value of D, the corresponding theorem does not hold in this simple form; we should have to make a special assumption about the position of the points.

Area of Rectilinear Polygons

We have a more fruitful field if we proceed to *applications of our triangle formula*. Let us first select somehow a point 0 in the interior of a triangle (1, 2, 3) and let us join 0 to each of the vertices (see Fig. 9). Then the sum of the areas, thought of in the elementary sense as absolute magnitudes, of the three partial triangles is equal to the area of the original triangle. Thus we may write

$$|(1,2,3)| = |(0,2,3)| + |(0,3,1)| + |(0,1,2)| \,.$$

Given the positions in the figure, the order of the vertices, in all the triangles, as they appear in the above equation, is counterclockwise. Hence the areas $(1,2,3)$,
[8] $(0,2,3)$, $(0,3,1)$, $(0,1,2)$ – signed in the sense of our general definition – are all positive so that we may write our formula in the form

$$(1,2,3) = (0,2,3) + (0,3,1) + (0,1,2) \,.$$

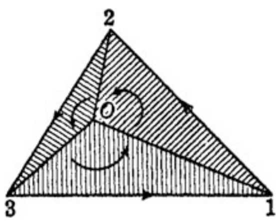

Figure 9

Now I assert *that the same formula also holds when* 0 *lies outside the triangle*, *and*, *further*, *when* 0, 1, 2, 3 *are any four points whatever in the plane*. If we take the position of Fig. 10, for example, we see that the boundaries of $(0, 2, 3)$ and $(0, 3, 1)$ are travelled around in a counterclockwise sense, but that of $(0, 1, 2)$ is in the clockwise sense, so that our formula for the areas, calculated as absolute quantities, would give

$$|(1,2,3)| = |(0,2,3)| + |(0,3,1)| - |(0,1,2)|\,.$$

The figure verifies the correctness of this equation.

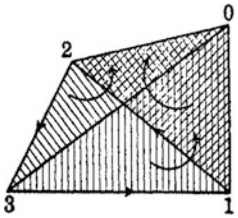

Figure 10

We shall give a general proof of our theorem by means of the *analytic definition*, whereby we shall recognise in our formula a well-known theorem of algebra, respectively of the theory of determinants. For convenience, let us take the point *0* as our origin $x = 0$, $y = 0$, which is obviously no essential specialization, and let us substitute for each of the four triangle areas the appropriate determinant. Then, omitting everywhere the factor $\frac{1}{2}$, it is left to prove that, for arbitrary values $x_1, \ldots, y_3$, the following relation holds:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}.$$

The value of each of the determinants on the right will remain unchanged if we replace the second and third 1 of the last column by zeros, since these elements enter only those minors, which are multiplied by zero, when we expand according to the top row. If we now make a cyclic interchange of rows in the last two determinants, which is permissible in determinants of the third, or, in fact, of any odd order, we can write our equation in the following form:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & 0 \\ 0 & 0 & 1 \\ x_3 & y_3 & 0 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

But this is an identity, for on the right there are only the minors of the last column of the left determinant, so that we have merely the well-known expansion of this determinant according to the elements of a column. Thus, at one stroke, we have proved our theorem for all possible positions of the four points.

[9] We can now generalise this formula so that it will give the *area of any polygon*. Imagine that you had, say, the following problem in surveying: To determine the area of a rectilinear field after having measured the coordinates of the vertices $1, 2, \ldots, n-1, n$ (see Fig. 11). One who is not accustomed to operate with signs would then sketch the shape of the polygon, divide it up into triangles by drawing diagonals, perhaps, and then according to the particular shape of the field, paying especial regard to whether some of the angles are re-entrant, find the area as the sum or difference of the areas of the partial triangles. However, we can give at once a general formula, which will give the correct result quite mechanically without any necessity of looking at the figure: If 0 is any point in the plane, say the origin, then the area of our polygon, being travelled around in the sense $1, 2, \ldots, n$, will be

$$(1,2,3,\ldots,n) = (0,1,2) + (0,2,3) + \cdots + (0,n-1,n) + (0,n,1)\,,$$

whereby each triangle is to be taken with the sign determined by the sense in which the circuit about it is made. *The formula yields the area of the polygon positively or negatively according as the circuit of the polygon in the sense* $1, 2, \ldots, n$ *is counterclockwise or not.* It will suffice to write this formula. You yourselves can easily supply the proof.

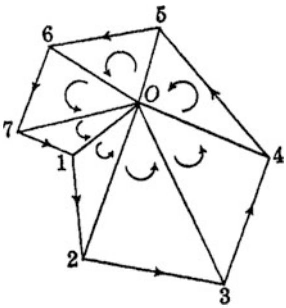

Figure 11

Instead of pursuing this example further, I prefer to take up some especially interesting cases, which, to be sure, could not arise in surveying, namely, cases of *polygons which are twisted upon themselves* as in the adjoining quadrilateral (see Fig. 12). If we wish here to talk at all about a definite area, it can only be the value which our formula yields. Let us consider what this value means geometrically. At the outset we notice that this must be independent of the particular location of the point 0. Let us place 0, as conveniently as possible, at the point where the twisting occurs. Then the triangles $(0,1,2)$ and $(0,3,4)$ will be zero and there remains:

$$(1,2,3,4) = (0,2,3) + (0,4,1)\,.$$

The first triangle has negative area, the second positive area; hence the area of our quadrilateral, if we ascribe it a circuit in the sense $(1, 2, 3, 4)$, is equal to the absolute **[10]**
value of the area of the part $(0, 4, 1)$ that was travelled around in counterclockwise sense, *diminished* by that of the part $(0, 2, 3)$ that was travelled around in a clockwise sense.

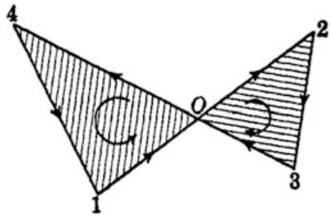

Figure 12

As a second example, let us examine the adjoined *star pentagon* (see Fig. 13). If we take O in the middle part, all the partial triangles in the sum

$$(0, 1, 2) + (0, 2, 3) + \cdots + (0, 5, 1)$$

are travelled in the positive sense; their sum covers the kernel, having five vertices, of the figure twice, and each of the five tips once. If we again compare a circuit around our polygon, done one-time along (1, 2, 3, 4, 5, 1), we see that every part of the boundary is travelled around counterclockwise and that, namely, we have the portion of the polygon which is doubly counted for determining the area, will be travelled twice around, but only once for the portion, which has to be counted only once.

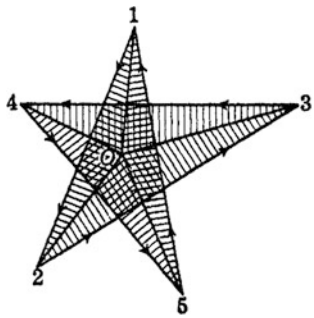

Figure 13

From these two examples we can infer the following *general rule*: *For any rectilinear polygon with* an *arbitrary number of twistings*, *our formula yields*, *as total area*, *the algebraic sum of the separate partial areas bounded by the polygonal path*, *whereby each of these partial areas is counted as often as we travel around* *its boundary when the circuit* $(1, 2, 3, \ldots, n, 1)$ *is made once*, *this counting to be made positively or negatively according as we travel around the partial area in counterclockwise or clockwise sense*. You will have no difficulty in establishing the truth of this general theorem. The more I am recommending you to entirely appropriate these interesting area formulas by some examples.

Areas with Curvilinear Boundaries

Let us now pass from polygons to *areas with curvilinear boundaries*. We shall consider any closed curve whatever, which may twist upon itself any number of times. We assign a *definite sense of direction along this curve* and ask for the area bounded by the curve. We find this area in a natural manner if we approximate the curve by polygons having an increasing number of shorter and shorter sides (see Fig. 14) and calculate the limit of the areas of these polygons, found in the way we have just described. If

$$P(x, y) \quad \text{and} \quad P_1(x + dx, y + dy)$$

are two neighbouring vertices of such an approximating polygon, then its area consists of a sum of elementary triangles (OPP_1), that is of summands:

[11]

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ x + dx & y + dy & 1 \end{vmatrix} = \frac{1}{2}(x\,dy - y\,dx)\,.$$

In the limit, this sum becomes the line integral

$$\frac{1}{2}\int (x\,dy - y\,dx)$$

taken along the curve in the given direction, which, therefore, defines the area bounded by the curve. If we wish to interpret this definition geometrically, we can transfer right away to the new case the result just given for polygons: *Each*

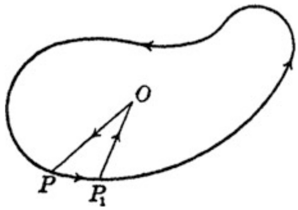

Figure 14

partial area enclosed by the curve is counted positively as many times as it is travelled around in a *counter-clockwise sense and negatively as many times as this is done in* a *clockwise sense while the given curve is traversed once in the prescribed sense*. For a simple curve, such as that of Fig. 14, the integral yields, accordingly, the exact area bounded by the curve, taken positively. In Fig. 15, the outer part is counted once positively, the inner part twice; in Fig. 16, the left-hand part is negative, while the right-hand part is positive, so that, altogether, a negative area results; in Fig. 17, one part is not counted at all, since it is encircled once positively and once negatively. Of course, curves can arise which, in this sense, bound a zero area. We obtain such a curve if we take the curve in Fig. 16 symmetric with respect to the point of twisting. Such a case presents nothing absurd when we recall that our determination of area rests upon a convenient assumption.

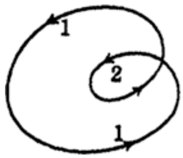

Figure 15

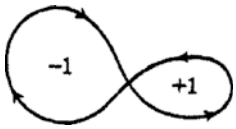

Figure 16

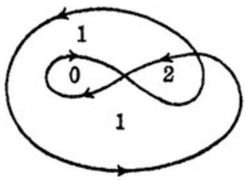

Figure 17

I shall now show you how appropriate these definitions are by considering the

Amsler's Polar Planimeter

This highly ingenious apparatus, very useful in practice, constructed in 1854 by the mechanic *Jacob Amsler* of Schaffhausen, effects the determination of areas precisely in the sense of our discussion above. Let me consider, first, the *theoretical basis of the construction*.

We think of a rod A_1A_2 (see Fig. 18) of length l moved in the plane in such
[12] a way that A_1 and A_2 describe separate closed curves and the rod itself returns to its initial position. We wish to find the area, which the rod sweeps over, counting the several parts of this area as positive or negative, according as they are swept over in one sense or in the other. To this end, we replace- according to the limit process to be realised for any integration – the continuous motion of the rod by a succession of arbitrarily small jerkily "elementary motions" from one position $1\,2$ to a neighbouring one $1'\,2'$. The actual area swept out by the rod will be the limit of the sum of all the "elementary quadrilaterals" $(1, 1', 2', 2)$ described during these elementary motions, and it is easy to see that the sense of the motion of the rod is taken into account properly if we give to each elementary quadrilateral the sign corresponding to a circuit in the sense $1, 1', 2', 2$. Now we can compose each elementary motion of the rod A_1A_2 from three steps (see Fig. 19):

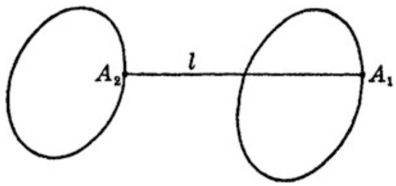

Figure 18

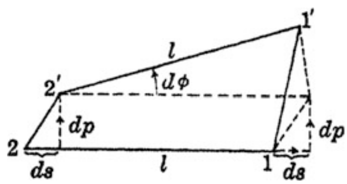

Figure 19

(1) A translation in the direction of the rod by an amount ds.

(2) A translation *normal* to its direction by an amount dp.

(3) A rotation about the end A_2 through an angle $d\phi$.

In this way the areas $0 \cdot ds$, $l \cdot dp$, $(l^2/2)d\phi$, respectively, will be swept out. We can replace the area of the elementary quadrilateral by the sum of these three areas, since the error thus made would be an infinitesimal of higher order and would disappear in the limit process (which is, indeed, a simple process of integration). It is essential to note that this sum

$$l \cdot dp + \frac{l^2}{2} \cdot d\phi$$

agrees in sign with the area of the quadrilateral $(1, 1', 2', 2)$, if we measure $d\phi$ positively in a counterclockwise sense and dp positively for translation toward the side of increasing ϕ.

Integration along the path of motion yields for the area swept out by A_1A_2 the value

$$J = l \int dp + \frac{l^2}{2} \int d\phi \, .$$

The integral $\int d\phi$ represents the entire angle through which the rod turns with respect to its initial position. Since we returned the rod to its initial position, $\int d\phi = 0$, unless the rod has made a complete revolution, so that the area is **[13]**

$$J = l \int dp \, . \tag{1}$$

If, however, the rod made one or more complete turns before returning to its original position, which is possible with suitably chosen paths for A_1 and A_2, then $\int d\phi$ is a multiple of 2π, and we must add to the right-hand side $+\pi l^2$ for each complete turn in the positive sense and $-\pi l^2$ for each one in the negative sense. For the sake of simplicity we shall pass over this slight complication.

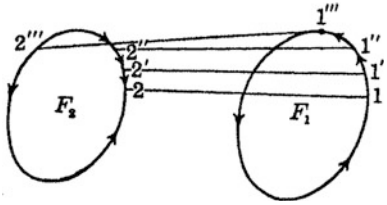

Figure 20

Now we can determine this same area J in a somewhat different way (see Fig. 20). In the succession of elementary motions let the rod take, one after another, the positions $1\,2$, $1'\,2'$, $1''\,2''$, ... Then J will be the sum of the elementary quadrilaterals

$$J = (1, 1', 2', 2) + (1', 1'', 2'', 2') + (1'', 1''', 2''', 2'') + \ldots,$$

or, more exactly, the integral which represents the limit of this sum, whereby each quadrilateral is to be travelled around in the sense here indicated, just as before. Using our earlier polygon formula, we now have, where *0* is the arbitrarily chosen origin of coordinates,

$$\begin{aligned} J = {} & (0, 1, 1') + (0, 1', 2') + (0, 2', 2) + (0, 2, 1) \\ & + (0, 1', 1'') + (0, 1'', 2'') + (0, 2'', 2') + (0, 2', 1') \\ & + (0, 1'', 1''') + (0, 1''', 2''') + (0, 2''', 2'') + (0, 2'', 1'') \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$