Resistivity Modeling
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Propagation, Laterolog and Micro-Pad Analysis

Wilson C. Chin, Ph.D., M.I.T.
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Preface

Those familiar with the author’s early work in fluid mechanics will find mathematical rigor tempered with healthy skepticism in formulating and solving physical problems correctly. Validations proliferate in his books and papers. This is not unusual for engineers trained in physics and applied math, and this approach has served well as interests turned from one fluid-dynamic specialty to another; then, still more, leading to diverse activities in reservoir engineering, annular flow, formation testing, MWD design and telemetry, and so on. However the author was, for years, a “closet resistivity man” trained in electrodynamics at M.I.T.’s physics department, long a bastion of experts in astrophysics, plasma dynamics, string theory, and so on.

To this author, electromagnetic simulation for oilfield applications had always ranked high among these Herculean tasks: the dream was much too large to resist. Interestingly, understanding what had been done and what was really available actually proved to be the greater challenge. Research in the oil service industry is shrouded in secrecy. “Validations” are log examples that lead to oil discoveries and corporate revenue. Whether or not numerical models are actually consistent with Maxwell’s equations and the handful of analytical solutions developed by top classical physicists over the past century is irrelevant. Technical training in university and industry short courses simply amounts to studying marketing literature and log analysis papers focused more on differences between competitor tools than with rigorous mathematical results. All of this would not be relevant except that, after years of service to fluid mechanics, the author was asked by multiple organizations to develop suites of electromagnetic simulators that would address modern applications for hardware and interpretation development. These would be available to new competitors and old, and they would, naturally, need to be properly formulated and rigorously validated. Mathematical correctness and real equations were in demand at last.
The author’s recent book *Electromagnetic Well Logging: Models for MWD/LWD Interpretation and Tool Design*, from John Wiley & Sons in 2014, would be a first step in delivering the new models. The work provided a full three-dimensional formulation for “non-dipolar” transmitters in heterogeneous layered anisotropic media with dip. By “non-dipolar” we meant finite circular transmitters, elliptic coils, and in fact, any open or closed antenna geometry with or without embedded drill collar mandrels, plus coil sizes that might extend across multiple formation layers. Effects like charge radiation at layer interfaces, borehole invasion and eccentricity, and the like, were permitted, with algorithms running stably and rapidly converging within fifteen seconds on Intel Core i5 machines. Complementary “receiver design” methods were added to post-processing capabilities; no longer were users restricted to conventional coils wound in circular fashion – more general formulations allowed a variety of antenna designs which would ideally “see” more accurately in very realistic formations, using any range of probing frequencies from induction to dielectric.

Such problems are by no means simple. One might have thought that, since the pioneering work of Coope, Shen and Huang (1984) for axisymmetric AC analysis in vertical boreholes, numerous models would be available to study interpretation schemes or to design prototype hardware in formations with simple radial and horizontal layers. However, this is unfortunately not so. The analytical work in Coope *et al.*, while correct, is highly mathematical and incomprehensible; to this author, the formulation was lacking because it could not be extended numerically to model complicated geologies, fluid invasion, plus other real-world effects. The reasons are numerous and esoteric: well known limitations of complex variables formulations, computational techniques that inaccurately modeled Dirac delta function sources, and methods that could not simulate rapidly varying fields for all frequencies. In this book, we will address complicated AC axisymmetric problems and fields very generally.

Analogous issues are found in DC laterolog applications. For instance, Li and Shen (1992) note, in their widely-read numerical analysis paper, that focusing conditions were inferred from the literature. Also, assumptions underlying proprietary simulators were subject to speculation – for instance, one “known” focusing model could not be disclosed because of a confidentiality agreement. But the authors’ own work was equally cryptic – their “finite element analysis” is not described at all, but presumably available only to consortium participants. The paper employed arbitrary methods. Upon convergence, the total current $I_m = I_0 + I_1 + I_1'$ and the corresponding voltage $V_m = 0.5 (V_{m0} + V_{m0'}) - 0.25 (V_{m1} + V_{m1'} + V_{m2} + V_{m2'})$ are
defined and apparent resistivity is further defined as \( R_a = KV_m / I_m \). And, at the risk of even more definition, “K is a tool constant that will make \( R_a \) equal to the true formation resistivity when the tool is in a standard medium [our italics].” Real solutions are neither simple nor arbitrary. And of course, real formations may be anisotropic, but that’s another story – until now, anyway, secrecy has prevailed.

Direct current laterolog and pad devices are by no means simple. With modern emphases on “low resistivity pay” and anisotropy, one would expect that industry publications would address the roles of \( R_v \) and \( R_h \). Yet, literature searches conducted as recently as 2016 disclosed few modeling results let alone basic theory. Those that were available showed current lines that were orthogonal to potential surfaces, a clear indication that isotropic media was assumed, additionally with planar flow underlying assumptions. General issues in streamline tracing should have been discussed decades ago. A current source that probes effectively in one direction may be ineffective when turned ninety degrees, and vice-versa. It is clear that interpretation in anisotropic media requires different methods in vertical versus horizontal wells – needless to say, so does tool design. “Streamline tracing,” the description of paths taken by electrical current, is developed rigorously here. In the published literature, these paths are typically carelessly sketched by hand – but accurate tracing is essential to understanding which part of the formation is actually being probed, if at all. When it gets down to details, answers to critical questions are needed. Here, we develop streamfunction methods pioneered by this author in the aerospace industry to problems in resistivity logging tool design and interpretation.

Solving for voltage distributions and current paths in fields with prescribed resistivity is one thing. But understanding what constitutes resistivity is another – an issue that raises more profound questions. What is resistivity? A simple analogy highlights the subtleties. Draw two “dots” on a solid surface an inch apart. Now measure that distance with a standard ruler – the answer, of course, is one inch. Repeat the measurement with a ruler, say, \( 10^{-100} \) inch long – because you’ll traverse every mountain and valley about every electron and proton, your answer might be, well, a thousand times that of the original. A similar situation arises, for example, with cross-bedded sands, which are treated in Chapter 6. Rock grains may be isotropic in a microscopic sense, but taken in the aggregate over multiple dipping layers, a direct current measurement may perceive anisotropy. An alternating current device may “see” events differently, e.g., are six-inch receiver spacings inherently different from thirty-inch spacings” tools? Quite clearly, the resistivity found depends on the “ruler” used.
Archie, of “Archie’s law” renown, long ago postulated an empirical relationship connecting resistivity to water saturation. Its application is universal and simple: determine farfield “virgin” resistivity from electrical measurements and his well known law gives saturation immediately. This recipe has dominated log analysis and reserves estimation for decades but it is overly simplified. All petrophysicists are familiar with the classic Schlumberger sketch for axisymmetric resistivity problems showing borehole fluid, mudcake, invaded zone (with spatially varying properties) and virgin rock. Correction charts proliferate which allow users to adjust predictions to account for idealizations that do not apply. But all of this is now unnecessary and antiquated given recent advances in resistivity and fluid-dynamical simulation.

Our approach is simple. The spatially variable water saturation field, which also evolves in time, is one that is easily calculated and found independently of resistivity. This fluid distribution depends on mudcake properties, which control invasion rates by virtue of extremely low cake permeabilities, wellbore and reservoir pressures, and relative permeability and capillary pressure (in the case of immiscible displacements) and molecular diffusion (for miscible flow). Now imagine that we have calculated \( S_w(r,t) \) in its entirety. Then, via Archie’s law, the corresponding resistivity distribution \( R(S_w(r,t)) \) is available for “plug in” to any of the general resistivity codes developed here and in Chin (2014) for various tools. Receiver responses are calculated. But, naturally, they are unlikely to agree with measured values. Of course, we recognize that multiphase properties are typically unknowns subject to guess work and refinement, so parameters related to, say, diffusion or relative permeability, are adjusted. Resistivity calculations are performed again and the process repeated until a parameter set consistent with receiver data is found.

This type of iterative analysis is no different from “history matching” in well testing (which matches to pressure transient response) or reservoir engineering (which utilizes production rate to gauge correctness). Our approach differs from the conventional use of Archie’s law in one significant detail: distributions of resistivity are used for history matching rather than single values. This topic is introduced in Chapter 7 by way of a simple example, but clearly, other permutations and possibilities quickly suggest themselves. Finally, Chapter 8 examines more sophisticated examples for “simpler, plug flow” fluid-dynamics models using algebraic as opposed to differential equations. These approaches will be useful in future developments of the “time lapse logging” methods introduced in Chapter 7 and in Chin et al. (1986).
So, it is with personal satisfaction that the author has solved, and has disclosed in this third volume of John Wiley & Sons’ new *Advances in Petroleum Engineering* series, those difficult resistivity problems not considered in Chin (2014). The process of “telling all” is not without risk – one wrong claim or equation can derail a consulting practice built over perspiration and time. The validations presented here reduce this risk. Furthermore, they are designed to encourage acceptance by an industry accustomed to endorsing marketing claims with minimal justification. Why is one coil configuration better than another? Why are certain (arbitrary) depth of investigation definitions used? Why use “apparent resistivities” related to fictitious isotropic reference media when real formations are anisotropic with $R_v >> R_h$? And why should amplitude and phase resistivities “see” different depths of investigation even though their coupled solution follows from a single formulation? Are there better ways to use Archie’s law? Can we find improved methods that couple electromagnetic and fluid analyses which create additional value to petroleum engineering? This book provides tools which facilitate research and software design. It raises questions. It promotes an understanding of the physics and an appreciation for mathematics with all its limitations. Finally it hopes, through a number of new ideas introduced, to elevate what has been a profession dominated by empirical service company equations and borehole correction charts into a scientific discipline that nurtures even more principled approaches. The research in this volume sets the stage for more comprehensive integration between electromagnetic analysis and fluid-dynamics in future publications – a work in progress that will continue despite the oil economy.

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The author gratefully acknowledges the efforts of several generations of petroleum physicists who have endeavored to bring rigor and understanding to very complicated geological applications of modern electromagnetism. Also, many of the problems successfully addressed here and in *Electromagnetic Well Logging: Models for MWD/LWD Interpretation and Tool Design* could not have been were it not for the Boeing Commercial Airplane Company in Seattle, Washington. It was here, during the author’s formative years just out of M.I.T., where exciting ideas related to complex Helmholtz partial differential equations, distributed sources, sinks and vortexes, three-dimensional streamline tracing, functions with discontinuous values or derivatives, and so on, were discussed and debated with enthusiasm and turned into software productively used to design modern aircraft. Many thanks go to Boeing, and in particular, to Paul Rubbert, Edward Ehlers, Donald Rizzetta and other colleagues. As usual, the author is indebted to Phil Carmical, Acquisitions Editor and Publisher, for his support and encouragement in disseminating his highly technical research monographs, together with equations, cryptic Greek symbols, formal algorithms and more. In times of uncertainty, such as the economic turmoil now facing all of us, it is even more important to “solve problems right” and work more productively. What our industry needs is more math and not less, more questioning and less acceptance, and it is through this latest volume that the author hopes to stimulate thought and continuing research in an important engineering endeavor central to modern exploration for oil and gas.
1

Physics, Math and Basic Ideas

1.1 Background, Industry Challenges and Frustrations

The author’s recent 2014 book *Electromagnetic Well Logging: Models for MWD/LWD Interpretation and Tool Design* from John Wiley & Sons solved a rigorously posed AC formulation for general coil transmitters in heterogeneous, anisotropic, layered media without the physical limitations associated with dipole, integral equation, mode matching, Born approximation and other models (see Figure 1.1). Detailed validations were given that showed the degree of quality control used to assure agreement in the spirit of Maxwell’s equations.

![Recent electromagnetic well logging book.](image)

The present monograph addresses additional topics, e.g., $R_h$ and $R_v$ determination from three-dimensional receiver amplitude and phase measurements, more limited (but rapid) axisymmetric AC coil simulations, direct current laterolog and pad resistivity modeling, and streamline tracing in highly anisotropic media. We also raise a profound question, “What is resistivity?” in Chapter 6, followed by introducing new ideas in “time lapse logging,” and also in time-dependent changes to resistivity and saturation fields coupled by Archie’s law, in Chapter 7. And finally, in Chapter 8, we develop new “plug flow” fluid mechanics models which will ultimately be used to infer permeability, porosity, viscosity and pore pressure from resistivity log data. These items remain works in progress, but optimistically speaking, it is hoped that a foundation has been developed to support and guide further research.
1.2 Iterative Algorithms and Solutions

Suppose we wished to solve \( \frac{d^2y(x)}{dx^2} = 0 \), with \( y(0) = 0 \) and \( y(10) = 100 \). In this example, the answer is simple: \( y = 10x \). But what if we didn’t know the solution was a straight line – or, thinking ahead, what if we had a partial differential equation that satisfied non-trivial boundary conditions, for which the answer or solution method was not-at-all obvious? How would we attack the problem? What formalism is best? Is it possible to design an iterative strategy whose initial guess is irrelevant and still obtain a correct solution?

Let us use the above example as a test. To begin, we discretize the set of \( x \) values by points \( x_1, x_2, x_3 \) and so on, denoting it by \( x_i \) where \( i \) varies, say, from 1 to \( i_{\text{max}} \). Corresponding to each \( x_i \) is a value \( y_i \), and for now, assume that all \( x_i \)'s are separated uniformly by a distance \( \Delta \). Then, the first derivative at \( "i" \) is just \( \frac{dy}{dx}|_i = \frac{(y_{i+1} - y_{i-1})}{2\Delta} \), while the second derivative takes the form \( \frac{d^2y}{dx^2}|_i = \frac{\{y_{i+1} - y_i\} - \{y_i - y_{i-1}\}}{\Delta^2} \approx 0 \). The equation \( \frac{d^2y(x)}{dx^2} = 0 \), using the latter approximation, leads to \( y_i = \frac{(y_{i+1} + y_{i-1})}{2} \), in this case stating that any value \( y \) is simply the arithmetic average of its left and right neighbors. Next, we suppose that a guess to \( y_i \) exists, not necessarily a suitable one, and we attempt to improve it by systematically “smoothing” it with \( y_i = \frac{(y_{i+1} + y_{i-1})}{2} \). This is performed in cycles, with each applied to internal points \( i = 2 \) to \( i_{\text{max}-1} \) followed by an update at the end points \( y(0) = 0 \) and \( y(10) = 100 \). We wish to follow the evolution of \( y_i \) as the number of cycles, or iterations, increases.

This algorithm is programmed in Fortran in Figure 1.2a, but any other computer or spreadsheet language is suitable. Although we know that the solution \( y = 10x \) varies between 0 and 100, we have assumed an absurd initialization (of 345 for the first half of the interval and 789 for the second) which is not even close to the solution. Figure 1.2b shows computed results at iterations 10, 50 and 100. It is clearly seen that, while the solution at 10 is poor, this unacceptable value has converged to the proper result at the 100th iteration!

This example demonstrates several important points we wish to convey. First, the approximation of derivative terms in a differential equation model is a “no brainer,” requiring only a rudimentary understanding of calculus. Second, the development of “recursion formulas” like \( y_i = \frac{(y_{i+1} + y_{i-1})}{2} \), which simply diffuse information throughout the computational domain, is just as elementary. And third, the coding of this formula is trivial in Fortran, or in any other computer language, where \( y_i \) is replaced by an intuitive \( Y(I) \). This type of discretization approach is called “finite differences,” while the iterative solution procedure is called “relaxation.” The method is extensively illustrated in the reservoir engineering book of Chin (2002), in which the two-dimensional partial differential equation \( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \) is solved using “point relaxation” (such as the one just used) and more rapidly convergent “line relaxation” methods. In general, the function \( y_i \) or its derivative \( dy/dx|_i \) may be discontinuous at internal boundaries, and special matching conditions will be needed (these auxiliary relationships will be considered as they arise).
POINT-RELAX.FOR

Point relaxation method for $d^2Y/dx^2 = 0$, with $Y(1) = 0$ and $Y(10) = 100$.

DIMENSION Y(10)

Number of iterations
NMAX = 100

Number of nodal points
IMAX = 10
IMAXM1 = IMAX-1

Assume constant grid where index I=1 defines X=0. Then exact solution satisfying $Y(1) = 0$, $Y(10) = 100$ is $Y = 10x$ or $Y(I) = 100*(I-1)/9$ where I varies from 1 to 10.

Initialize $Y(I)$ to something, any guess! Can be meaningless, but good choice reduces computation. Following is completely irrelevant first approximation.

DO 100  I=1,IMAX
IF(I.GE.1.AND.I.LE.5) Y(I) = 345.
IF(I.GT.5) Y(I) = 789.
100  CONTINUE

Perform NMAX iterations ...
DO 200  N = 1,NMAX
C Here each applies the simple recursion formula
DO 150  I = 2,IMAXM1
Y(I) = (Y(I-1) + Y(I+1))/2.
150  CONTINUE
C Include boundary conditions $Y(1) = 0$, $Y(10) = 100$
Y(1) = 0.
Y(IMAX) = 100.
200  CONTINUE

WRITE(*,205) NMAX
WRITE(*,210)
205  FORMAT(' Number of iterations: ',I3)
210  FORMAT(' I COMPUTED EXACT! %ERROR')
DO 300  I=1,IMAX
C Exact solution is straight line
EXACT = 100.*(I-1)/9.
C Compute % error, can add convergence criterion here
to terminate calculations at desired accuracy
ERROR = 100.*Y(I)-EXACT)/EXACT
IF(I.EQ.1) WRITE(*,245) I,Y(I),EXACT
IF(I.GT.1) WRITE(*,250) I,Y(I),EXACT,ERROR
245  FORMAT(1X,I4,2F10.1)
250  FORMAT(1X,I4,3F10.1)
300  CONTINUE
C
STOP
END

Figure 1.2a. One-dimensional algorithm with poor initial guesses (that is, “345” and “789” at middle of page in “Do 100” loop.)
### Resistivity Modeling

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<th>EXACT!</th>
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Figure 1.2b. Convergent solutions from bad initial guess.

Finite differences, combined with iterative solutions, have been the author’s preference for a range of applications from reservoir flow, to annular rheology, to borehole electromagnetics, over the years. Numerical models are straightforward to formulate and program, and software is readable and easily debugged. By contrast, finite element methods require host variational principles, e.g., “energy minimization,” if a model is developed from first principles. In some problems, such principles may not exist and subtleties arise. Very often, finite element models are run using commercial simulators to obtain quick answers; however, this compromises the portability and speed that custom numerical solvers allow. In this book, we develop finite difference algorithms from first principles that run quickly on all platforms.
1.3 Direct Current Focusing from Reservoir Flow Perspective

Developing new ideas, and particularly computer simulation code, is always simpler when use can be made of physical analogies. Here we demonstrate how laterolog-type mechanisms, where streamlines from a source point are focused by guard electrodes, can be viewed from a steady fluid mechanics or reservoir engineering perspective and programmed using the robust “dummy proof” approach discussed in the previous section. In order to develop ideas quickly, we omit the details of the well, simply identifying it with a “100” pressure specification and locating it in a rectangular reservoir.

Consider a point source isolated at the center of the reservoir. Obviously, lines of constant pressure are circular contours drawn about the source near the source, while the streamlines flow radially into the well. In our case, the pressure “100” falls monotonically to “0” in the farfield and cylindrical symmetry is found. Next suppose that this well is bounded at one side by a producing fracture open to the same “100” pressure and an identical fracture specification at the opposite side of the well. For this symmetric arrangement of well and fracture flows, it suffices to consider a half-plane formulation.

What are the governing equations? For pressure, the steady, isotropic Darcy model gives $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0$ in single-phase liquid flow. For its second derivatives, we infer from $\frac{\partial^2 y}{\partial x^2} = (y_{i+1} - 2y_i + y_{i-1})/\Delta^2$ developed previously that $(P_{i+1,j} - 2P_{i,j} + P_{i-1,j}) + (P_{i,j+1} - 2P_{i,j} + P_{i,j-1}) = 0$ where we have assumed equal $x$ and $y$ mesh widths, so that $P_{i,j} = (P_{i-1,j} + P_{i+1,j} + P_{i,j-1} + P_{i,j+1})/4$. This serves as our recursion formula for iterative pressure calculation. Chin (2002) shows that the “dummy proof” approach taken above applies to the present problem and provides meaningful convergence to actual solutions. This problem is solved with $P = 100$ at the source point subject to symmetry conditions at the top and bottom and $P = 0$ in the distant rectangular farfield.

Once convergence is achieved, streamlines can be obtained by post-processing the computed pressure field. In general, a direct integration of the streamline definition $\frac{dy}{dx} = \frac{\partial P}{\partial y}/(\partial P/\partial x)$ starting with the coordinates at the source is extremely inaccurate because of rapid gradients and turns – for these reasons, published current lines are often provided as approximate hand-drawn sketches. In planar problems with isotropic properties, steady streamlines can alternatively be constructed as orthogonals to lines of constant pressure; however, this procedure does not apply to axisymmetric or anisotropic problems, so that tracing regions of dependence and influence at first seems forbidding. It turns out that special algorithms we have developed involving the “streamfunction” $\Psi$ will prove useful in general streamline tracing. For now, we will illustrate its use for the planar isotropic problem at hand.

It turns out that, for such problems, $\Psi$ satisfies $\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$, identical in form to that for pressure. Thus, the same iterative algorithm applies. We discuss general boundary conditions later. At the fractures, located along
portions of \( x = 0 \), a required normal derivative \( \partial \Psi / \partial x \) is computed from the pressure solution \( \partial P / \partial y \), that is, the tangential derivative of pressure. Lines of constant \( \Psi(x,y) \) provide the required streamlines. These are easily plotted using contour plotting software packages that are widely available. The relations connecting Laplace equations \( \partial^2 P / \partial x^2 + \partial^2 P / \partial y^2 = 0 \) and \( \partial^2 \Psi / \partial x^2 + \partial^2 \Psi / \partial y^2 = 0 \) are available from the theory of complex variables: the existence of one equation guarantees that of the other, where \( P \) and \( \Psi \) are connected by the Cauchy-Riemann conditions \( \partial \Psi / \partial x = \partial P / \partial y \) and \( \partial \Psi / \partial y = - \partial P / \partial x \). Thus, by knowing the solution of either \( P \) or \( \Psi \), the solution to the other is available “free of charge.”

It is often claimed that streamfunctions are restricted to, in the context of petroleum engineering, planar, isotropic problems. However, this is not true. In fact, we will show how the general theory (which requires a background in complex variables) can be extended to applications which are anisotropic, nonplanar axisymmetric, or both, using simple calculus. For now, though, we merely illustrate its usage to demonstrate basic ideas in streamline focusing.

A schematic for the computational domain is given in Figure 1.3a. The Fortran source code used is given in Figure 1.3b, while computed solutions appear in Figure 1.3c. Streamlines are plotted in Figure 1.3d. We remind the reader that we have solved for a well in a rectangular (and not circular) reservoir, so that our results will differ slightly from those in reservoir engineering books. However, Figure 1.3d serves its purpose as it does illustrate how fluid elements move away from the well in all directions.

![Figure 1.3a. Pressure formulation and computational domain.](image-url)
OPEN(UNIT=7,FILE='RESERVOIR.DAT',STATUS='UNKNOWN')
OPEN(UNIT=8,FILE='GRAPH-DATA.DAT',STATUS='UNKNOWN')

Initialize solutions to zero everywhere
DO 100 I=1,11
DO 100 J=1,11
P(I,J) = 0.
SF(I,J) = 0.
100 CONTINUE

Find pressure field
DO 300 N=1,100
DO 200 I=2,10
DO 150 J=2,10
C Apply recursion formula ...
150 CONTINUE
C Update solutions along I=1, leave P=0 at other boundaries
C located at I=11, J=1 and J=11
DO 175 J=2,10
C Assume flow barrier (electrical insulator analogy) ...
P(1,J) = P(2,J)
C Except source at J=6 with pressure of 100 ...
P(1,6) = 100.
C Do not implement statement below yet ...
C IF(J.LE.4.OR.J.GE.8) P(I,J) = P(I,6)
175 CONTINUE
200 CONTINUE
300 CONTINUE

Find flow streamlines
DO 500 N=1,100
DO 400 I=2,10
DO 350 J=2,10
SF(I,J)=(SF(I-1,J)+SF(I+1,J)+SF(I,J-1)+SF(I,J+1))/4.
C Set antisymmetry condition SF = 0, but use small nonzero
C value so color plotter does not draw a "hole"
IF(J.EQ.6) SF(I,J) = 0.01
350 CONTINUE
400 CONTINUE
Figure 1.3b. Reservoir pressure and streamline calculation.

We now ask, what happens when we replace the “insulator” or symmetry condition (implemented by zero normal derivatives $\partial P/\partial x$ above) by one which specifies pressures identical to that in the well along most of the bottom right boundary in Figure 1.3d? This is the situation encountered in a LL3-type direct current logging tool – the “guard electrodes” usually used are analogous to the pressure-specified fractures in the present flow example. Note the convenience afforded by finite difference methods – only a single line of source code needs to be changed, which is highlighted in red below.
In fact, we replace

\[ P(1, 6) = 100. \]

\[
\text{C} \quad \text{Do not implement statement below yet ...}
\]
\[
\text{C} \quad \text{IF}(J \leq 4 \text{ OR } J \geq 8) \quad P(1, J) = P(1, 6)
\]

by the following “uncommented” code

\[
P(1, 6) = 100.
\]
\[
\text{IF}(J \leq 4 \text{ OR } J \geq 8) \quad P(1, J) = P(1, 6)
\]

**Figure 1.3d.** Streamlines from unfocused source point.

Comparison of Figures 1.3d (with lines emanating from a point) and 1.4c (with lines originating from a line boundary) clearly shows how “guard fractures” reduce the streamline divergence encountered in well-alone problems, although by no means have we attempted to optimize this effect here (physical dimensions are chosen for printed page display purposes only).

* Software reference: reservoir-1.for.
Our purpose in this section is to illustrate close physical analogies between fluid pressure and electrical fields, in particular, “fluid fracture focusing” and “laterolog guard electrode focusing,” common elements in computational methods, as well as ideas in finite difference analysis and iterative solutions. Note that we have implemented a “point relaxation” algorithm here, using a simple algorithm with $P_{ij} = (P_{i-1,j} + P_{i+1,j} + P_{i,j-1} + P_{i,j+1})/4$ applied point-by-point. This formula is easily programmed and useful in exploring ideas quickly. In the methods of Chapters 2, 3 and 4, we will, in fact, employ much faster “successive line over-relaxation” methods which converge with rapid speed and numerical stability. Finally, we have laid the groundwork for analogies connecting steady fluid pressure fields to electrical voltage applications, allowing us to use efficient methods originally developed in computational fluid dynamics for DC laterolog and pad micro-resistivity design and log interpretation.