Harnessing Bistable Structural Dynamics
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For Vibration Control, Energy Harvesting and Sensing

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This book describes a recent perspective that seeks to strategically harness the unique dynamics of bistable structural systems for engineering advances, with focus on the three technical areas of vibration control, energy harvesting, and sensing.

When a structural system exhibits two statically-stable configurations, it is said to be bistable. This class of structures has been employed in various mechanical, civil, marine, and aerospace engineering applications for many years. The bistable components include arches and post-buckled beams/columns, panels or shells having a shallow curvature, and curved microbeam or dome diaphragm transducers, to name just a few examples. In many historical assessments, it was undesirable for the bistable structure to “snap” to the second state of static equilibrium – a phenomenon referred to as snap-through – because the consequence might be unfavorable to the health and performance of the overall engineered system. Thus, the structures or materials were used in ways to avoid static or dynamic loading (e.g., pressure on a shell) that could cause the bistable system to switch from the original, stable configuration to the other stable equilibrium (e.g., an inverted shell).

It is from such a perspective that the focus of this book departs. Recently, researchers have been challenged to reconsider bistable structural systems within a variety of emerging engineering contexts. Many scientists and engineers have discovered and explored the dynamics of bistable structures that may be deliberately exploited to the advantage of certain applications. Innovative ideas have been proposed to intelligently induce snap-through behaviors such that the performance of the overall system is enhanced and/or new functionality is realized. This new spirit of engineering system development is the foundational viewpoint of this book: harnessing bistable structural dynamics.

The new ideas have been found to be well-suited for many applications across a variety of engineering disciplines. Among them, researchers have particularly investigated the exploitation of bistable structural dynamics in the areas of (i) vibration control, (ii) vibration energy harvesting, and (iii) sensing and detection. In the first area, the dynamics of bistable devices integrated with the structural system are used so as to isolate, dissipate, or reactively attenuate the input energies that excite the system. Depending on the performance objective, the multi-faceted dynamics of the bistable members are strategically employed to best control the vibrations and improve the operational integrity of the system.
The aim of energy harvesting is to electromechanically convert ambient vibrations into usable electrical power resources. To this end, maintaining persistent snap-through behaviors is a common aim because of the large dynamic mechanical and electrical response amplitudes that they induce in the energy harvesting systems. Hence, the energetics of snap-through promotes a significant potential for energy conversion. In the context of sensing and detection, the ability to recognize small changes in monitored parameters, which represent time-varying system characteristics, is critical in providing the earliest indicator of structural change. Thus, by harnessing the sudden transition between low amplitude oscillations around a stable equilibrium to energetic snap-through vibrations spanning the static equilibria, bistable dynamics-based detection strategies have been found as promising, novel approaches well-suited for a variety of sensing contexts.

The recent interest in harnessing bistable structural dynamics for engineering advances in vibration control, energy harvesting, and sensing has led to a flourishing research development that ranges from rigorous theoretical formulations to exploratory experimental studies. The comprehensive investigations have drawn a variety of conclusions regarding the exploitation of bistable dynamics that may best promote the aims of the respective technical area and applications, and the continued active research engagements suggest that much is left to be discovered.

The insights on the effective exploitation of bistable dynamics to enhance the aims of the aforementioned three technical areas are currently scattered amongst numerous scientific publications and proceedings, including a selection of works by the authors. To derive the greatest benefits from the assorted findings, there is a need to consolidate the results and, with the organized evidence, to derive coherent conclusions that are informed by the broad range of research. This book seeks to meet this need through the presentation of extensive topical reviews and detailed case studies which complement the overall conclusions. For the benefit of the ongoing efforts, this book also seeks to identify the emerging areas and outstanding needs that require future attention before the exploitation of bistable dynamics meets its fullest potential for vibration control, energy harvesting, and sensing and detection.

A large body of scientists and engineers is represented among these technical areas. The approaches and conclusions described in detail in this book will inspire researchers investigating the exploitation of bistability in engineering systems, as well as enlightening a broad range of readers interested in vibration control, energy harvesting, and sensing to the attractive opportunities that may be engendered via harnessing bistable structural dynamics. By reviewing the developments and presenting specific studies as examples, a goal of this book is to provide an accessible avenue for a large readership to an appreciation of the recent findings from the engineering objective lens. We assume that readers have a college level undergraduate education in an engineering curriculum including engineering mathematics and structural dynamics or vibrations. Exposure to more advanced topics such as electromechanical systems, nonlinear dynamics, and stochastic vibrations is beneficial but not essential. Additionally, this book focuses on all-mechanical, electromechanical, or all-electrical bistable systems,
which may be realized or adequately modeled as one-dimensional systems; such platforms represent a large number and wide variety of bistable devices that have been investigated. For readers interested in more details on a specific topic or concept, references to many books and papers are provided throughout the book.

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Background and Introduction

This chapter provides a background to the common realizations and dynamics of bistable structures which have recently been harnessed to advance the aims of the three technical areas of interest in this book: vibration control, vibration energy harvesting, and sensing and detection. The structural forms and dynamical behaviors of bistable systems are first introduced to demonstrate the broad versatility of design and response which are commonly exploited, and to highlight the dynamics which are enabled via leveraging bistability. Then, two aspects of bistable structural dynamics are identified as trademark features which serve as common rationales for exploiting bistability in engineering applications. These aspects are elaborated upon through concise descriptions of example implementations within the technical areas considered here. Finally, an outline of the remaining chapters is provided.

1.1 Examples of Bistable Structures and Systems

To introduce the essential static characteristics of a bistable structure, a schematic of an example one-dimensional, mechanical bistable system is shown in Figure 1.1. To set a clear convention, hereafter the terms bistable structures and systems will be used interchangeably, irrespective of whether the object considered is all-mechanical, electromechanical, or all-electrical. For the bistable system shown in Figure 1.1, two identical springs of undeformed lengths $l_o$ connect a lumped mass to a surrounding frame of span $2d$. It is assumed that all displacements of the system are in a horizontal direction, such that the frame motions $z$ and mass displacements $X$ move along one axis. When the undeformed spring length is less than or equal to half of the span of the frame, $l_o \leq d$, the system is monostable and the mass will come to rest at the zero displacement position, $X = 0$. In contrast, when the undeformed spring length is greater than half of the span of the frame, $l_o > d$, the springs exert a force on the mass such that the mass cannot be easily maintained in the central configuration. The zero displacement configuration is unstable, while two stable positions of the mass are adjacent (and, here, symmetric) to the central, unstable state. As a result of the geometric design condition $l_o > d$, the mass-spring and frame system is said
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Figure 1.1 Schematic of a bistable system composed of springs, mass, and frame. Here \( l_0 > d \) which makes the central configuration \( X = 0 \) an unstable position of static equilibrium.

The stable equilibria of the structure are shown to be configurations of the mass such that the displacements are \( X = \pm a \).

Figure 1.2a,b illustrates the force, \( F(X) \), and potential energy, \( U(X) \), of the bistable system, respectively, as functions of the mass displacement. Potential energy is determined by \( U = \int F \cdot dX \). In this example, the bistable structure is symmetric and the only forces resisting the horizontal mass displacements are due to the identical springs. Figure 1.2a shows that the total restoring force in the \( X \)-axis is zero when the inertial mass is positioned at any of the equilibria. On the other hand, Figure 1.2b shows that the potential energy is locally maximized at the unstable central configuration of the inertial mass \( X = 0 \), while the adjacent, stable equilibria at \( X = \pm a \) locally minimize the potential energy of the system. Therefore, based on the principle of minimum total potential energy, disturbances to the inertial mass when it is originally positioned at the unstable equilibrium will propel the mass towards one of the stable equilibria.

An instructive analogy is that of a ball on a terrain with elevation profile shaped like the potential energy plot in Figure 1.2b. While situated precisely at the peak of the central hill (at \( X = 0 \)), the ball will remain stationary even though the gravitational potential energy is high. But if given a slight perturbation, the ball will roll into the nearest valley where it settles into the displacement position which minimizes potential energy (specifically, gravitational potential energy in this analogy).

By Hooke’s law, the stiffness of a spring element is determined by the spatial derivative of the restoring force, \( \frac{dF}{dX} \). Considering the total spring force profile at the unstable equilibrium in Figure 1.2a, it is apparent that the bistable force profile is characterized as having a negative stiffness for this mass position. In contrast to a spring which resists the motion of the mass in a given direction, a spring exhibiting negative stiffness over a range of displacements will assist the motion of the mass. As a result, the small perturbation to the inertial mass when precisely positioned at the unstable equilibrium will lead the bistable spring to propel the mass away from the central location to one of the stable system configurations.

All the bistable systems considered in this book share these fundamental, static characteristics illustrated using the mechanical example in Figure 1.1. In fact, the existence of two statically-stable equilibria configurations and one unstable configuration make it straightforward to identify bistable structures or systems. However, the type of geometrical constraints exemplified in the mechanical
device shown in Figure 1.1 are just one possible approach to effect bistability. For the technical areas of vibration control, energy harvesting, or sensing, numerous and diverse methods are employed to realize bistability.

To sample the many approaches, Figure 1.3 shows recently investigated engineering systems that utilize one of the various methods to effect bistability. In Figure 1.3a, modules of series- and parallel-assembled double-beam units are compressed within a housing support near the threshold of buckling. When the

**Figure 1.2** Dependence of (a) spring force and (b) stored potential energy on the displacement position of the inertial mass.
Figure 1.3 (a) Damping module of buckling beams arranged in series and parallel within a housing frame for energy attenuation between the ends of the module. (b) Bistable device for vibration isolation of suspended top bearing mass. (c) Cantilevered beam with piezoelectric PVDF patches and magnetic attraction induced bistability for energy harvesting. (d) Bistable circuitry attached to host beam structure via piezoelectric transducer for sensing structural change.
Background and Introduction

A statically compressed modular structure is excited with periodic dynamic loads, the modules provide high damping due to the transitions among the various stable topologies. Such transition phenomena result in a large dissipation of the input energy according to the number of state changes that occur. In Figure 1.3b, a two degrees-of-freedom (DOF) vibration suspension concept is shown [1]. Here, bistability is effected between the DOF (the moving frame and top bearing mass) via geometric relations, comparing the length of the interfacing spring to the distance between the frame and top bearing mass. Due to the activation of unique bistable dynamics, the investigations of the two DOF suspension system uncover an exceptional reduction of motion transmissibility as compared to the counterpart two DOF linear suspension [1]. Note that the geometrical constraints in the two DOF architecture shown in Figure 1.3b which lead to bistability between the two moving bodies are similar to those constraints employed in the translational single DOF bistable system example shown in Figure 1.1.

In some vibration energy harvesting applications [2], a composite plate with attached piezoelectric patches is made bistable through the generation of a static stress for the flattened plate configuration, such that the plate maintains one of two stable equilibria shapes having finite curvatures. There are numerous design and fabrication parameters which may be adjusted to tailor the two stable plate curvatures, including the composite material layer selection, lay-up order, layer relative rotations, and thermal conditions under which the layers are stacked and cured [2,3]. When the plate is excited at its center by external vibrations, the energetic snap-through actions of the plate from one stable curvature to the next greatly strain piezoelectric materials attached to the plate surface. In consequence, the input vibrational energy is converted by the piezoelectric material to an oscillating flow of current which can be exploited for energy storage purposes (e.g., battery charging) or conditioned for direct utilization as a supply for low-power microelectronics. In another vibration energy harvesting investigation, bistability is realized using a combination of elastic and magnetic restoring forces on a cantilever beam, as shown in Figure 1.3c [4]. Motions of the ferromagnetic cantilever beam tip are resisted via the beam’s inherent elasticity but the attractive influences of the magnet pair are tuned so as to draw the beam away from the original cantilevered configuration. The piezoelectric PVDF (polyvinylidene fluoride) patches attached to the beam surface are significantly strained as ground motions excite the cantilever base and cause the beam to oscillate between the stable beam positions. Similar to the case of the bistable plate, the charge generation on the piezoelectric material electrodes may then be harnessed for energy harvesting purposes. The broad frequency bandwidth and high sensitivity at low frequencies of snap-through dynamics set bistable energy harvesters apart from other harvester platform designs and strongly justify the recent attention given to bistable structures for energy harvesting applications [5,6].

Bistability may also be effected via electromechanical or electrical means. For instance, due to electrostatic actuation including bias and oscillating voltages, a microbeam having an initial curvature may be excited to oscillate between two stable beam shapes [7]. In this example, the beam may or may not be inherently bistable due to elastic influences alone. Nevertheless, the electrostatic forces provide a means to ensure bistability and then to actuate the beam between the
two stable shapes. The applications for micro- and nanoscale bistable systems are numerous, and include their implementation as electromechanical signal filters, switches, actuators, or as novel mechanical memory elements, to name only a few of the functions for which they have been explored [7–10]. Additionally, the circuit schematic shown in Figure 1.3d realizes bistability simply by the circuit design [11]. In the absence of an input voltage $V_i$, the output voltage $V_o$ across the capacitor $C$ will settle on a finite positive or negative value. Bistable circuitry is a well-established tool in the study of many physical sciences [12], but the exploitation of such circuit designs to advance the aims in engineering contexts is an emerging area of technical interest. For example, when the circuit input is connected to a structural transducer, such as a piezoelectric patch as shown in Figure 1.3d, the activation of the high amplitude snap-through response of the bistable circuit output voltage $V_o$ can be harnessed as an indicator of change in the structural system. By this strategy, one may monitor shifts in important parameters, such as a reduction in stiffness which could indicate damage. Thus, small variations in structural responses of the beam are tracked using large changes in bistable circuit voltage dynamics. Through a relation between the changing structural model parameters and the critical conditions that activate the circuit snap-through dynamics, one realizes a novel pathway for robust and sensitive detection of change [11].

As exemplified by the cross-disciplinary sample described above, the bistable systems considered in this book represent diverse mechanical, electromechanical, and electrical platforms that span a vast range of length scales. Yet, regardless of the specific manifestation and inducement of bistability, the existence of two statically-stable configurations and one unstable configuration is the common denominator for all bistable systems. However, the mechanics of bistable structures are historically-established science [13]. Thus, following the previous introduction of the essential static features of bistable systems and how they have been realized in structural or system forms, the next section introduces the numerous, characteristic dynamics of bistable structures. These are the behaviors which have been recently harnessed to the advantage of the technical areas of interest in this book.

### 1.2 Characteristics of Bistable Structural Dynamics

Regardless of the method by which bistability is effected, each platform shown in Figure 1.3 exemplifies dynamics common to all bistable systems. Moreover, these dynamics have particularly unique manifestations as compared to the responses exhibited by linear or monostable nonlinear systems, which justifies the recent research attention given to bistable structures in the technical areas of interest in this book. The following sections review the distinct dynamic characteristics of bistable structures and provide illustrative and exemplary plots of the behaviors. Before elaborating on the unique dynamical features, a brief description is provided below regarding the means to model and predict such dynamics and the interpretations of the parameters that have been employed to generate the illustrative results.
The illustrative plots presented in the following sections are computed as the 
dynamic solutions to the governing equation of motion of a bistable system. 
The analytical and numerical techniques to solve the equation, so as to generate 
the following plots, are described in detail in Chapter 2. A conventional governing 
equation construction is used which is known to accurately model the dynamics 
of numerous bistable structure realizations that have been examined across a 
wide range of sciences and engineering fields. The equation may be expressed in 
a normalized form as

$$\frac{d^2x}{d\tau^2} + \gamma \frac{dx}{d\tau} - x + \beta x^3 = p \cos \omega \tau$$  \hspace{1cm} (1.1)$$

where $x(\tau)$ is considered to be the non-dimensional, generalized displacement of 
the bistable structural system which is a function of non-dimensional time $\tau$; $\gamma$ is 
a damping factor; $\beta$ is a degree of nonlinearity; $p$ is the excitation level. The term 
$\omega$ is the excitation frequency which is normalized according to the system’s natu-
ral frequency computed using the magnitude of the linear stiffness and the mass 
of the non-normalized system [14]; thus, $\omega = 1$ indicates a resonance-like excita-
tion, although the analogy is imperfect. The transient solution to Eq. (1.1) also 
dePENDs on the initial conditions of displacement and velocity, $x(0)=x_0$ and 
$\dot{x}(0)=\dot{x}_0$, respectively, where an overdot indicates differentiation with respect to 
the normalized time, $d/d\tau$.

Although the following examples describe the characteristic dynamics 
computed from Eq. (1.1) in regards to a bistable mechanical system response, in 
which case terms such as displacement and velocity will be used, it is noted for 
completeness that Eq. (1.1) also models bistable systems realized in electrical 
domains. Thus, the coordinate $x$ may represent a normalized voltage or current 
and the corresponding system and excitation parameters $(p, \omega, \gamma, \beta)$ may be 
described in terms of the appropriate counterpart electrical components.

The unforced, static solution to Eq. (1.1) leads to the determination of the 
equilibria configurations of the bistable system. In this case, one solves an 
equation expressed by

$$-x + \beta x^3 = 0$$  \hspace{1cm} (1.2)$$

The solutions to the polynomial in Eq. (1.2) are $x=\pm1$ and 0. Further mathematical treatment of these results, detailed in Chapter 2, reveals that the 
equilibras at $x=\pm1$ are stable whereas the equilibrium at $x=0$ is unstable.

Using the analytically or numerically computed solutions to the governing 
equation and by taking into consideration the bistable system equilibria configu-
rations, the following sections elaborate on the characteristics of bistable struc-
tural dynamics. While some of the general dynamical features are exhibited by 
other types of nonlinear systems, the unique ways in which they are realized by 
bistable structures are detailed to highlight the important distinctions. For brevity, 
the following sections describe and elucidate the plotted results while the 
figure captions detail the specific parameter set combinations and relevant 
computational aspects to generate the data.
1.2.1 Coexistence of Single-periodic, Steady-state Responses

In contrast to linear systems, when driven by single-frequency excitations many nonlinear systems may potentially exhibit one or more forms of steady-state response that coexist as a result of the same excitation conditions. These vibrations occur primarily at the excitation frequency. The coexistence of dynamic responses means that for a set of design parameters and prescribed harmonic excitation level and frequency, the nonlinear system may undergo different dynamics in time, often one of two distinct harmonic response amplitudes. The specific dynamic that occurs is dependent upon the system initial conditions at the starting time of the single-frequency excitation.

In the case of bistable systems, there are numerous types of single-periodic, steady-state dynamics which may coexist. First, one may classify the dynamics into two regimes: intrawell oscillations which occur around one of the two stable equilibria, or interwell oscillations which vibrate across the unstable equilibrium twice per excitation cycle. Because the potential energy profile of bistable systems, shown in Figure 1.2b, is conventionally referred to as the double-well potential, the terms intrawell and interwell denote that the oscillations of the inertial mass remain confined to one of the local wells of potential energy or cross back-and-forth between them, respectively. One may then separate these two dynamic regimes into low and high amplitude versions of the intra- and interwell dynamics. As a result, there are four forms of single-periodic, steady-state dynamics exhibited by a bistable structure, some of that may coexist for the same excitation and system design parameters.

As examples of the two forms of intrawell dynamics, Figure 1.4a presents numerically computed displacement responses using Eq. (1.1). Here, the different initial conditions lead to unique steady-state dynamics while all other parameters remain the same. It is clear that the intrawell dynamics, plotted as the black or gray solid curves, oscillate around one of the two stable equilibria shown as thick dotted lines. Although not able to be observed using one example, the low and high amplitude oscillations, as predicted from Eq. (1.1), may occur around either stable equilibrium. In the example presented in Figure 1.4a, the initial conditions are such that the high amplitude oscillations vibrate around the stable equilibrium at $x = -1$ while the low amplitude responses occur around $x = +1$.

Figure 1.4b plots the displacement amplitude across all frequencies as a consequence of the excitation at frequency $\omega$. The data are computed using the fast Fourier transform (FFT) of the displacement time series shown in Figure 1.4a; the final 50 of 100 excitation periods are used in the computation. The spectra shown in Figure 1.4b plainly indicate that the different intrawell dynamics are distinct in their amplitude at the excitation frequency, shown as the circle points. The input energy at the frequency of $\omega$ is apparently diffused to other harmonics; further discussion of this feature is deferred until Section 1.2.6.

Figure 1.5a plots an example in which the low and high amplitude interwell dynamics are found to coexist for the same excitation frequency and level. The interwell behaviors are distinct from the intrawell responses in that the prior oscillate across the unstable equilibrium position, which was identified above to be the normalized position of $x = 0$. Apart from the difference in the amplitudes
Figure 1.4 (a) For an excitation frequency at which both the low and high intrawell dynamics coexist, the time series of the displacement responses are presented. The thick dotted lines indicate the statically-stable equilibria. (b) The displacement magnitude spectral responses of the low and high amplitude intrawell dynamics are shown as computed from the time series data plotted in (a). The energy diffusion to other harmonics of the excitation frequency is apparent and distinct between the response forms. In (a,b), the parameters are $(\rho, \omega, \gamma, \beta) = (0.1, 1.1, 0.09, 1)$, with initial conditions $(x_0, \dot{x}_0) = (1.0041, 0.0961)$ for the low amplitude oscillations and $(2.2858, -0.9302)$ for the high amplitude dynamics.
Figure 1.5 (a) For an excitation frequency at which both the low and high interwell dynamics coexist, the time series of the displacement responses is presented. (b) The displacement magnitude spectral responses of the low and high amplitude interwell dynamics are shown as computed from the time series data plotted in (a). The energy diffusion to other harmonics of the excitation frequency is apparent and distinct between the response forms. In (a, b), the parameters are \((p, \omega, \gamma, \beta) = (8, 2.5, 0.09, 1)\), with initial conditions \((-1.9297, -0.0944)\) for the low amplitude interwell responses and \((x_0, \dot{x}_0) = (-2.9443, -0.1791)\) for the high amplitude interwell vibrations.
of the two dynamic forms, it is clear that the responses are approximately 180 degrees out-of-phase. Although not shown in Figure 1.5a, the high amplitude snap-through dynamics occur with phase lags no greater than 90 degrees with respect to the excitation, while the low amplitude interwell oscillations lag the excitation typically by around 180 degrees. The unique phase relationships enable a practical means to distinguish the two forms of interwell dynamics during experimentation in which case they both oscillate back-and-forth across the unstable equilibrium but clearly have different phase relationships with respect to the excitation. Similar to the intrawell dynamic spectral features, Figure 1.5b shows that the two interwell dynamics of bistable systems are different in terms of response amplitude at the excitation frequency of $\omega$ (shown as the circle points) as well as in terms of the energy diffusion to other integer multiple harmonics of the excitation.

By evaluating Eq. (1.1) over a broad range of excitation frequencies using an analytical strategy, Figure 1.6a plots the normalized displacement amplitudes of a bistable system (shown as solid curves) as compared to the linear system governed by the corresponding normalized governing equation of motion $\ddot{x} + \gamma \dot{x} + x = p \cos \omega t$ (dash-dot curve). Additionally, Figure 1.6a shows thin gray curves which represent unstable dynamics of the bistable system. In other words, these are responses which are mathematical solutions to the governing equations but which are not physically realizable dynamics.

The amplitudes of the bistable system intrawell dynamics show resonance-like frequency dependence similar to the linear system, but the prior are “bent” towards lower frequencies. This is an indicator of a “softening” type of nonlinearity, further details of which are provided in later chapters of this book. It is also observed in Figure 1.6a that the high amplitude interwell dynamics exhibit large displacement magnitudes over a broad range of frequencies. In fact, the large amplitude snap-through dynamics are predicted to occur even for a progressively vanishing excitation frequency, in other words as $\omega \to 0$. This prediction may be intuitively appreciated through a basic consideration of the mechanics involved. Note that as the frequency approaches zero, the motions of the linear system mass are confined to how greatly the linear spring may be statically deformed for the applied load. In contrast, as the excitation frequency approaches quasi-static conditions, the displacements of a bistable system mass will still undergo a large stroke from one stable equilibrium to the other so long as the system is “pushed” with a large enough load level to induce a snap-through dynamic. As such, it may be stated that snap-through is a non-resonant dynamic because there is no reliance on resonance-like features to excite the energetic motions. The exceptional sensitivity to broadband and low frequency inputs is a repeatedly exploited feature of bistable systems in the technical areas described in this book.

A major difference between the responses of the bistable structure and the counterpart linear oscillator are the bifurcations in the bistable system responses, observed in consequence to variation in the excitation frequency, $\omega$. A bifurcation is a large qualitative change in a system state (static and/or dynamic) due to an infinitesimally small change in an individual parameter (whether internal or external to the system) [15,16]. Three bifurcations in the dynamics of the bistable
system are apparent in Figure 1.6a. The first is seen to occur where the low amplitude intrawell oscillations destabilize if the excitation frequency exceeds $\omega \approx 1.1$. While the linear system exhibits smooth and continuous variation of displacement amplitudes as the excitation frequency is varied, the bistable system dynamics will undergo a sudden jump from low (point B) to high amplitude

![Diagram](image)

**Figure 1.6** Depending on the excitation level, the single-periodic, steady-state response of a bistable system may exhibit non-unique dynamic forms. (a) A high amplitude interwell and both low and high amplitude intrawell responses may occur depending on the excitation frequency. Here, the parameters are $(p, \gamma, \beta) = (0.18, 0.09, 0.7)$. The corresponding linear system stationary response computed using the counterpart equation $\ddot{x} + \gamma \dot{x} + x = p \cos \omega t$. (b) A greater excitation level than that used in (a) may induce low or high interwell responses or an intrawell response. Here, the parameters are $(p, \gamma, \beta) = (8, 0.09, 1)$. (c) Responses of softening and hardening Duffing oscillators contrasted against the corresponding linear system. Here, the parameters are $(p, \gamma, \beta) = (0.18, 0.09, 0.7)$. The thin gray curves indicate unstable response forms.
intrawell oscillations when the frequency exceeds \( \omega \approx 1.1 \). Another bifurcation is evident in the high amplitude intrawell response if the excitation frequency decreases below \( \omega \approx 0.95 \); if the frequency is swept below this critical point, the system is predicted to undergo either low amplitude intrawell oscillations or snap-through. The final bifurcation featured in the plot occurs in this bandwidth when the bistable structure is initialized in the high amplitude interwell dynamic state and the excitation frequency exceeds \( \omega \approx 1 \). This latter bifurcation from an exceptionally large amplitude response (snap-through) to an intrawell dynamic is a distinct characteristic of bistable systems, and the significant response amplitude difference involved can be favorably exploited in several scientific and engineering contexts. For the previous two bifurcations where three, coexistent steady-state responses are reduced to two, the initial conditions at the time of bifurcation activation determine into which dynamic response the bistable system settles.

Figure 1.6b shows results predicted using different system and excitation parameters, and indicates that the low and high interwell dynamics may coexist near an excitation frequency of \( \omega \approx 2.5 \). It is found that the low amplitude interwell oscillations occur across a narrow bandwidth of excitation frequencies. The trend for the high amplitude interwell dynamics is a nearly linear increase in displacement magnitude as the excitation frequency progressively increases. This trend continues up to a critical frequency at which the single-periodic snap-through behaviors become destabilized: a similar bifurcation feature as that observed in Figure 1.6a. Finally, the results plotted in Figure 1.6b show that a high amplitude intrawell dynamic is possible for excitation frequencies greater than \( \omega \approx 4 \). The phase relationship of these intrawell oscillations with respect to the excitation is the characteristic factor to identify the dynamic regime.

The analytically predicted results in Figure 1.6 provide a clear example of the coexistence of numerous, steady-state dynamic forms of bistable structures as well as the possibility for sudden transitions in the dynamic regime due to small
changes in system or excitation parameters. The multiplicity of the response forms, characterized by their unique displacement amplitudes and excitation phase lags, makes bistable structural dynamics particularly distinct from those of other linear and nonlinear systems. Indeed, snap-through is truly a distinguishing feature of bistable systems.

It is worthwhile to compare the steady-state, single-periodic dynamics of bistable structures to those of the traditional hardening and softening Duffing oscillators, which are regularly implemented as archetypal models for engineering systems in the real-world [17,18]. These oscillators are governed by the equations

\[ \ddot{x} + \gamma \dot{x} + x \pm \beta x^3 = p \cos \omega t \]

where the positive (negative) nonlinear restoring force term denotes hardening (softening) behavior. It is evident that this governing equation is similar to Eq. (1.1) for the bistable system, repeated here for convenience:

\[ \ddot{x} + \gamma \dot{x} - x + \beta x^3 = p \cos \omega t. \]

On the other hand, the resulting static and dynamic behaviors are considerably distinct. Note that the only static solution to the equations \( \ddot{x} + \gamma \dot{x} + x \pm \beta x^3 = p \cos \omega t \) is \( x = 0 \), indicating that the Duffing systems have just one static equilibrium and are thus termed monostable. Using the methods presented in Chapter 2, the steady-state dynamics of the forced Duffing equations may be approximately solved and representative results are shown in Figure 1.6c using parameters of \( (p, \gamma, \beta) \) identical to those used in Figure 1.6a. Here again, the corresponding linear system dynamics are also shown, and light gray curves denote the unstable behaviors of the Duffing systems. Figure 1.6c illustrates that the hardening Duffing oscillator exhibits a “bending” or “leaning” of the resonance curve to higher frequencies with respect to linear system trends. In contrast, the softening Duffing oscillator shows response amplitude curves that lean to lower frequencies. Figure 1.6c indicates that the Duffing systems may exhibit dynamic bistability, in that more than one steady-state oscillation regime may occur for a given excitation frequency (similar results are obtained by varying the other parameters \( \beta, \gamma, \) and \( p \)). In comparing the different responses of bistable and these monostable Duffing oscillators between Figures 1.6a and 1.6c, respectively, it is evident that the intrawell oscillation regime of the bistable system is comparable to the softening Duffing response, while the Duffing oscillators exhibit no such dynamic behavior as snap-through since the Duffing systems do not possess static bistability. Truly, it is static bistability that enables snap-through and draws a dramatic distinction in the resulting, potential dynamic behaviors of bistable systems which the following sections continue to detail.

### 1.2.2 Sensitivity to Initial Conditions

It was shown in Figures 1.4 and 1.5 that system and excitation parameters as well as the initial conditions influence the resulting dynamic state of a bistable system. In fact, this is a characteristic of all nonlinear systems which may exhibit multiple coexistent steady-state dynamics. On the other hand, due to the two stable equilibria, there are important implications of the initial condition sensitivity demonstrated by bistable structures. A clear example of this feature is shown in Figure 1.7 where a difference in initial normalized displacement of \( 10^{-6} \) leads to completely different results as time elapses. It is seen that, based on the